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The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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METHODS OF PROOF IN LINEAR ALGEBRA

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1. Introduction. In a problem in linear algebra, one has, generally speaking, a definite field of coefficients given in advance. Often the first question is to find an appropriate linear space so that the problem reduces to one concerning linear transformations on that space. This in turn may be handled in one of several ways. For example the older approach is to consider the space of n-tuples $(a_1, \cdots, a_n)$ as the linear space, in which case one is committed to a definite coordinate basis $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$, $\cdots$. The modern approach is to use invariant basis-free methods. Of course bases cannot be completely discarded. In many cases the essential finite dimensionality of the situation is exploited precisely by using a basis. In such problems the best technique is to select a basis which most fits the given data. At all times however one should be perfectly willing to change the basis if a more natural choice appears, to change the space itself if that can help matters, and even to change the field of coefficients for a more useful one.(1)*

In this article we shall emphasize the idea of abandoning what has already been selected for something better. We shall give proofs of several known theorems in linear algebra which will show the various possibilities. These proofs, while probably not the shortest or most elegant possible, at least seem to us to bring out the reasons for the validity of the theorems they demonstrate. Some of these theorems have perhaps not appeared heretofore to be as elementary as they really are.

In Section 2 we give the familiar characterization of automorphisms of a total matric algebra. In the proof one sees at the outset that one basis is as good as another, but after a basis is fixed one soon sees that there is another definite one to be reckoned with. In Section 3 we give a result on fields of matrices. It is pretty clear in this situation what the right linear space is, but the point of the proof is to consider the space over a new field of coefficients. The remaining sections are centered around the idea of superdiagonalization, which is a technique of selecting a basis naturally suited to a problem. This is introduced in Section 4, applied to a single transformation in Section 5, to a commutative set of transformations in Section 6, and to a nil associative algebra in Section 8; some basic concepts of linear associative algebra theory are given in Section 7. A combination of the results of Sections 6 and 8 is given in Section 9. One of the main tools in superdiagonalization is the fact that if M is a subspace of a space L which reduces a given transformation, then that transformation induces transformations on both M and on the quotient or coset space L/M. Thus attention

* See the notes at the conclusion of the paper.

is shifted to other linear spaces. In order to study this further we introduce the idea of a Lie algebra in Section 10 and then prove the basic theorems of Engel and Lie in Sections 11–14. A final remark on characteristic roots is given in Section 15.

2. Automorphisms of matrices. We begin with the following theorem.[2]

THEOREM 1. *Let $\mathfrak{M}_n$ be the algebra of all $n \times n$ matrices over a field k and let σ be an automorphism of this algebra. Then σ is an inner automorphism, that is, there exists a non-singular matrix P such that $\sigma(A) = PAP^{-1}$ for all A.*

First of all, we turn this into a linear space problem by noting that $\mathfrak{M}_n$ is isomorphic to the algebra of all linear transformations on any n-dimensional linear space L over k. Let us restate the proposition this way.

THEOREM 1′. *Let $\mathfrak{A} = \text{Lin } (L)$ be the algebra of all linear transformations on an n-dimensional linear space L over a field k and let σ be an automorphism of $\mathfrak{A}$. Then there exists a non-singular linear transformation $\mathbf{P}$ of L such that $\sigma(\mathbf{A}) = \mathbf{PAP}^{-1}$ for all $\mathbf{A}$ in $\mathfrak{A}$.*

The proof consists of little more than a precise formulation of the possible isomorphisms between $\mathfrak{A} = \text{Lin } (L)$ and $\mathfrak{M}_n$ plus the observation that one is not, after all, forced to stick to any particular one of these isomorphisms.

We select *any* basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ of L. Each $\mathbf{A} \in \text{Lin } (L)$ yields a matrix $A = \|a_{ij}\|$ according to the equations

$$\mathbf{v}_i\mathbf{A} = \sum_{j=1}^{n} a_{ij}\mathbf{v}_j,$$

and this correspondence $\mathbf{A} \leftrightarrow A = \|a_{ij}\|$ is the kind of isomorphism we deal with between Lin (L) and $\mathfrak{M}_n$. The basic matric units E_{ij} (with 0 everywhere except for 1 in i-th row and j-th column) correspond to transformations $\mathbf{E}_{ij}$ defined as follows:

$$\mathbf{v}_i\mathbf{E}_{jk} = \delta_{ij}\mathbf{v}_k$$

on the basis of L we have selected. They satisfy the important algebraic relations

$$\mathbf{E}_{ij}\mathbf{E}_{kl} = \delta_{jk}\mathbf{E}_{il}. \tag{1}$$

Here is the crux of the matter.

LEMMA. *Let $\mathfrak{A} = \text{Lin } (L)$ as before and let $\mathbf{F}_{ij}$ be n^2 elements of $\mathfrak{A}$, not all 0 and satisfying the relations*

$$\mathbf{F}_{ij}\mathbf{F}_{kl} = \delta_{jk}\mathbf{F}_{il}.$$

Then there exists a basis $\mathbf{w}_1, \cdots, \mathbf{w}_n$ of L such that

$$\mathbf{w}_i\mathbf{F}_{jk} = \delta_{ij}\mathbf{w}_k. \tag{2}$$

Proof. Some $\mathbf{F}_{ij} \neq 0$ and $\mathbf{F}_{ij} = \mathbf{F}_{i1}\mathbf{F}_{11}\mathbf{F}_{1j}$, hence $\mathbf{F}_{11} \neq 0$. We select $\mathbf{w} \in L$ so that $\mathbf{w}\mathbf{F}_{11} \neq 0$ and set

$$\mathbf{w}_1 = \mathbf{w}\mathbf{F}_{11}, \mathbf{w}_2 = \mathbf{w}\mathbf{F}_{12}, \cdots, \mathbf{w}_n = \mathbf{w}\mathbf{F}_{1n}.$$

These vectors $\mathbf{w}_1, \cdots, \mathbf{w}_n$ form a basis of L; all we need show is that they are linearly independent. But a dependence relation $\sum a_i \mathbf{w}_i = 0$ yields $\sum a_i \mathbf{w}\mathbf{F}_{1i} = 0$. We simply multiply on the right by $\mathbf{F}_{j1}$ to obtain $a_j \mathbf{w}\mathbf{F}_{11} = 0$, hence $a_j = 0$ for $j = 1, \cdots, n$. Having this, one readily verifies that the $\mathbf{w}_j$ satisfy the relations (2).

Now we can complete the proof of Theorem 1′. We start with any basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ of L and the corresponding basis $\mathbf{E}_{ij}$ of $\mathfrak{A}$ $(i, j = 1, \cdots, n)$. We set $\mathbf{F}_{ij} = \sigma(\mathbf{E}_{ij})$. Since σ is an algebra automorphism of $\mathfrak{A}$, the relations (1) satisfied by the $\mathbf{E}_{ij}$ imply the relations for the $\mathbf{F}_{ij}$ in the hypothesis of the lemma, hence there is a basis $\mathbf{w}_1, \cdots, \mathbf{w}_n$ of L such that $\mathbf{w}_i\mathbf{F}_{jk} = \delta_{ij}\mathbf{w}_k$.

Now we have two bases of L: $\mathbf{v}_1, \cdots, \mathbf{v}_n$; $\mathbf{w}_1, \cdots, \mathbf{w}_n$. We define an automorphism (non-singular linear transformation) $\mathbf{P}$ of L by sending one basis into the other: $\mathbf{w}_1\mathbf{P} = \mathbf{v}_i$; hence $\mathbf{v}_i\mathbf{P}^{-1} = \mathbf{w}_i$. We now have

$$\mathbf{w}_i\mathbf{P}\mathbf{E}_{jk}\mathbf{P}^{-1} = \mathbf{v}_i\mathbf{E}_{jk}\mathbf{P}^{-1} = \delta_{ij}\mathbf{v}_k\mathbf{P}^{-1} = \delta_{ij}\mathbf{w}_k = \mathbf{w}_i\mathbf{F}_{jk}.$$

Hence $\mathbf{P}\mathbf{E}_{jk}\mathbf{P}^{-1} = \mathbf{F}_{jk}$, for they agree on a basis of L. We have proved that $\mathbf{P}\mathbf{E}_{jk}\mathbf{P}^{-1} = \sigma(\mathbf{E}_{jk})$. Since the $\mathbf{E}_{jk}$ form a linear basis of $\mathfrak{A}$, we finally have $\mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \sigma(\mathbf{A})$ for all $\mathbf{A} \in \mathrm{Lin}\ (L)$.

We might mention that this result has some fairly deep consequences in associative algebra theory; for example, it follows rather easily that every automorphism of a central simple associative algebra is inner.[3]

3. Fields of matrices. In this section we consider a result which at first sight is rather surprising.[4]

THEOREM 2. *Let $\mathfrak{M}_n$ be the algebra of all $n \times n$ matrices over a field k and consider k as a subalgebra of $\mathfrak{M}_n$ by identifying elements a of k with the corresponding scalar multiples aI of the identity. Let k' be another field such that $k \leqq k' \leqq \mathfrak{M}_n$ and suppose $[k' : k] = m$ is the linear dimension of k' over k. Let $\mathfrak{C}$ be the set of all elements of $\mathfrak{M}_n$ which commute with every element of k'. Then m divides n so that $n = mq$ and $\mathfrak{C}$ is isomorphic to $\mathfrak{M}_q(k')$, the algebra of $q \times q$ matrices over k'.*[5]

The result looks somewhat far-fetched and complicated; indeed quite complicated proofs have been given. Nevertheless when looked at in the proper invariant light it is transparent.

Again we start with an n-dimensional linear space L over k and write $n = \dim_k (L)$. As before, $\mathfrak{A} = \mathrm{Lin}\ (L)$ and now we have $k \leqq k' \leqq \mathfrak{A}$. Since k' is a field of linear transformations on L, we may consider L as a linear space over k' (and this is the crucial observation). As such it is obviously finite dimensional, say of dimension $q = \dim_{k'}(L)$. That $n = mq$ is a consequence of the plausible relation

$$\dim_k(L) = \dim_{k'}(L)\cdot\dim_k(k')$$

which is quickly proved by showing that a basis $\mathbf{v}_1, \cdots, \mathbf{v}_q$ of L over k' and a basis $\mathbf{A}_1, \cdots, \mathbf{A}_m$ of k' over k make up a basis $\{\mathbf{v}_i\mathbf{A}_j; i=1, \cdots, q, j=1, \cdots, m\}$ of qm elements of L over k. Finally, for a linear transformation on L over k to commute with all of the elements of k' is exactly the same as its being a linear transformation on L over k'. Hence the subalgebra $\mathfrak{C}$ of $\mathfrak{A}=\mathrm{Lin}_k\,(L)$ of all linear transformations which commute elementwise with the transformations in k' may be identified with $\mathrm{Lin}_{k'}\,(L)$ which, of course, is isomorphic (over k') to $\mathfrak{M}_q(k')$.

4. Superdiagonalization. There is a powerful technique for handling individual linear transformations or sets of transformations which is closely related to the idea of invariant subspace. Let L be an n-dimensional linear space over a field k and let $\mathbf{A}$ be a linear transformation on L into L. Suppose M is a subspace of L; we say that M *reduces* $\mathbf{A}$ if whenever $\mathbf{v}\in M$ then $\mathbf{vA}\in M$. By using the notation

$$M\mathbf{A} = \{\mathbf{vA}\,|\,\mathbf{v}\in M\},$$

the condition is simply $M\mathbf{A}\leqq M$. Similarly if $\mathfrak{S}$ is a set of linear transformations, we say that M reduces $\mathfrak{S}$ if $M\mathfrak{S}\subseteq M$, meaning $M\mathbf{A}\leqq M$ for each $\mathbf{A}$ in $\mathfrak{S}$.

Since the sets of transformations also include the sets consisting of a single transformation, we may restrict attention to sets. Suppose then that $\mathfrak{S}$ is a set of transformations on L and that we have found subspaces $M_1, M_2, M_3, \cdots, M_n$ of L, each of which reduces $\mathfrak{S}$ and such that

$$L = M_1 > M_2 > M_3 > \cdots > M_n > 0 = M_{n+1}$$

so that (since $\dim\,(L)=n$) $\dim\,(M_2)=n-1$, $\dim\,(M_3)=n-2, \cdots$. We fit a basis of L to this chain of subspaces by selecting vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ where $\mathbf{v}_i\in M_i$ but $\mathbf{v}_i\notin M_{i+1}$. If $\mathbf{A}\in\mathfrak{S}$, then $\mathbf{v}_i\mathbf{A}\in M_i\mathbf{A}\leqq M_i$, hence $\mathbf{v}_i\mathbf{A}$ is a linear combination of $\mathbf{v}_i, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_n$; that is,

$$\begin{cases} \mathbf{v}_1\mathbf{A} = a_{11}\mathbf{v}_1 + \cdots + a_{1n}\mathbf{v}_n \\ \mathbf{v}_2\mathbf{A} = a_{22}\mathbf{v}_2 + \cdots + a_{2n}\mathbf{v}_n \\ \mathbf{v}_3\mathbf{A} = a_{33}\mathbf{v}_3 + \cdots + a_{3n}\mathbf{v}_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{v}_n\mathbf{A} = a_{nn}\mathbf{v}_n \end{cases}$$

so that the matrix of $\mathbf{A}$ with respect to the basis $(\mathbf{v}_i)$ is in superdiagonal (or triangular) form; and this is true for each $\mathbf{A}$ in $\mathfrak{S}$.

The remaining sections deal with some cases in which this superdiagonalization of a set of transformations is possible. An important tool in this theory is the following. If the subspace M reduces a set $\mathfrak{S}$ of transformations, then $\mathfrak{S}$ induces a set of linear transformations on the subspace M. Also, $\mathfrak{S}$ induces a set of transformations on the quotient space L/M, the space of cosets of L modulo

M. In fact, if $\mathbf{A}\in\mathfrak{S}$ and $\mathbf{v}\in L$, then the induced transformation, also denoted $\mathbf{A}$, on L/M has the effect

$$(\mathbf{v}+M)\mathbf{A}=(\mathbf{v}\mathbf{A})+M$$

on the coset $\mathbf{v}+M$. One readily checks that this is well defined, *i.e.*, independent of the particular representative $\mathbf{v}$ of the coset, exactly because $M\mathbf{A}\leqq M$. Speaking in a general way, algebraic properties of the original set $\mathfrak{S}$ carry over to the induced sets both on M and on L/M.

5. Single linear transformation. We begin by showing that if the ground field k is sufficiently large then a single transformation $\mathbf{A}$ may be superdiagonalized. The field in fact must contain all of the characteristic roots of $\mathbf{A}$. We take care of this for *all* $\mathbf{A}$ by assuming that k is algebraically closed, *i.e.*, that every polynomial $t^n+a_1t^{n-1}+\cdots+a_n$ with coefficients in k has a root in k.

THEOREM 3. *Let* $\mathbf{A}$ *be linear on a space* L *of dimension* $n>1$ *over an algebraically closed field* k. *Then there is a subspace* M *reduced by* $\mathbf{A}$ *such that* $0<M<L$.

We simply set $M=k\mathbf{v}$, the space generated by a single characteristic vector $\mathbf{v}$. Since $\mathbf{v}\mathbf{A}=c\mathbf{v}$ we have $M\mathbf{A}\leqq M$.

COROLLARY. *The transformation* $\mathbf{A}$ *may be superdiagonalized.*

Proof. We proceed by induction on $n=\dim(L)$. Since $\mathbf{A}$ induces a single linear transformation on M and $\dim(M)<n$, we have a sequence

$$0<M_1<M_2<\cdots<M_r=M$$

of subspaces reduced by $\mathbf{A}$, each of dimension one greater than the preceding; since $\mathbf{A}$ induces a single linear transformation on L/M and $\dim(L/M)=\dim(L)-\dim(M)<n$, we have another such sequence

$$M<M_{r+1}<\cdots<M_n=L.$$

Together they yield a sequence

$$0<M_1<M_2<\cdots<M_n=L$$

which provides the superdiagonalization of $\mathbf{A}$.

We give the proof in this manner as a model to be followed in more complicated situations.

6. Commutative sets. Our next result also requires a restriction on k.

THEOREM 4. *Let* $\mathfrak{S}$ *be a commutative set of linear transformations on a space* L *of dimension* $n>1$ *over an algebraically closed field* k. *Then there is a subspace* M *reduced by* $\mathfrak{S}$ *such that* $0<M<L$.

Proof. If each $\mathbf{A}$ in $\mathfrak{S}$ is a scalar multiple of the identity $\mathbf{I}$, then any proper subspace M of L will reduce $\mathfrak{S}$. If this is not the case, we select an $\mathbf{A}\in\mathfrak{S}$ such that $\mathbf{A}\neq c\mathbf{I}$ for any c. Let c be a characteristic root of $\mathbf{A}$ and set

$$M = \{\mathbf{v} \mid \mathbf{vA} = c\mathbf{v}\}.$$

Then $0 < M < L$. If $\mathbf{B}$ is an arbitrary transformation in $\mathfrak{S}$, then for each $\mathbf{v} \in M$ we have

$$(\mathbf{vB})\mathbf{A} = (\mathbf{vA})\mathbf{B} = (c\mathbf{v})\mathbf{B} = c(\mathbf{vB}),$$

hence $\mathbf{vB} \in M$. It follows that $M\mathbf{B} \leqq M$, *i.e.*, that M reduces $\mathbf{B}$ and consequently M reduces $\mathfrak{S}$.

As a corollary, *any commutative set may be superdiagonalized.*

Proof. Exactly as for the corollary to Theorem 3. It must only be checked that the sets induced by $\mathfrak{S}$ on M and on L/M are also commutative; this follows immediately from their definitions.[6]

7. Algebras and their representations.[4] An *associative algebra* is the structure consisting of a (finite dimensional) linear space $\mathfrak{A}$ over a field k which is also an associative ring and for which the identities $(ax)y = x(ay) = a(xy)$ hold, connecting the multiplication by a scalar a with the ring multiplication of algebra elements x, y. A *subalgebra* $\mathfrak{B}$ of $\mathfrak{A}$ is a subset of $\mathfrak{A}$ which is also an algebra under the operations of $\mathfrak{A}$. If $\mathfrak{B}$ and $\mathfrak{C}$ are two subalgebras, then $\mathfrak{B}\mathfrak{C}$ denotes the totality of finite sums $\sum y_i z_i$ where $y_i \in \mathfrak{B}$, $z_i \in \mathfrak{C}$. Clearly $\mathfrak{B}\mathfrak{C}$ is a linear subspace of $\mathfrak{A}$, but it is not necessarily a subalgebra itself. It is not even true in general that $\mathfrak{B}\mathfrak{C}$ includes $\mathfrak{B}$ or $\mathfrak{C}$. A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is an *ideal* if $\mathfrak{A}\mathfrak{B} \leqq \mathfrak{B}$ and $\mathfrak{B}\mathfrak{A} \leqq \mathfrak{B}$. If $\mathfrak{B}$ and $\mathfrak{C}$ are ideals, then $\mathfrak{B}\mathfrak{C}$ is also an ideal. Also, in general $(\mathfrak{B}\mathfrak{C})\mathfrak{D} = \mathfrak{B}(\mathfrak{C}\mathfrak{D})$.

A *homomorphism* is an operation preserving mapping on an algebra $\mathfrak{A}_1$ into an algebra $\mathfrak{A}_2$. The *kernel*, consisting of all elements of $\mathfrak{A}_1$ mapping onto 0 is an ideal of $\mathfrak{A}_1$; indeed each ideal of an algebra is the kernel of some homomorphism. The image of $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{A}_2$.

Especially interesting are the *matric algebras*, or *algebras of linear transformations*. These are the subalgebras of Lin (L), the algebra of all linear transformations on a finite-dimensional space L over k. Each algebra $\mathfrak{A}$ is homomorphic to an algebra of linear transformations on $L = \mathfrak{A}$ given by the *regular representation*: each $y \in \mathfrak{A}$ is associated to the linear transformation $\mathbf{R}_y$, called the *right multiplication* of y, and defined by

$$x\mathbf{R}_y = xy.$$

An element x of an algebra $\mathfrak{A}$ is *nilpotent* if some power of x vanishes: $x^n = 0$. The algebra $\mathfrak{A}$ is called a *nil algebra* if each of its elements is nilpotent. The algebra $\mathfrak{A}$ is called *nilpotent* if some power of the whole algebra vanishes: $\mathfrak{A}^n = 0$. Here $\mathfrak{A}^n = \mathfrak{A}\mathfrak{A}\mathfrak{A} \cdots$ to n factors. Thus $\mathfrak{A}^n$ is the set of all sums $\sum x_1 y_2 \cdots z_n$, and $\mathfrak{A}^n = 0$ is the same as $x_1 y_2 \cdots z_n = 0$ for every n elements $x_1, y_2, \cdots, z_n$ of $\mathfrak{A}$. Clearly a nilpotent algebra is nil.

8. Nil algebras. The following result presupposes no special properties of the ground field.

THEOREM 5. *Let $\mathfrak{A}$ be a nil algebra of linear transformations on a space L of*

dimension $n>1$. Then there is a subspace M of L, $0<M<L$, and M reduces $\mathfrak{A}$.

Proof. If $\mathfrak{A}=0$, it is clear. Assuming $\mathfrak{A}\neq 0$, there is an element $\mathbf{v}\in L$ such that $M=\mathbf{v}\mathfrak{A}\neq 0$. M is a subspace of L simply because $\mathfrak{A}$ is a linear space. Suppose $M=L$. Then $\mathbf{v}\in M$, hence $\mathbf{v}=\mathbf{v}\mathbf{A}$ for some $\mathbf{A}\in\mathfrak{A}$. Hence $\mathbf{v}=\mathbf{v}\mathbf{A}=\mathbf{v}\mathbf{A}^2=\mathbf{v}\mathbf{A}^3=\cdots$. But $\mathbf{A}$ is nilpotent, hence for some r, $\mathbf{A}^r=0$. This implies $\mathbf{v}=0$ which contradicts $M\neq 0$. Finally $M\mathfrak{A}=(\mathbf{v}\mathfrak{A})\mathfrak{A}=\mathbf{v}(\mathfrak{A}\mathfrak{A})\leqq\mathbf{v}\mathfrak{A}=M$ since $\mathfrak{A}$ is a ring (closed under multiplication).

It quickly follows that there is a basis of L with respect to which $\mathfrak{A}$ is superdiagonalized. But each $\mathbf{A}\in\mathfrak{A}$ is nilpotent, hence the diagonal elements in the matrix $A=\|a_{ij}\|$ representing $\mathbf{A}$ must all vanish.

COROLLARY 1. *Each nil algebra $\mathfrak{A}$ of linear transformations on an n-dimensional linear space L is nilpotent, in fact*

$$\mathfrak{A}^n = 0.$$

For we have seen that there is a basis of L such that each element $\mathbf{A}$ of $\mathfrak{A}$ has the representation

$$A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

But the product of any n such matrices vanishes.

Surprisingly enough, we can now generalize this corollary to include *all* algebras, not just the algebras of linear transformations.

COROLLARY 2. *Let $\mathfrak{A}$ be a nil algebra of dimension n over k. Then $\mathfrak{A}$ is nilpotent, in fact $\mathfrak{A}^{n+1}=0$.*

Proof. The regular representation of $\mathfrak{A}$ yields a nil algebra $\mathfrak{A}_1$ of linear transformations on $L=\mathfrak{A}$. By Corollary 1, $\mathfrak{A}_1{}^n=0$. Now if $x_0, x_1, \cdots, x_n\in\mathfrak{A}$, then

$$x_0x_1\cdots x_n = x_0(x_1\cdots x_n) = x_0(\mathbf{R}_{x_1}\mathbf{R}_{x_2}\cdots\mathbf{R}_{x_n}) = x_00 = 0.$$

Thus $\mathfrak{A}^{n+1}=0$.

9. A generalization. In a certain sense, the following result includes both Theorems 4 and 5.[7]

THEOREM 6. *Let k be an algebraically closed field and let $\mathfrak{A}$ be an algebra of linear transformations on a linear space L over k of dimension $n>1$. Suppose there is a nil ideal $\mathfrak{B}$ of $\mathfrak{A}$ such that the quotient algebra $\mathfrak{A}/\mathfrak{B}$ is commutative. Then there exists a subspace M of L, $0<M<L$, and M reduces $\mathfrak{A}$.*

Proof. If $\mathfrak{B}=0$, then $\mathfrak{A}$ is commutative and the result follows from Theorem 4. If $\mathfrak{B}\neq 0$, there is a $\mathbf{v}$ in L such that $M=\mathbf{v}\mathfrak{B}\neq 0$. Exactly as in the proof of Theorem 5 we conclude that $0<M<L$. Finally $M\mathfrak{A}=(\mathbf{v}\mathfrak{B})\mathfrak{A}=\mathbf{v}(\mathfrak{B}\mathfrak{A})\leqq\mathbf{v}\mathfrak{B}$ since $\mathfrak{B}$ is an ideal.

The hypotheses on $\mathfrak{A}$—that $\mathfrak{A}$ has a nil ideal and that the quotient algebra modulo this ideal is commutative—go over to any homomorphic image of $\mathfrak{A}$. But both the restriction of $\mathfrak{A}$ to M and the algebra induced by $\mathfrak{A}$ on L/M are homomorphic images of $\mathfrak{A}$, hence we may apply induction to superdiagonalize $\mathfrak{A}$ in the usual manner.

10. Lie algebras. Theorems 4 and 5 allow very broad generalizations which must be expressed in the language of Lie algebras. A *Lie algebra* over a field k consists of a linear space $\mathfrak{A}$ over k in which is defined a multiplication $[x, y]$. Thus if $x, y \in \mathfrak{A}$, then $[x, y] \in \mathfrak{A}$. We assume $[x, y]$ is linear in each variable, that $[x, x] = 0$, $[x, y] + [y, x] = 0$, and the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

We particularly note that a Lie algebra is not usually an associative algebra. A *subalgebra* (or *Lie subalgebra*) of a Lie algebra $\mathfrak{A}$ is a subset $\mathfrak{B}$ of $\mathfrak{A}$ which is itself a Lie algebra with respect to the operations of $\mathfrak{A}$.

If $\mathfrak{A}$ is an associative algebra, we may turn it into a Lie algebra by defining the *Lie product*

$$[x, y] = xy - yx$$

in terms of the associative product. This applies in particular to the algebra Lin (L) of linear transformations on a linear space L. A Lie subalgebra of this particular algebra is called a *Lie algebra of linear transformations*. Via the usual matric representation, we obtain a *Lie algebra of matrices*. (Example: the $n \times n$ matrices of trace zero. Note that this is closed under the Lie product but not under the ordinary product.)

If $\mathfrak{B}$ and $\mathfrak{C}$ are Lie subalgebras of a Lie algebra $\mathfrak{A}$, then $[\mathfrak{B}, \mathfrak{C}]$ denotes the set of all sums $\sum x_i y_i$ where $x_i \in \mathfrak{B}$, $y_i \in \mathfrak{C}$. A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is called an *ideal* if $[\mathfrak{A}, \mathfrak{B}] \leqq \mathfrak{B}$. (It is unnecessary to assume $[\mathfrak{B}, \mathfrak{A}] \leqq \mathfrak{B}$ as in the associative case because $[y, x] = -[x, y]$.) If $\mathfrak{B}$ and $\mathfrak{C}$ are ideals, then so is $[\mathfrak{B}, \mathfrak{C}]$. This follows from the Jacobi identity as we may indicate symbolically

$$[\mathfrak{A}, [\mathfrak{B}, \mathfrak{C}]] \leqq [\mathfrak{B}, [\mathfrak{C}, \mathfrak{A}]] + [\mathfrak{C}, [\mathfrak{A}, \mathfrak{B}]] \leqq [\mathfrak{B}, \mathfrak{C}] + [\mathfrak{C}, \mathfrak{B}] \leqq [\mathfrak{B}, \mathfrak{C}].$$

In particular, $\mathfrak{A}' = [\mathfrak{A}, \mathfrak{A}]$ is the *derived algebra* of $\mathfrak{A}$. If $\mathfrak{A}^{(r)}$ has been defined, then we define $\mathfrak{A}^{(r+1)} = [\mathfrak{A}^{(r)}, \mathfrak{A}^{(r)}]$. Evidently $\mathfrak{A}', \mathfrak{A}'', \cdots$ are all ideals of $\mathfrak{A}$ and

$$\mathfrak{A} \geqq \mathfrak{A}' \geqq \mathfrak{A}'' \geqq \cdots \geqq \mathfrak{A}^{(r)} \geqq \cdots.$$

The Lie algebra $\mathfrak{A}$ is called *solvable* if $\mathfrak{A}^{(r)} = 0$ for some r.

11. Engel's theorem.[8] The theorem of Engel we now discuss is the generalization of Theorem 5 we have in mind.

THEOREM 7. *Let $\mathfrak{A}$ be a Lie algebra of linear transformations on a linear space L over k such that each $\mathbf{A}$ in $\mathfrak{A}$ is nilpotent. Then there exists a non-zero vector $\mathbf{v}$ in L such that $\mathbf{v}\mathfrak{A} = 0$.*

Here is the difference between this result and that of Theorem 5. In Theorem 5 we assumed not only that each $\mathbf{A}$ in $\mathfrak{A}$ is nilpotent, but that $\mathfrak{A}$ is an associative algebra, hence closed under ordinary multiplication: if $\mathbf{A}$ and $\mathbf{B}\in\mathfrak{A}$, then $\mathbf{AB}\in\mathfrak{A}$. Now we are only assuming $\mathbf{AB}-\mathbf{BA}=[\mathbf{A},\mathbf{B}]\in\mathfrak{A}$.

Proof. We proceed by induction on $m=\dim(\mathfrak{A})$. When $m=1$, the result is obvious since $\mathfrak{A}=k\mathbf{A}$ where $\mathbf{A}$ is nilpotent. We henceforth assume $m>1$ and that the result is true for all lower dimensions. Since $\mathfrak{A}$ contains a one-dimensional Lie subalgebra,[9] there exists a subalgebra $\mathfrak{B}$, $0<\mathfrak{B}<\mathfrak{A}$, of *maximal dimension.* By the induction hypothesis, there is a non-zero $\mathbf{w}$ in L such that $\mathbf{w}\mathfrak{B}=0$. We set $\mathfrak{B}_1=\{\mathbf{A}\in\mathfrak{A}\,|\,\mathbf{wA}=0\}$. Then $\mathfrak{B}_1$ is a Lie subalgebra of $\mathfrak{A}$ and $\mathfrak{B}\leqq\mathfrak{B}_1\leqq\mathfrak{A}$. Since $\mathfrak{B}$ has maximal dimension there are only two possibilities: $\mathfrak{B}_1=\mathfrak{A}$ or $\mathfrak{B}_1=\mathfrak{B}$. If $\mathfrak{B}_1=\mathfrak{A}$, then $\mathbf{w}\mathfrak{A}=\mathbf{w}\mathfrak{B}_1=0$ and $\mathbf{v}=\mathbf{w}$ is the sought-for vector. We may consequently suppose that the contrary case $\mathfrak{B}_1=\mathfrak{B}$ prevails.

Thus *if* $\mathbf{A}\in\mathfrak{A}$ *and* $\mathbf{wA}=0$, *then* $\mathbf{A}\in\mathfrak{B}$. We set $M=\mathbf{w}\mathfrak{A}$ so that M is a subspace of L (since $\mathfrak{A}$ is a linear space). Clearly $M>0$ since $\mathbf{wA}\neq 0$ for each $\mathbf{A}$ not in $\mathfrak{B}$. Also $\mathbf{w}\mathfrak{A}<L$, for if $\mathbf{w}\in\mathbf{w}\mathfrak{A}$, then $\mathbf{w}=\mathbf{wA}$ for some $\mathbf{A}\in\mathfrak{A}$; but each $\mathbf{A}$ in $\mathfrak{A}$ is by hypothesis nilpotent. Thus $0<M<L$.

We next prove that $M\mathfrak{B}\subseteq M$. For if $\mathbf{wA}\in M$ and $\mathbf{B}\in\mathfrak{B}$, then

$$(\mathbf{wA})\mathbf{B}=\mathbf{w}(\mathbf{AB})=\mathbf{w}[\mathbf{A},\mathbf{B}]+\mathbf{wBA}=\mathbf{w}[\mathbf{A},\mathbf{B}]\in\mathbf{w}\mathfrak{A}=M.$$

Thus $\mathfrak{B}$ *induces a Lie algebra of linear transformations on* M, each element of which is nilpotent since that is so of $\mathfrak{B}$. We apply the induction hypothesis once again to find a non-zero vector $\mathbf{z}$ in M such that $\mathbf{z}\mathfrak{B}=0$. Thus $\mathbf{z}=\mathbf{wA}_0$ for some $\mathbf{A}_0\in\mathfrak{A}$.

We next prove that $[\mathbf{A}_0,\mathfrak{B}]\subseteq\mathfrak{B}$. For if $\mathbf{B}\in\mathfrak{B}$, then

$$\mathbf{w}[\mathbf{A}_0,\mathbf{B}]=\mathbf{wA}_0\mathbf{B}-\mathbf{wBA}_0=\mathbf{zB}=0,$$

hence $[\mathbf{A}_0,\mathbf{B}]\in\mathfrak{B}$. It follows from this and the identity $[\mathbf{A}_0,\mathbf{A}_0]=0$, that the set

$$\mathfrak{A}_1=\mathfrak{B}+k\mathbf{A}_0$$

is a Lie subalgebra of $\mathfrak{A}$ and that $\mathfrak{B}$ is an ideal in $\mathfrak{A}_1$. By the maximality of $\mathfrak{B}$ we must have $\mathfrak{A}_1=\mathfrak{A}$,

$$\mathfrak{A}=\mathfrak{B}+k\mathbf{A}_0,$$

and $\mathfrak{B}$ is an ideal in $\mathfrak{A}$.

Now let $N=\{\mathbf{u}\in L\,|\,\mathbf{u}\mathfrak{B}=0\}$ so that $\mathbf{w}\in N$, hence $0<N\leqq L$. If $\mathbf{u}\in N$ and $\mathbf{B}\in\mathfrak{B}$, then

$$(\mathbf{uA}_0)\mathbf{B}=\mathbf{u}(\mathbf{A}_0\mathbf{B})=\mathbf{u}[\mathbf{A}_0,\mathbf{B}]+\mathbf{uBA}_0=0,$$

hence $\mathbf{uA}_0\in N$, $N\mathbf{A}_0\leqq N$, *i.e.*, N reduces $\mathbf{A}_0$. Since $\mathbf{A}_0$ is nilpotent, there exists a non-zero vector $\mathbf{v}\in N$ such that $\mathbf{vA}_0=0$. But $\mathbf{v}\mathfrak{B}=0$ since $\mathbf{v}\in N$, hence finally $\mathbf{v}\mathfrak{A}=0$.[10]

The following corollaries are now easy consequences.

COROLLARY 1. *Any Lie algebra of nilpotent transformations may be super-diagonalized.*

COROLLARY 2. *If $\mathfrak{A}$ is a Lie algebra of nilpotent transformations on an n-dimensional space L, and if $\mathbf{A}_1, \cdots, \mathbf{A}_n \in \mathfrak{A}$, then*

$$\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n = 0.$$

12. A lemma of Jacobson. As preparation for the proof in Section 14 we need the following result.[11]

LEMMA. *Let L be an n-dimensional linear space over a field k of characteristic* 0. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *be linear transformations on L such that*

$$\mathbf{C} = [\mathbf{A}, \mathbf{B}] \quad \textit{and} \quad [\mathbf{A}, \mathbf{C}] = 0.$$

Then $\mathbf{C}$ *is nilpotent.*

Characteristic zero means that k contains the field of rationals or, what is the same thing, that if

$$n \cdot a = a + a + \cdots (n \text{ summands}) = 0,$$

then $a = 0$ (for $n = 1, 2, 3, \cdots$).[12]

Proof. The proof is based on a trace argument. The trace function S is linear and satisfies $S(\mathbf{XY}) = S(\mathbf{YX})$. We shall show that $S(\mathbf{C}^r) = 0$ for $r = 1, 2, \cdots$. In fact

$$S(\mathbf{C}) = S([\mathbf{A}, \mathbf{B}]) = S(\mathbf{AB}) - S(\mathbf{BA}) = 0,$$

and

$$\mathbf{C}^r = \mathbf{C}^{r-1}[\mathbf{A}, \mathbf{B}] = \mathbf{C}^{r-1}\mathbf{AB} - \mathbf{C}^{r-1}\mathbf{BA}.$$

But $[\mathbf{A}, \mathbf{C}] = \mathbf{AC} - \mathbf{CA} = 0$, hence $\mathbf{AC} = \mathbf{CA}$ and this implies $\mathbf{AC}^{r-1} = \mathbf{C}^{r-1}\mathbf{A}$. We have

$$S(\mathbf{C}^r) = S(\mathbf{AC}^{r-1}\mathbf{B}) - S(\mathbf{C}^{r-1}\mathbf{BA}) = S(\mathbf{AC}^{r-1}\mathbf{B}) - S(\mathbf{AC}^{r-1}\mathbf{B}) = 0.$$

The lemma now follows from the known trace criterion (over characteristic zero) for nilpotence: $\mathbf{C}$ is nilpotent when the first n powers of $\mathbf{C}$ have trace zero. For completeness we include a discussion of this result.

13. The trace argument. The precise result is this.

THEOREM 8. *Let* $\mathbf{C}$ *be a linear transformation on an n-dimensional space L over a field k of characteristic zero and suppose that*

$$S(\mathbf{C}) = S(\mathbf{C}^2) = S(\mathbf{C}^3) = \cdots = S(\mathbf{C}^n) = 0.$$

Then $\mathbf{C}$ *is nilpotent.*

We may give an irrational proof based on Theorem 3. We first must enlarge the field k so that it includes the characteristic roots of $\mathbf{C}$. Then we may take a basis of L so that the matrix $C=\|c_{ij}\|$ of $\mathbf{C}$ is superdiagonal, the diagonal elements being the characteristic roots of $\mathbf{C}$. Each power C^r of C is also superdiagonal and we realize from this that if $c_1, \cdots, c_n$ are the diagonal elements of C, then $c_1^r, \cdots c_n^r$ are those of C^r, hence the trace conditions become

$$\begin{cases} c_1 + c_2 + \cdots + c_n = 0 \\ c_1^2 + c_2^2 + \cdots + c_n^2 = 0 \\ c_1^n + c_2^n + \cdots + c_n^n = 0. \end{cases}$$

We deduce either by the Vandermonde determinant or by the Newton identities (on symmetric functions) that $c_1=c_2= \cdots =c_n=0$, and hence that C is nilpotent.

More in keeping with the spirit of this paper is a rational proof of Theorem 8 based on the technique rather than the result of Theorem 3. We begin with the characteristic polynomial

$$f(t) = \det(tI - \mathbf{C}) = t^n + a_1 t^{n-1} + \cdots + a_n$$

of $\mathbf{C}$. By the Cayley-Hamilton theorem,

$$\mathbf{C}^n + a_1\mathbf{C}^{n-1} + \cdots + a_n\mathbf{I} = 0.$$

Taking traces:

$$na_n = 0.$$

Hence $a_n=0, f(0)=0$. It follows that 0 is a characteristic root of $\mathbf{C}$, hence there is a non-zero vector $\mathbf{v}_n$ in L such that $\mathbf{v}_n\mathbf{C}=0$. We make this the last element in a basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_n$ of L so that the corresponding matrix of $\mathbf{C}$ is of the form

$$C = \begin{pmatrix} C_1 & * \\ 0 & 0 \end{pmatrix}$$

where C_1 is an $(n-1)$-rowed square matrix and the asterisk denotes some $(n-1)$-element column. Clearly

$$C^r = \begin{pmatrix} C_1^r & * \\ 0 & 0 \end{pmatrix}$$

so that $S(C^r)=S(C_1^r)=0$ for $r=1, 2, \cdots, (n-1)$ (and also n), *i.e.*, the hypotheses are satisfied for C_1. An induction argument suffices to finish the proof.

14. The theorems of Lie. We return to the study of Lie algebras. We recall that a Lie algebra $\mathbf{A}$ is *solvable* if the derived sequence of ideals $\mathfrak{A} \supseteq \mathfrak{A}' \supseteq \mathfrak{A}'' \supseteq \cdots$ terminates with 0.

LEMMA. *Each Lie subalgebra and each homomorphic image of a solvable Lie algebra* $\mathfrak{A}$ *is solvable.*

Proof. If $\mathfrak{B} \leqq \mathfrak{A}$, then $\mathfrak{B}' = [\mathfrak{B}, \mathfrak{B}] \leqq [\mathfrak{A}, \mathfrak{A}] = \mathfrak{A}'$, $\mathfrak{B}'' \leqq \mathfrak{A}''$, *etc.* Hence $\mathfrak{A}^{(r)} = 0$ implies $\mathfrak{B}^{(r)} = 0$. The solvability of a homomorphic image is proved similarly.

The basic theorem of Lie shows that solvability implies a strong property of the derived algebra.

THEOREM 9. *Let* $\mathfrak{A}$ *be a solvable Lie algebra of linear transformations on an n-dimensional space L. Then each element of* $\mathfrak{A}'$ *is nilpotent.*

Proof. The solvability of $\mathfrak{A}$ implies

$$\mathfrak{A} > \mathfrak{A}' > \mathfrak{A}'' > \cdots > \mathfrak{A}^{(r)} = 0,$$

for equality at any step would entail equality forever afterwards. We set $m = \dim(\mathfrak{A})$ and select a linear subspace $\mathfrak{B}$ of $\mathfrak{A}$ such that

$$\mathfrak{A}' \leqq \mathfrak{B} < \mathfrak{A}$$

and $\dim(\mathfrak{B}) = m - 1$. Then

$$[\mathfrak{B}, \mathfrak{B}] \subseteq [\mathfrak{A}, \mathfrak{A}] = \mathfrak{A}' \leqq \mathfrak{B},$$

hence $\mathfrak{B}$ is a subalgebra. Even more:

$$[\mathfrak{A}, \mathfrak{B}] \subseteq [\mathfrak{A}, \mathfrak{A}] = \mathfrak{A}' \leqq \mathfrak{B},$$

hence $\mathfrak{B}$ is an ideal in $\mathfrak{A}$. We can actually do a little better than this. For if $\mathbf{A}_0$ is any element of $\mathfrak{A}$ not in $\mathfrak{B}$, then $\mathfrak{A} = \mathfrak{B} + k\mathbf{A}_0$; since $[\mathbf{A}_0, \mathbf{A}_0] = 0$, we have

$$\mathfrak{A}' = [\mathfrak{A}, \mathfrak{A}] = [\mathfrak{B} + k\mathbf{A}_0, \mathfrak{B} + k\mathbf{A}_0] \subseteq [\mathfrak{B}, \mathfrak{B}] + [\mathfrak{B}, \mathbf{A}_0] + [\mathbf{A}_0, \mathfrak{B}] \subseteq [\mathfrak{A}, \mathfrak{B}],$$

consequently

$$\mathfrak{A}' = [\mathfrak{A}, \mathfrak{B}].$$

We proceed with the proof of the theorem by double induction on (m, n). If $m = 1$, then $\mathfrak{A} = k\mathbf{A}_0$, $\mathfrak{A}' = 0$. If $n = 1$, then $\dim(L) = n = 1$, $\mathfrak{A}$ is a commutative set, $\mathfrak{A}' = 0$. Thus we may assume the theorem whenever either m or n (or both) are decreased, but neither is increased.

In particular we may assume that each element of $\mathfrak{B}'$ is nilpotent.

Case 1. $\mathfrak{B}' = 0$ Then if $\mathbf{A} \in \mathfrak{A}$ and $\mathbf{B} \in \mathfrak{B}$ we have $\mathbf{C} = [\mathbf{A}, \mathbf{B}] \in \mathfrak{B}$, $[\mathbf{C}, \mathbf{B}] \in \mathfrak{B}'$, hence $[\mathbf{C}, \mathbf{B}] = 0$. By the Jacobson Lemma $\mathbf{C}$ is nilpotent. Each element of $\mathfrak{A}' = [\mathfrak{A}, \mathfrak{B}]$ is a sum of such elements $\mathbf{C}$. Any two such, $\mathbf{C}_1$, $\mathbf{C}_2$ commute since $[\mathbf{C}_1, \mathbf{C}_2] \in \mathfrak{B}' = 0$, hence each element of $\mathfrak{A}'$, $\sum \mathbf{C}_j$, is nilpotent.[13]

Case 2. $\mathfrak{B}' \neq 0$. By Theorem 7, if $M = \{\mathbf{v} \in L \mid \mathbf{v}\mathfrak{B}' = 0\}$, then $0 < M < L$. We shall prove that M reduces $\mathfrak{A}$. To do this we must prove that if $\mathbf{v} \in M$ and $\mathbf{A} \in \mathfrak{A}$, then $(\mathbf{vA})\mathfrak{B}' = 0$. But each element of $\mathfrak{B}'$ is a sum of elements of the form $[\mathbf{B}_1, \mathbf{B}_2]$ where $\mathbf{B}_1, \mathbf{B}_2 \in \mathfrak{B}$. Now

$$(\mathbf{vA})[\mathbf{B}_1, \mathbf{B}_2] = \mathbf{v}[\mathbf{A}, [\mathbf{B}_1, \mathbf{B}_2]] + \mathbf{v}[\mathbf{B}_1, \mathbf{B}_2]\mathbf{A}$$
$$= -\mathbf{v}[\mathbf{B}_1, [\mathbf{B}_2, \mathbf{A}]] - \mathbf{v}[\mathbf{B}_2, [\mathbf{A}, \mathbf{B}_1]].$$

Since $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, $[\mathbf{B}_2, \mathbf{A}] \in \mathfrak{B}$, $[\mathbf{B}_1, [\mathbf{B}_2, \mathbf{A}]] \in \mathfrak{B}'$. Likewise $[\mathbf{B}_2, [\mathbf{A}, \mathbf{B}_1]] \in \mathfrak{B}'$. Hence

$$(\mathbf{vA})[\mathbf{B}_1, \mathbf{B}_2] = 0.$$

Since dim $(M) <$ dim (L), the induction hypothesis implies that each element of $\mathfrak{A}'$ induces a nilpotent linear transformation on M. By the theorem of Engel, Theorem 7, we find a non-zero vector $\mathbf{w}$ in M such that $\mathbf{w}\mathfrak{A}' = 0$.

We now set $N = \{\mathbf{w} \in L \mid \mathbf{w}\mathfrak{A}' = 0\}$ so that $0 < N < L$. (If $N = L$, then $\mathfrak{A}' = 0$, $\mathfrak{B}' = 0$; but we are still in Case 2.) The same argument we used on M may now be used on N to prove that $\mathfrak{A}$ reduces N: $N\mathfrak{A} \subseteq N$. Consequently $\mathfrak{A}$ acts on the quotient space L/N which is of lower dimension than L. By induction once again we deduce that each element of $\mathfrak{A}'$ induces a nilpotent transformation on L/N. Thus if $\mathbf{C} \in \mathfrak{A}'$, then for some s, $L\mathbf{C}^s \leqq N$, hence $L\mathbf{C}^{s+1} \leqq N\mathbf{C} = 0$. Thus each element of $\mathfrak{A}'$ is nilpotent on L.[14]

The superdiagonalization theorem for solvable Lie algebras holds, as we shall now see. This generalizes Theorem 6.

THEOREM 10. *Let $\mathfrak{A}$ be a solvable Lie algebra of linear transformations on a linear space L of dimension $n > 1$ over an algebraically closed field k of characteristic zero. Then there is a subspace M of L such that $0 < M < L$ and M reduces $\mathfrak{A}$.*

Proof. We set

$$N = \{\mathbf{v} \in L \mid \mathbf{v}\mathfrak{A}' = 0\}.$$

By Theorems 9 and 7 we have $0 < N \leqq L$. If $N = L$, then $\mathfrak{A}' = 0$ which means that $\mathfrak{A}$ is a commutative set; we simply apply Theorem 4. If $0 < N < L$, then we take $M = N$ and must prove that N reduces $\mathfrak{A}$. But $\mathbf{v} \in N$, $\mathbf{A} \in \mathfrak{A}$, and $\mathbf{C} \in \mathfrak{A}'$ implies

$$(\mathbf{vA})\mathbf{C} = \mathbf{v}[\mathbf{A}, \mathbf{C}] + \mathbf{vCA} = 0$$

since $[\mathbf{A}, \mathbf{C}] \in \mathfrak{A}'$.

COROLLARY. *Under the hypotheses of Theorem* 10, *there exists a basis of L with respect to which each transformation $\mathbf{A}$ of $\mathfrak{A}$ has a superdiagonal matrix.*

By using the regular representation, these results may be applied to abstract Lie algebras. If $\mathfrak{A}$ is a Lie algebra over k and $y \in \mathfrak{A}$, we set $L = \mathfrak{A}$ and define $\mathbf{R}_y$ on L by

$$x\mathbf{R}_y = [x, y].$$

We have

$$\begin{aligned} x\mathbf{R}_{[y,z]} &= [x, [y, z]] = -[z, [x, y]] - [y, [z, x]] \\ &= [[x, y], z] - [[x, z], y] \\ &= [x\mathbf{R}_y, z] - [x\mathbf{R}_z, y] \\ &= x\mathbf{R}_y\mathbf{R}_z - x\mathbf{R}_z\mathbf{R}_y \\ &= x(\mathbf{R}_y\mathbf{R}_z - \mathbf{R}_z\mathbf{R}_y) = x[\mathbf{R}_y, \mathbf{R}_z], \end{aligned}$$

hence

$$\mathbf{R}_{[y,z]} = \mathbf{R}_y, \mathbf{R}_z.$$

This means that the mapping $y \to \mathbf{R}_y$ is a homomorphism on the Lie algebra $\mathfrak{A}$ into a Lie algebra of linear transformations on L. (This may be interpreted as the content of the Jacobi identity.) Here is a sample of how this is used.

THEOREM 11. *Let $\mathfrak{A}$ be a solvable Lie algebra over a field of characteristic zero. Then there is an integer r such that whenever $x \in \mathfrak{A}$ and $y_1, y_2, \cdots, y_r \in \mathfrak{A}'$, then*

$$[\cdots [[x, y_1], y_2] \cdots, y_r] = 0.$$

This follows by applying the theorems of Lie and Engel to the regular representation of $\mathfrak{A}$.

15. Characteristic roots. Here we make a further remark on super-diagonal sets. If $\mathfrak{S}$ is a set of superdiagonal $n \times n$ matrices, each $A \in \mathfrak{S}$ has the form

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

and the i-th diagonal element may be denoted by $\lambda_i(A)$ to denote its dependence on A. Thus $\lambda_1(A), \cdots, \lambda_n(A)$ are the characteristic roots of A. It is clear that if $B \in \mathfrak{S}$, then

$$\begin{aligned} \lambda_i(aA + bB) &= a\lambda_i(A) + b\lambda_i(B), \\ \lambda_i(AB) &= \lambda_i(A)\lambda_i(B). \end{aligned}$$

Thus if $F(X_1, \cdots, X_r)$ is any (non-commutative) polynomial in r variables with scalar coefficients, if $A_1, \cdots, A_r \in \mathfrak{S}$, and if $B = F(A_1, \cdots, A_r)$, then

$$\lambda_i(C) = F(\lambda_i(A_1), \cdots, \lambda_i(A_r)).$$

Notes

Numbers in square brackets refer to the bibliography below.

1. The basic material on linear spaces is found in the very readable book of Halmos [3]. We differ from the notation there in writing $\mathbf{vA}$ for the effect of a linear transformation $\mathbf{A}$ on a vector $\mathbf{v}$, instead of the customary $\mathbf{Av}$. We use the notation $M \leqq L$ for "M is a subspace of L" and $M < L$ for "M is a proper subspace of L."

2. Proofs will also be found in [1, p. 90] and [6, p. 302].

3. See [2, especially Ch. IV].

4. The necessary elementary material on associative algebras is given in [2, Ch. I] and [7, Ch. XVI].

5. See [1, p. 244].

6. The superdiagonalization of commutative sets is due to Frobenius. There is an interesting application to complex euclidean space (also a corresponding one for real euclidean space, but we restrict interest to the complex case.) Suppose that L is an n-dimensional complex space with an inner product denoted $(\mathbf{v}, \mathbf{w})$. From a sequence $L = M_1 > M_2 > \cdots > M_n > 0 = M_{n+1}$ of subspaces we now extract an orthonormal basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ by selecting $\mathbf{v}_i$ in M_i of unit length and orthogonal to M_{i+1}. Suppose $\mathfrak{S}$ is a commutative set of normal transformations. Thus $\mathbf{A} \in S$ implies $\mathbf{AA}^* = \mathbf{A}^*\mathbf{A}$ where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$ defined by $(\mathbf{vA}, \mathbf{w}) = (\mathbf{v}, \mathbf{wA}^*)$. Then by Theorem 4 and the remark we have just made, there is an orthonormal basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ of L with respect to which the matrix $A = \|a_{ik}\|$ of each $\mathbf{A}$ in $\mathfrak{S}$ is superdiagonal. But A is a normal matrix since $\mathbf{A}$ is normal and the basis is orthonormal. This means $AA^* = A^*A$ where A^* is the transposed conjugate of A. It quickly follows, by computing the diagonal elements of AA^* and A^*A, that A must be in diagonal, or spectral, form. Thus *a commutative set of normal transformations may be simultaneously brought to spectral form.*

7. See [5].

8. See [4], [8].

9. This is quite general. If $\mathfrak{A}$ is any Lie algebra over k and x is any non-zero element of $\mathfrak{A}$, then $\mathfrak{B} = kx$ is a (one-dimensional) Lie subalgebra of $\mathfrak{A}$. We must prove that $[\mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{B}$. But $[x, x] = 0$, hence $[\mathfrak{B}, \mathfrak{B}] = 0$.

10. This is not an easy result to prove by any means. The closure of $\mathfrak{A}$ under $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ together with the nilpotence of each element of $\mathfrak{A}$ may be used to systematically derive identities amongst the elements of $\mathfrak{A}$: this is the basis of the interesting proof in [8] which, however, considers only the special case in which $L = \mathfrak{A}$ and $\mathfrak{A}$ acts on L through the regular representation (called the *adjoint* representation in Lie algebra parlance). The proof we have given minimizes the use of identities by the device of selecting a maximal $\mathfrak{B}$ with the desired property.

11. See [4].

12. The lemma and proof are also valid if k is a field of characteristic p provided that $p > n$. The same applies to the applications in Section 14.

13. If $\mathbf{C}_1$, $\mathbf{C}_2$ are nilpotent, $\mathbf{C}_1^p = \mathbf{C}_2^q = 0$, and $\mathbf{C}_1\mathbf{C}_2 = \mathbf{C}_2\mathbf{C}_1$, then $(\mathbf{C}_1 + \mathbf{C}_2)^{p+q-1} = 0$ since the binomial expansion of the power leads to a sum of terms $\mathbf{C}_1^\alpha \mathbf{C}_2^\beta$ where $\alpha + \beta = p + q - 1$. Hence either $\alpha \geqq p$ or $\beta \geqq q$. The same reasoning applies to any sum of commutative nilpotent transformations.

14. This proof is closely modeled on that of [4]. By using double induction we avoid the enveloping algebra that Jacobson employs.

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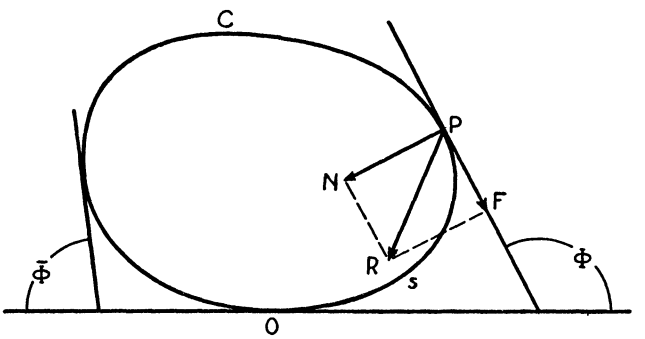
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THE MOTION OF A PARTICLE CONSTRAINED TO MOVE ON A ROUGH CONVEX CURVE

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The boundary of a bounded convex body* in the plane will be called a convex curve. A convex curve at each of whose points the curvature exists is called a *differentiable convex curve*, and we denote it by C. In this paper we restrict ourselves to such curves, and it will be helpful to refer to the diagram below in the following discussion.



Assumptions and notations. We assume that C corresponds to a rough wire whose coefficient of friction is μ. A bead or particle p of mass m is constrained to move on C. We assume the only force acting on p is the constraint C; *no* other external forces including gravity are acting on the particle. Let R denote the reaction of C on p. This force is resolved into the normal component N and the tangential component F. The law of friction states that $F=\mu N$. Moreover, ϕ is the conventional angle measured counterclockwise which the tangent to C makes with the positive x-axis. Let s be the arc length of that portion of C between the points O and P (see diagram). We assume the following initial conditions at time $t=0$ for the particle p,

$$(1) \qquad t = 0, \quad s = s_0, \quad \phi(s_0) = \phi_0, \quad \frac{ds}{dt} = v_0 > 0.$$

Objective. It is our purpose to determine results about the motion of p which depend solely upon the fact that C is a *differentiable convex curve*. Although the subject is an old and interesting one, I have not seen the theorems below elsewhere in print. Theorem 3 is truly delightful.

THEOREM 1. *If C is a differentiable convex curve, then the terminal velocity with which the particle p returns to its initial position is independent of C* (no gravity acts on p; see assumptions).

* A convex body is a convex set having interior points.

Proof. The differential equations for the motion of p are

$$m\frac{d^2s}{dt^2} = -F, \qquad \frac{mv^2}{\rho} = N, \tag{2}$$

where $1/\rho = d\phi/ds \geqq 0$, and where $v = ds/dt$. Since C is a differentiable convex curve, $\phi(s)$ is a continuous monotone non-diminishing function of s. Hence, $d\phi/ds$ is summable on $0 \leqq \phi \leqq 2\pi$ [1; p. 166], [2; p. 590]. This, together with the existence of $d\phi/ds$ on $0 \leqq \phi \leqq 2\pi$ implies $\phi(s) - \phi(s_1) = \int_{s_1}^{s} \phi'(s)ds$. [3, p. 143]. Equations (2) and $F = \mu N$ imply

$$\frac{d^2s}{dt^2} = \frac{-\mu v^2}{\rho}. \tag{3}$$

Since $v_0 > 0$, the unique solution of equation (3) subject to conditions (1) satisfies the equation

$$\frac{dv}{ds} = \frac{-\mu v}{\rho}, \tag{4}$$

as long as $v \neq 0 (d^2s/dt^2 = (dv/ds)v)$. The solution of (4) subject to conditions (1) is

$$v = v_0 e^{\mu(\phi_0 - \phi)}, \tag{5}$$

which can be easily verified by substitution. Equation (5) implies that $v \to 0$ and $s \to \infty$ as $t \to \infty$. Hence

$$v(\phi_0 + 2\pi) = v_0 e^{-2\pi\mu}, \tag{6}$$

and theorem 1 is established.

DEFINITION 1. *Let $T(\phi_0)$ and $T_1(\phi_0)$ denote the periods of time required for p to complete journeys about C in the counterclockwise and clockwise directions respectively, starting with an initial speed $v_0 > 0$, and $\phi_0 = \phi(s_0)$.*

THEOREM 2. *The period $T(\phi_0)$ satisfies the equation*

$$T(\phi_0) = \frac{e^{-\mu\phi_0}}{v_0} \int_{s_0}^{s_0+L} e^{\mu\phi} ds, \tag{7}$$

where L is the length of C, and where $\phi = \phi(s)$ on C.

Proof. Equation (7) is a direct consequence of equation (5) since $v \neq 0$ for $0 \leqq t < \infty$. In the event that $d\phi/ds \neq 0$ on C, then equation (7) has the interesting form

$$T(\phi_0) = \frac{e^{-\mu\phi_0}}{v_0} \int_{\phi_0}^{\phi_0+2\pi} \rho e^{\mu\phi} d\phi \tag{8}$$

where $\rho = \rho(\phi)$ is defined at all points of C.

THEOREM 3. *The period $T(\phi_0)$ is a constant for all ϕ_0 on the interval $0 \leqq \phi_0 \leqq 2\pi$ if and only if C is a circle. If r is the radius of the circle, then*

$$T \equiv T(\phi_0) = \frac{r}{v_0\mu}(e^{2\pi\mu} - 1).$$

Proof. From (7) we see that dT/ds_0 has the form

$$\frac{dT}{ds_0} = \frac{e^{-\mu\phi_0}}{v_0}\left[e^{\mu\phi(s_0+L)} - e^{\mu\phi(s_0)} - \mu\frac{d\phi(s_0)}{ds_0}\int_{s_0}^{s_0+L} e^{\mu\phi}ds\right]. \tag{9}$$

Equations (7) and (9), together with the fact $\phi(s_0+L) = \phi(s_0)+2\pi$, imply

$$\frac{dT}{ds_0} = -\mu T\frac{d\phi_0}{ds_0} + \frac{1}{v_0}(e^{2\pi\mu} - 1). \tag{10}$$

To prove the "only if" part, since $T(\phi_0) = \text{constant}$ on C, equation (10) implies

$$\frac{d\phi(s_0)}{ds_0} = \text{constant} \qquad (0 \leqq s_0 \leqq L).$$

Hence C is a circle, since C is a convex curve. The "if" part is obviously true, and so is the final equality in the theorem.

THEOREM 4. *The average value of $T(\phi_0)$ for the interval $0 \leqq \phi_0 \leqq 2\pi$ is equal to that of a circle of equal length L.*

Proof. I am indebted to my colleague Professor R. Steinberg for calling the above theorem to my attention. Equation (10) implies

$$\frac{1}{L}\int_0^L \frac{dT}{ds_0}ds_0 = \frac{-\mu}{L}\int_0^L T\frac{d\phi(s_0)}{ds_0}ds_0 + \frac{1}{v_0}(e^{2\pi\mu} - 1).$$

Since $T(s_0+L) = T(s_0)$, the left member is zero, so that we get

$$\frac{1}{2\pi}\int_0^{2\pi} Td\phi_0 = \frac{r}{\mu v_0}(e^{2\pi\mu} - 1),$$

where $L = 2\pi r$.

THEOREM 5. *If $T(\phi_0) = T_1(\phi_0)$ for all ϕ_0 on the interval $0 \leqq \phi_0 \leqq 2\pi$ (see definition 1), then C is a circle, and conversely.*

Proof. Measuring in a clockwise direction on C, let $\bar{\phi}$, $\bar{s}$ correspond to ϕ, s respectively, so that for a position of p on C, we have $\bar{s}+s = L$, $\bar{s}_0+s_0 = L$, $\bar{\phi}+\phi = 2\pi$, $\bar{\phi}_0+\phi_0 = 2\pi$ (see figure). The expression for $T_1 = T_1(\phi_0)$ corresponding to (7) is

$$T_1 \equiv T_1(\phi_0) = \frac{e^{-\mu\bar{\phi}_0}}{v_0}\int_{\bar{s}_0}^{\bar{s}_0+L} e^{\mu\bar{\phi}}d\bar{s},$$

and

$$\frac{dT_1}{d\bar{s}_0} = -\mu T_1 \frac{d\bar{\phi}_0}{d\bar{s}_0} + \frac{1}{v_0}(e^{2\pi\mu} - 1). \tag{11}$$

Subtracting (11) from (10), we get

$$\frac{dT}{ds_0} - \frac{dT_1}{d\bar{s}_0} = -\mu\left(T\frac{d\phi_0}{ds_0} - T_1\frac{d\bar{\phi}_0}{d\bar{s}_0}\right).$$

Since $d\bar{s}_0 = -ds_0$, $d\bar{\phi}_0 = -d\phi_0$, we get

$$\frac{d(T+T_1)}{ds_0} = -\mu\frac{d\phi_0}{ds_0}(T - T_1). \tag{12}$$

Since $T = T_1$ for $0 \leqq \phi_0 \leqq 2\pi$, equation (12) implies $T+T_1$ is a constant for $0 \leqq \phi_0 \leqq 2\pi$. Hence T is a constant, and, by Theorem 3, C is a circle. The converse is obvious.

THEOREM 6. *Let p and p_1 be two particles which leave the same initial point of C in opposite directions but with the same numerical speed v_0. If p and p_1 meet again for the first time with the same speed v, and if this is true for all initial points of departure, then C is a circle. The converse is also true.*

Proof. Equation (5) implies that p and p_1 meet again for the first time when $\phi = \phi_0 + \pi$. Hence, by the same token, p and p_1 meet again the second time at the original point of departure. Hence $T(\phi_0) = T_1(\phi_0)$. Since this is true for $0 \leqq \phi_0 \leqq 2\pi$, theorem 5 implies the above conclusion. The converse is obvious.

DEFINITION 2. *Let C_r be a parallel curve to C, where r is the distance between corresponding parallel tangent lines. An outward normal to C at P meets C_r at a corresponding point P_r.*

THEOREM 7. *Let $T(P_r, \phi_0)$ be the time to complete a journey about C_r starting at P_r with a speed v_0 and initial tangential angle ϕ_0, where $\phi_0 = \phi(s_0)$ on C. Then*

$$T(P_r, \phi_0) = T(P, \phi_0) + \frac{r}{\mu v_0}(e^{2\pi\mu} - 1). \tag{13}$$

(Observe that the last term is the time required to traverse a circle of radius r, starting with a speed v_0.)

Proof. Let s_r be the arclength on C_r measured in the same manner as in the figure. Equation (7) for C_r has the equivalent form

$$T(P_r, \phi_0) = \frac{e^{-\mu\bar{\phi}_0}}{v_0}\int_{s_{r0}}^{s_{r0}+L_r} e^{\mu\bar{\phi}(s_r)}ds_r, \tag{14}$$

where $\bar{\phi}(s_r)$ is the tangential angle at P_r for C_r, and where L_r is the length of C_r. Since $s_r = s + r\phi(s)$, $\bar{\phi}(s_r) = \phi(s)$, equation (14) implies equation (13). Equation

(13) also follows immediately from (8) when $d\phi/ds \neq 0$ on C.

In conclusion I will mention a few other questions, some of which are thought-provoking.

(1) What is the differentiable convex curve of length L for which min $T(\phi_0)$ $(0 \leqq \phi_0 \leqq 2\pi)$ is a minimum? Theorem 4 implies that it cannot be a circle. In fact, does such a minimizing curve exist?

(2) If a convex curve has a corner, can one judiciously replace the corner by a small circular arc of radius r and then let $r \to 0$ so as to get the theory for the case with a corner? Where would the coefficient of restitution appear?

(3) Can the theory leading to equations (5) and (7) be applied to space curves of positive curvature? In Theorem 4, the circle would be replaced by a closed curve of constant curvature. What are they?

(4) What is the function $\mu = \mu(\rho)$ so that $T(\phi_0)$ is a constant $(0 \leqq \phi_0 \leqq 2\pi)$ for all differentiable convex curves? Professor Steinberg knows the answer.

(5) It is quite clear that the above theory still holds if C possesses a curvature at all but a countable number of its points, and if C is smooth. Now suppose C is a convex curve at each of whose points a tangent line exists, and assume nothing more about the curvature of C. What can one say about the motion of p? Are the laws of motion defined in this situation?

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RESTRICTED OCCUPANCY THEORY—A GENERALIZATION OF PASCAL'S TRIANGLE*

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1. Introduction. One of the most elementary problems of occupancy theory is to determine the number of ways in which r indistinguishable objects can be distributed in k cells if *at most one* object is allowed in each cell. It is well known that the number of ways in which this can be done is given by the binomial coefficients $\binom{k}{r}$ which can be pictured in the form of Pascal's triangle where they are usually calculated by means of the recursion formula

$$\binom{k-1}{r-1} + \binom{k-1}{r} = \binom{k}{r}. \tag{1}$$

This formula enables one to calculate the binomial coefficients in a triangular

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array in which each entry is obtained by adding the two appropriate entries in the line immediately above.

It is the purpose of this paper to show that by generalizing this occupancy theory to the case where r indistinguishable objects are distributed in k cells with the maximum number of objects allowed in any one cell restricted to an arbitrary number m, one obtains recursion relations very similar to (1). Furthermore, it will be shown that these recursion relations generate a class of number triangles, one for each value of m, constituting a generalization of Pascal's triangle. The latter is here merely a special case for $m=1$.

2. Some preliminary relationships. In order to permit a general treatment of restricted occupancy problems let us put

$N_m(r, k)$ = the number of distinct ways in which r indistinguishable objects can be distributed in k cells allowing at most m objects per cell, and

$F_m(r, k/i)$ = the number of distinct ways in which r indistinguishable objects can be distributed in k cells allowing at most m objects per cell, if i cells contain the maximum number, namely, m, objects.

In order to unify the notation let us also employ the customary convention of letting binomial coefficients $\binom{a}{b}$ equal zero if either a and b are positive and b is greater than a or if b is negative.

The following three preliminary relationships involving the $F_m(r, k/i)$ and $N_m(r, k)$ will be needed in our subsequent work:

$$F_m(r, k/i) = \binom{k}{i} N_{m-1}(r - mi, k - i) \tag{2}$$

$$N_m(r, k) = \sum_{i=0}^{\infty} F_m(r, k/i) \tag{3}$$

$$N_m(r, k) = \sum_{i=0}^{\infty} \binom{k}{i} N_{m-1}(r - mi, k - i). \tag{4}$$

Equation (2) states that the number of arrangements in which exactly i cells contain m objects is the product of $\binom{k}{i}$, the number of ways in which one can select these i cells, times the number of ways in which the remaining $r-mi$ objects can be distributed in the remaining $k-i$ cells so that at most $m-1$ objects are in any one of these cells. Equation (3) expresses the fact that $N_m(r, k)$ equals the *sum* of the number of arrangements where *no* cell contains as many as m objects, where exactly *one* cell contains m objects, where exactly *two* cells contain m objects, *etc.* Equation (4) was obtained by simply substituting the righthand member of (2) for the $F_m(r, k/i)$ in equation (3). The summations on i in equations (3) and (4) are taken from 0 to infinity. This is merely a matter of convenience as there are only a finite number of non-zero terms. Evidently, we will get zero for $\binom{k}{i}$ when i is greater than k and we will get zero for $N_{m-1}(r-mi, k-i)$ whenever no such arrangements are possible.

3. The general recursion formula. We shall now prove that

$$N_m(r, k) = \sum_{j=0}^{m} N_m(r - j, k - 1) \tag{5}$$

for all positive integers k and m and non-negative integers r. When $k=0$ we have $N_m(r, 0) = \binom{0}{r}$ and in view of the convention agreed upon previously it follows that $N_m(r, 0)$ is equal to 1 for $r=0$ and equal to 0 for all other values of r.

In order to prove (5) by mathematical induction on m let us demonstrate first that it holds for $m=1$. Substituting $m=1$ we get

$$N_1(r, k) = N_1(r, k - 1) + N_1(r - 1, k - 1),$$

which can also be written as

$$\binom{k}{r} = \binom{k-1}{r} + \binom{k-1}{r-1}$$

in view of the results pointed out in the introduction. This last equation is so well known, that there is no need here for a proof.

Assuming now that (5) holds for $m=n$, where n is an arbitrary positive integer, we shall have to show that (5) also holds for $m=n+1$. Using (4) we can write

$$N_{n+1}(r - j, k - 1) = \sum_{i=0}^{\infty} \binom{k-1}{i} N_n(r - [n+1]i - j, k - i - 1).$$

Summing both sides of the equation on j from 0 to n we get

$$\begin{aligned}\sum_{j=0}^{n} N_{n+1}(r - j, k - 1) &= \sum_{j=0}^{n}\sum_{i=0}^{\infty} \binom{k-1}{i} N_n(r - [n+1]i - j, k - i - 1) \\ &= \sum_{i=0}^{\infty} \binom{k-1}{i} \sum_{j=0}^{n} N_n(r - [n+1]i - j, k - i - 1) \\ &= \sum_{i=0}^{\infty} \binom{k-1}{i} N_n(r - [n+1]i, k - i).\end{aligned}$$

The last step was based on the assumption that (5) holds for $m=n$, replacing r and k in (5) by $r-[n+1]i$, and $k-i$ respectively. Making use of the fact that

$$\binom{k-1}{i} = \frac{k-i}{k}\binom{k}{i}$$

we can now write

$$\sum_{j=0}^{n} N_{n+1}(r-j,\, k-1) = \sum_{i=0}^{\infty} \frac{k-i}{k}\binom{k}{i} N_n(r-[n+1]i,\, k-i)$$

$$= \sum_{i=0}^{\infty}\binom{k}{i} N_n(r-[n+1]i,\, k-i)$$

(6)

$$- \frac{1}{k}\sum_{i=0}^{\infty} i\binom{k}{i} N_n(r-[n+1]i,\, k-i)$$

$$= N_{n+1}(r,k) - \frac{1}{k}\cdot\sum_{i=0}^{\infty} i\binom{k}{i} N_n(r-[n+1]i,\, k-i),$$

the last step being based on equation (4). In order to evaluate the second term of the righthand member of the last equation let us use the relationship

$$i\binom{k}{i} = k\binom{k-1}{i-1}$$

and write

$$\sum_{i=1}^{\infty} i\binom{k}{i} N_n(r-[n+1]i,\, k-i) = k\cdot\sum_{i=1}^{\infty}\binom{k-1}{i-1} N_n(r-[n+1]i,\, k-i),$$

where the summation is taken from $i=1$ rather than $i=0$ since for $i=0$ the quantity summed is equal to zero.

If we now put $j=i-1$ and $K=k-1$, the righthand member of the last equation can be written as

$$k\cdot\sum_{j=0}^{\infty}\binom{K}{j} N_n(r-[n+1]-[n+1]j,\, K-j),$$

and using again relation (4) this becomes

$$k\cdot N_{n+1}(r-[n+1],\, K) = k\cdot N_{n+1}(r-[n+1],\, k-1).$$

Substituting this result into (6) we get

$$\sum_{j=0}^{n} N_{n+1}(r-j,\, k-1) = N_{n+1}(r,k) - N_{n+1}(r-[n+1],\, k-1)$$

and adding $N_{n+1}(r-[n+1],\, k-1)$ to both sides of the equation

$$\sum_{j=0}^{n+1} N_{n+1}(r-j,\, k-1) = N_{n+1}(r,k).$$

This completes the proof by induction, since we have shown that (5) must hold for $m=n+1$ assuming that it holds for $m=n$. An alternate derivation of this recursion formula may be given with the use of a generating function.

4. Illustrations. For $m=1$ equation (5) becomes

$$N_1(r, k) = N_1(r, k-1) + N_1(r-1, k-1)$$

and the total number of arrangements for small values of r and k are shown in the table below

k \ r	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
	. . . . .						

This is, of course, Pascal's triangle in which each entry (except for $k=0$) can be obtained as the sum of the entry immediately above and the one to the left of the one immediately above.

For $m=2$ equation (5) becomes

$$N_2(r, k) = N_2(r, k-1) + N_2(r-1, k-1) + N_2(r-2, k-1)$$

and the total number of arrangements for some small values of r and k are given below:

k \ r	0	1	2	3	4	5	6	7	8	
0	1									
1	1	1	1							
2	1	2	3	2	1					
3	1	3	6	7	6	3	1			
4	1	4	10	16	19	16	10	4	1	
5	1	5	15	30	45	51	45	30	15	. . .
	. . . . .									

In the construction of this table each value (except for $k=0$) was obtained by taking the sum of the entry immediately above and the two entries to the left of the one immediately above. For example, for $r=7$ and $k=5$ we had $16+10+4=30$.

For $m=3$ equation (5) becomes

$$N_3(r, k) = N_3(r, k-1) + N_3(r-1, k-1) + N_3(r-2, k-1) + N_3(r-3, k-1)$$

and the number of ways in which r indistinguishable objects can be distributed in k cells so that there are at most 3 objects per cell are shown below for small values of r and k:

k \ r	0	1	2	3	4	5	6	7	8	9	
0	1										
1	1	1	1	1							
2	1	2	3	4	3	2	1				
3	1	3	6	10	12	12	10	6	3	1	
4	1	4	10	20	31	40	44	40	31	20	. . .
5	1	5	15	35	65	101	135	155	155	135	. . .
	.	.	.	.	.	.	.	.	.	.	

In this table each entry (except for $k=0$) was obtained by taking the sum of the value immediately above and the three entries to the left of the one immediately above. For example, for $r=7$ and $k=5$ we had $31+40+44+40=155$.

It should be noted that in each of these tables the first row always consisted of a 1 for $r=0$ and no other entries. This holds in general in view of what we said in Section 3.

If m equals an arbitrary positive integer n, the construction of the corresponding table by means of (5) consists of taking for each entry (except for $k=0$) the sum of the entry immediately above and the n entries to the left of the one immediately above. In all of these calculations, missing values, *i.e.*, cases where no arrangements are possible, are taken as zero.

5. A further generalization. In the development of the general recursion relation we have assumed that the maximum number of objects per cell was restricted to an arbitrary constant integer m. We shall now demonstrate that there exists a recursion formula analogous to (5) in the limiting case when there are no restrictions as to the maximum number of objects permitted per cell. Letting

$N(r, k)$ = the number of distinct ways in which r indistinguishable objects can be distributed in k cells if there is no restriction as to the maximum number of objects permitted per cell,

the recursion relation analogous to (5) may be written as

$$N(r, k) = \sum_{j=0}^{\infty} N(r - j, k - 1). \tag{7}$$

In view of the fact that there are no possible arrangements when j is greater than r, (7) can also be written as

$$N(r, k) = \sum_{j=0}^{r} N(r - j, k - 1). \tag{8}$$

This relationship is easy to prove as it is known that*

$$N(r, k) = \binom{r + k - 1}{r}. \tag{9}$$

Making this substitution it remains to be shown that

$$\binom{r + k - 1}{r} = \sum_{j=0}^{r} \binom{r + k - j - 2}{r - j}. \tag{10}$$

Repeatedly applying identity (1) it can be seen that

$$\begin{aligned}\binom{r + k - 1}{r} &= \binom{r + k - 2}{r} + \binom{r + k - 2}{r - 1} \\ &= \binom{r + k - 2}{r} + \binom{r + k - 3}{r - 1} + \binom{r + k - 3}{r - 2} \\ &= \binom{r + k - 2}{r} + \binom{r + k - 3}{r - 1} + \binom{r + k - 4}{r - 2} + \binom{r + k - 4}{r - 3} \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

which proves recursion formulas (8) and (10).

Using this recursion relation one can readily construct a table of the number of distinct ways of distributing r indistinguishable objects in k cells with no restrictions as to the maximum number allowed per cell by calculating each entry (except for $k=0$) as the sum of the value immediately above plus the sum of *all* of the entries to the left of the entry immediately above. For $k=0$ we get the same values as before. In the table shown below the number of such distinct arrangements are shown for some small values of r and k:

k \ r	0	1	2	3	4	5	6	7	8	
0	1									
1	1	1	1	1	1	1	1	1	1	. . .
2	1	2	3	4	5	6	7	8	9	. . .
3	1	3	6	10	15	21	28	36	45	. . .
4	1	4	10	20	35	56	84	120	165	. . .
5	1	5	15	35	70	126	210	330	495	. . .
	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	

This table constitutes a final generalization of the method used in the construc-

* See Feller, W., An Introduction to Probability Theory and Its Applications, John Wiley, New York, 1950, p. 52.

tion of Pascal's triangle although, of course, the entries in this table no longer form a triangle.

The values shown in this table can be calculated by means of the recursion relation indicated above although it is actually much simpler to calculate each entry by taking the sum of the value immediately above and the value immediately to the left using the recursion relation

$$\binom{r+k-1}{r} = \binom{r+k-2}{r} + \binom{r+k-2}{r}.$$

MATHEMATICAL NOTES

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Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

AN INEQUALITY OF G. BORG

H. O. CORDES, Göttingen

The inequality in question is

$$\int_0^l \left|\frac{y''}{y} + k\right| dx \geqq \begin{cases} 2\sqrt{k}\cot\dfrac{\sqrt{k}\,l}{2}, & \text{when } -\infty < k < \dfrac{\pi^2}{l^2} \\ k\left(l - \dfrac{\pi}{\sqrt{k}}\right), & \text{when } \dfrac{\pi^2}{l^2} \leqq k < \infty. \end{cases} \tag{1}$$

In his paper (Über die Stabilität gewisser Klassen von linearen Differentialgleichungen, Arkiv för Matematik, Astronomi och Fysik, vol. 31 (1945)) Mr. Borg proved this under the assumptions

$$\begin{aligned} &y(x) \text{ continuous, } y'(x) \text{ absolutely continuous within } 0 \leqq x \leqq l, \\ &y(x) > 0 \quad \text{within} \quad 0 < x < l, \qquad y(0) = y(l) = 0. \end{aligned} \tag{1a}$$

In the special case $k=0$ the inequality becomes $\int_0^l |y''/y|\,dx > 4/l$ and was found earlier by A. Beurling. Borg's proof contains essentially the concept of convex covering; therefore it is necessary to allow all functions with absolutely continuous first derivatives, and the differentiation-theory of Lebesgue must be applied.

In this note we shall give a simplified proof which is valid under the assumptions (1a) as well as under the more elementary assumptions (1b)

$$\begin{aligned} &y(x),\ y'(x) \text{ continuous, } y''(x) \text{ piecewise continuous within } 0 \leqq x \leqq l; \\ &y(x) > 0 \quad \text{within} \quad 0 < x < l, \qquad y(0) = y(l) = 0. \end{aligned} \tag{1b}$$

Proof. Let

$$\left(\operatorname{Max}\left(\frac{4k}{\pi^2}, \frac{1}{l^2}\right)\right)^{-1} = p.$$

In each of the intervals

$$I_1: 0 \leqq x \leqq p \quad \text{and} \quad I_2: l - p \leqq x \leqq l$$

there is a point x_1 (resp. x_2) where $y(x)$ has a maximum within this interval. It is easy to see that one can choose $x_1 \leqq x_2$, even if I_1 and I_2 have a common part. In the case $p = l$ then $x_1 = x_2$. Now we put

$$T(y, k) = \int_0^l \left|\frac{y''}{y} + k\right| dx = \int_0^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{l} = T_1 + T_2 + T_3$$

and define

$$u(x) = \begin{cases} k^{-1/2} \sin \sqrt{k}x & \text{within} \quad 0 \leqq x \leqq x_1 \\ k^{-1/2} \sin \sqrt{k}\,(l\text{-}x) & \text{within} \quad x_2 < x \leqq l. \end{cases}$$

It follows that

$$T_1 = \int_0^{x_1} \left|\frac{y''}{y} - \frac{u''}{u}\right| dx \geqq \frac{1}{y(x_1)u(x_1)} \left| \int_0^{x_1} (uy'' - u''y)dx \right|$$

$$= \left|\frac{y'(x_1)}{y(x_1)} - \frac{u'(x_1)}{u(x_1)}\right| \geqq \left|\frac{y'(x_1)}{y(x_1)} + \frac{u'(x_1)}{u(x_1)}\right|.$$

The last relation is valid because at least one of the two summands always vanishes. Likewise

$$T_3 \geqq \left|\frac{y'(x_2)}{y(x_2)} + \frac{u'(x_2+0)}{u(x_2+0)}\right|, \quad \text{whence} \quad T_1 + T_3 \geqq \left|\left(\frac{y'}{y}\Big|_{x_2}^{x_1} + \frac{u'}{u}\Big|_{x_2+0}^{x_1}\right)\right|.$$

Finally

$$T_2 \geqq \left| \int_{x_1}^{x_2} \left(\frac{y''}{y} + k\right) dx \right| = \left| k(x_2 - x_1) + \frac{y'}{y}\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{y'}{y}\right)^2 dx \right|.$$

Consequently

$$T(y, k) \geqq \left| k(x_2 - x_1) + \frac{u'}{u}\Big|_{x_2+0}^{x_1} + \int_{x_1}^{x_2} \left(\frac{y'}{y}\right)^2 dx \right| \geqq k(x_2 - x_1) + \frac{u'}{u}\Big|_{x_2+0}^{x_1}$$

because

$$\frac{u'}{u}\Big|_{x_2+0}^{x_1} \geqq 0, \qquad k(x_2 - x_1) \geqq 0.$$

Using the following inequalities the stated inequality is proved:

$$x_2 - x_1 \geqq l - \frac{\pi}{\sqrt{k}}, \quad \text{when} \quad k \geqq \frac{\pi^2}{l^2}$$

$$\frac{u'}{u}\Big|_{x_2+0}^{x_1} \geqq 2\sqrt{k}\cot\frac{\sqrt{k}\,l}{2}, \quad \text{when} \quad k < \frac{\pi^2}{l^2}$$

(the latter of these can be proved by elementary differentiation).

Remark. In the case $k<\pi^2/l^2$ one can see the inequality (1) perhaps still more simply in the following manner: Then a Green's function $G(x, z)$ of the problem $y''+ky=0$, $y(0)=y(l)=0$ exists and we have

$$y(x) = \int_0^l G(x, z)(y''(z) + ky(z))dz.$$

$G(x, z)$ does not change sign within $0<x, z<l$. One obtains by differentiation

$$M = \max_{0\leqq x,z\leqq l} |G(x, z)| = \frac{1}{2\sqrt{k}\cot\dfrac{\sqrt{k}\,l}{2}}.$$

If one defines $m=\max_{0\leqq x\leqq l}y(x)=y(x_1)$ it follows that

$$T(y, k) \geqq \frac{1}{mM}\left|\int_0^l G(x_1, z)(y''(z) + ky(z))dz\right| = \frac{1}{M} = 2\sqrt{k}\cot\frac{\sqrt{k}\,l}{2}.$$

THE n-DIMENSIONAL CUBE AND THE TOWER OF HANOI

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The purpose of this note is to point out a connection between a Hamiltonian circuit, abbreviated H-circuit, of the n-cube and the moves of the well known tower of Hanoi puzzle.*

With respect to a rectangular coordinate system in Euclidean n-space, let the vertices of an n-cube be defined by the 2^n ordered n-tuples, $(\epsilon_1, \cdots, \epsilon_n)$, $\epsilon_i=0, 1$ for all i. By the 1-skeleton of the n-cube is meant the linear graph whose vertices and lines are the vertices and edges of the n-cube. An H-circuit of a linear graph is a closed path containing each vertex exactly once. An H-circuit of the n-cube is defined to be an H-circuit of its 1-skeleton. A sequence of 2^n vertices, $v_1, \cdots, v_{2^n}$, of the n-cube is said to *describe* an H-circuit of the n-cube if the closed path $v_1v_2 \cdots v_{2^n}v_1$ is an H-circuit of the n-cube.

The following two remarks are easily verified. (1) Two vertices,

* See, for example, W. W. Rouse Ball, Mathematical Recreations and Essays, Rev. ed., New York, 1947, pp. 262–266 (Hamiltonian circuit), 303–305 (tower of Hanoi); H. S. M. Coxeter, Regular Polytopes, New York, 1949, pp. 8 (Hamiltonian circuit), 123 (n-cube).

$v=(\epsilon_1, \cdots, \epsilon_n)$ and $v'=(\eta_1, \cdots, \eta_n)$, of the n-cube are connected by an edge, vv', if and only if there is some k, $1 \leqq k \leqq n$, such that $\epsilon_k \neq \eta_k$ and $\epsilon_i = \eta_i$ for all $i \neq k$. (2) Any sequence of 2^n different n-tuples of 0's and 1's having the property that any two consecutive n-tuples differ in exactly one entry (where the first and last n-tuples are to be considered as consecutive) describes an H-circuit of the n-cube.

We now describe briefly the tower of Hanoi. This puzzle consists of three pegs, on one of which are piled n flat pierced disks, each smaller than the one below it. The problem is to use the fewest possible moves to transfer these disks to one of the other two pegs by removing one disk at a time from the top of any pile and placing it on any peg which does not already hold a smaller disk. It is easily seen that there is a (unique) solution to the problem requiring exactly 2^n-1 moves. That is, counting the initial position and the final position, there are 2^n different positions of the disks. The sequence of positions, in the order in which they occur, is called an H_n-sequence. We propose to identify these positions with the vertices of the n-cube. Let the disks be numbered according to increasing size with the numbers $1, \cdots, n$. Let the jth position of the disks in the solution of the puzzle be represented by the n-tuple of 0's and 1's, $(\epsilon_{1j}, \cdots, \epsilon_{ij}, \cdots, \epsilon_{nj})$, where ϵ_{ij} is the number of moves, modulo 2, which have been completed by the ith disk.

THEOREM. *The sequence of n-tuples in an H_n-sequence describes an H-circuit of the n-cube.*

Proof. We have only to show that an H_n-sequence has the properties given in (2). Except possibly for the first and last n-tuples of the sequence it is clear that any two consecutive n-tuples differ in exactly one entry, for any position is obtained from its predecessor by a move of exactly one disk. The first and last n-tuples are also so related, for the first consists entirely of 0's and the last consists entirely of 0's except for a final 1. It remains only to prove that all the terms of the sequence are different. For $n=1$ this is clearly true. The general case is proved by an inductive argument, using the fact that for $n+1$ disks one first solves the puzzle for n disks, then moves the largest disk (number $n+1$) and replaces the n disks on top of it by re-solving the puzzle for n disks. Consequently, the H_{n+1}-sequence contains (in a rearrangement) the 2^n different terms of the H_n-sequence augmented by a 0 in the $(n+1)$st place and the same 2^n different terms augmented by a 1 in the $(n+1)$st place. Thus the terms of the H_{n+1}-sequence are all different. This completes the proof of the theorem.

We conclude with the following remark. Suppose the jth step in solving the Hanoi puzzle consists in moving the kth disk, clockwise or counterclockwise according as k is even or odd. Then the successive values of k are 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, $\cdots$. In fact, k is one more than the number of factors 2 in j. That is, k is given by the formula $j=2^{k-1}l$, where l is odd. The sequence of k's facilitates the description of the H-circuit, for its jth edge is parallel to the kth coordinate axis.

AN EQUIVALENT CONDITION FOR UNIFORM CONVERGENCE

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Given a pointwise convergent sequence of functions from a compact space, it is known that a sufficient condition for uniform convergence is equicontinuity. For the usual type of countable sequence, equicontinuity is also necessary. (See [1], Theorems 24 and 25.) The main purpose of this note is to present another necessary and sufficient condition for uniform convergence of a pointwise convergent sequence of continuous mappings from a compact Hausdorff space. At the end of the note, some generalizations are stated.

One consequence of the following theorem is the Dini theorem on uniform convergence of a monotone sequence. ([1], Theorem 26.)

1. THEOREM. *Let S be a compact space, and R a space with a metric $d[\cdot, \cdot]$. Let $\{f_n\}$ be a sequence of continuous mappings on S to R, which converges pointwise to a mapping g. Then $\{f_n\}$ converges uniformly if and only if* (1) *g is continuous, and* (2) *for arbitrary $\epsilon>0$, there exists an integer $m\geqq 0$ and a $\delta>0$, such that whenever $n>m$ and $d[f_i(s), g(s)]<\delta$, then $d[f_{i+n}(s), g(s)]<\epsilon$.*

Proof: The necessity of (1) is well known, and the necessity of (2) is an immediate consequence of the definition of uniform convergence.

To prove sufficiency, choose $\epsilon>0$ and let m and δ be as given by (2). Since $\{f_n\}$ converges pointwise to g, for any $s_0\in S$, there exists an integer i so that $d[f_i(s_0), g(s_0)]<\delta$. Using continuity of f_i and of g, there exists a neighborhood of s_0, such that for s in this neighborhood, we have $d[f_i(s), g(s)]<\delta$. By compactness of S, there exists a finite set of integers $i_1, \cdots, i_p$, such that for every $s\in S$, one of these integers i will satisfy $d[f_i(s), g(s)]<\delta$. Define k to be the largest of the integers $m+i_1, \cdots, m+i_p$. Then for $n>k$ and $s\in S$, there will be an integer i such that $n>m+i$ and $d[f_i(s), g(s)]<\delta$. Consequently, for $n>k$ and all $s\in S$, we have $d[f_n(s), g(s)]<\epsilon$, and the convergence is uniform.

If $g(\cdot)$ of the above theorem is a constant mapping, then it is not required that the topology on R be metric (or have some other uniformity specified). This remark leads to the following corollary.

2. COROLLARY. *Let Q be a closed subset of a topological space R, and $\{f_n\}$ any sequence of continuous mappings from a compact space S into R. Then for each open set O containing Q, there exists an integer $k=k(O)$ independent of s such that $f_n(s)\in O$ for all s and all $n>k$, if and only if* (1) *for each $s\in S$, there exists an integer $k_s(O)$ such that for $n>k_s$, $f_n(s)\in O$, and* (2) *for each neighborhood N of Q, there exists a positive integer m and a neighborhood M of Q, such that whenever $n>m$ and $f_i(s)\in M$, then $f_{i+n}(s)\in N$.*

Proof: Identify the points of Q. Then let the mapping g be the constant mapping of S into the point Q, and apply the preceding remark.

It might be supposed that if the condition of compactness were removed from Theorem 1 that the theorem would still be true. However, consider the sequence of homomorphisms of the locally compact group of real numbers into the group of complex numbers as given by $f_n(x) = xe^{i/n}$. Although the condition holds here and the convergence is uniform on every bounded set of real numbers, the sequence $\{f_n\}$ does not converge uniformly.

If we consider a sequence $\{f_n\}$ which converges uniformly to g and for which $f_1 = g$ but $f_n \neq g$ for some $n > 1$, we see that the number m of the previous theorem must sometimes be greater than zero no matter how δ is chosen. The next theorem shows that for the case where the f_n's are iterations of a single mapping, m may always be taken to be zero.

3. THEOREM. *Let S be a compact space with a metric $d[\cdot, \cdot]$, and f a continuous mapping of S into S. Suppose that the sequence of iterations of f, $\{f^n\}$, converges pointwise to a mapping g. Then $\{f^n\}$ converges uniformly if and only if* (1) *g is continuous, and* (2) *for arbitrary $\epsilon > 0$, there exists $\delta > 0$, such that if $d[s, g(s)] < \delta$, then for $n = 1, 2, \cdots$, we have $d[f^n(s), g(s)] < \epsilon$.*

Proof: Sufficiency of the condition is easily established by applying Theorem 1 with $m = 0$.

To prove necessity, by Theorem 1 there is a $\delta_1 > 0$ and a positive integer m, such that for $n > m$ and $d[f^i(s), g(s)] < \delta_1$, we have $d[f^{i+n}(s), g(s)] < \epsilon$. Since f is a continuous function and $f^n(s)$ converges to $g(s)$ for $s \in S$, it follows that for $s \in S$, $f[g(s)] = g(s)$. By continuity of f^i for $i = 1, \cdots, m$, and by compactness of S, there is a δ with $0 < \delta < \delta_1$, such that for $s_1, s_2 \in S$, and $d[s_2, g(s_1)] < \delta$, $f^i(s_2)$ is in the ϵ-neighborhood of $g(s_1)$, for $i = 1, \cdots, m$. Therefore, if $0 < n \leqq m$ and $d[f^i(s), g(s)] < \delta$, we have that $f^{i+n}(s)$ is in the ϵ-neighborhood of $g(s)$. But since $\delta < \delta_1$, this is also true if $n > m$, and so the theorem is proved.

Again, if $g(\cdot)$ is a constant mapping, then the existence of a metric (or other uniformity) on R is not required.

4. COROLLARY. *Let Q be a closed subset of a compact space S, and f a continuous mapping of S into S. Then $\bigcap_{n=1}^{\infty} f^n(S) \subset Q$ if and only if* (1) *for each $s \in S$ and each neighborhood N of Q, there exists a positive integer m such that $f^n(s) \in N$ for $n > m$, and* (2) *for each neighborhood N_1 of Q there exists a neighborhood N_2 of Q such that $f^n(N_2) \subset N_1$ for all positive integers n.*

Proof: Identify the points of Q, and apply the preceding remark and the fact that $S \supset f(S) \supset f^2(S) \supset \cdots$.

Consider the case where S is the interval $0 \leqq s \leqq 1$ and $f(s) = s^2$. Then we have an example where the second part of the condition of Theorem 3 holds, but where g is not continuous. If the point $s = 0$ is identified with the point $s = 1$, then the function g is continuous but the second half of the condition of Theorem 3 does not hold.

5. Definition. Let J be a Moore-Smith sequence [2] and let C be a set of conditions on ordered pairs of elements of the sequence. Then let C be said to be a *uniformity* on J, if and only if C satisfies the following requirements:

a) Given j_1, j_2 and $j_3 \in J$ so that j_3 follows j_2, any condition of C which is satisfied by (j_1, j_2) must be satisfied by (j_1, j_3).

b) Given any element $j \in J$ and condition $c \in C$, there exists an element $j' \in J$ such that (j, j') satisfies c.

c) Given $j \in J$, there exists a condition in C such that whenever (j_1, j_2) satisfies the condition, then j_2 must follow j.

For J the set of positive integers, C may consist of the set of conditions $\{c_m\}$ where each c_m corresponds to the non-negative integer m such that (j_1, j_2) satisfies c_m if and only if $j_2 - j_1 > m$. In this case, Theorem 6, which follows, specializes to Theorem 1.

6. THEOREM. *Let J be a Moore-Smith sequence with a uniformity C. Let S be a compact Hausdorff space and R a uniform topological space* [3]. *For each $j \in J$, let f_j be a continuous mapping of S into R. Let the sequence of mappings $\{f_j\}$ converge pointwise to a mapping g. Then a necessary and sufficient condition that $\{f_j\}$ converges uniformly to g is that g is continuous and that given an index α, there exists a condition $c \in C$ and an index β such that whenever $f_{j_1}(s) \in \beta[g(s)]$ and (j_1, j_2) satisfies c, then $f_{j_2}(s) \in \alpha[g(s)]$.*

Proof: The proof of this theorem is entirely analogous to the proof of Theorem 1.

Corollary 2 may also be generalized to Moore-Smith sequences with a uniformity. Theorem 3 may be generalized to non-metric compact spaces.

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CLASSROOM NOTES

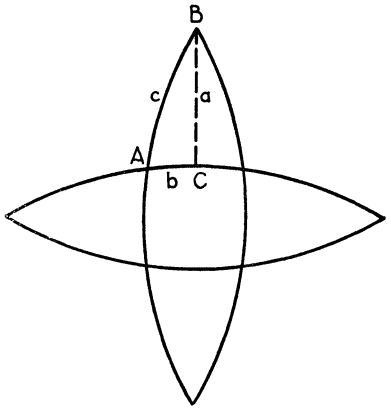
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AN APPROXIMATE REPRESENTATION OF A STERADIAN

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A steradian is defined as the solid angle subtended at the center of a sphere by a portion of the surface whose area is equal to the square of the radius. An excellent approximate representation can be made as follows: Take two 60° lunes whose median meridians bisect each other perpendicularly. In geographical terminology one lune might have its vertices at the north and south poles respectively and its median line in coincidence with the prime meridian, its sides being the 30° (east and west respectively) meridians. The other lune would then have its median line along the equator, one vertex at longitude 90° east, the other at 90° west. The sides of this second lune would intersect the prime meridian at latitude 30° north and south respectively. That part of the surface of the sphere common to the two lunes subtends a solid angle at the center of the sphere which approximates a steradian to a high degree of accuracy.



In the schematic representation of the two lunes, the dotted line is the great circle arc which bisects angle B. Thus, in the spherical triangle ABC, we have $B=30°$, $a=60°$, $C=90°$, and by using a simple formula from spherical trigonometry we find

$$\cos A = \cos a \sin B = \cos 60° \sin 30° = 1/4, \quad A = 75°31.36'.$$

Then $180°-A=104°\ 28.64'$, which is one of the angles of the regular spherical quadrilateral that purports to subtend the steradian. From this it is easy to calculate the spherical excess of the quadrilateral to be 57° 54.56′. Since the area of a spherical quadrilateral subtending a steradian is 57° 17.75′, our approximate representation is only 1 per cent too large.

ON UNIFORM AND ABSOLUTE CONVERGENCE

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The following question is of frequent occurrence: If a series $\sum a_n(x)$, $0 \leqq x \leqq 1$, is uniformly convergent and absolutely convergent, is it absolutely uniformly convergent, *i.e.*, is $\sum |a_n(x)|$ uniformly convergent?

To construct a counter example, take a series $\sum a_n(x)$ of non-negative terms which is non-uniformly convergent and study the series $\sum (-1)^n a_n(x)$ for uniform convergence.

Example. $x^n(1-x) = a_n(x)$.

THEOREM. *Let* $\sum_{n=1}^{\infty} a_n(x)$, $0 \leqq x \leqq 1$, *be a series of non-negative terms which is convergent but not uniformly, such that*

a) $a_n(x) \longrightarrow 0$ *uniformly in* x

b) $a_n(x) \geqq a_{n+1}(x)$.

Then $\sum_{n=1}^{\infty} (-1)^n a_n(x)$ *is uniformly convergent.*

Proof. By Leibniz' theorem on alternating series

$$|R_N(x)| = \left| \sum_{n=N+1}^{\infty} (-1)^n a_n(x) \right| \leqq |(-1)^N a_N(x)| = a_N(x)$$

and by a) $|R_N| \to 0$ uniformly in x.

REMARKS CONCERNING THE PROBABILITY INTEGRAL

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In a recent issue of this MONTHLY [1] the probability integral

$$\int_0^{\infty} e^{-u^2/2} du \tag{1}$$

was evaluated without the use of the conventional approach involving double integrals. In this article there will be presented another method of evaluation which does not employ double integration and which is perhaps a little more direct from the point of view of motivation, at least at the beginning. We shall restrict our attention to the evaluation of

$$S = \int_0^{\infty} e^{-t^2} dt \tag{2}$$

which may be transformed into (1).

By virtue of the well-known result

$$\lim_{n \to \infty} \left(1 + \frac{u}{n}\right)^n = e^u \tag{3}$$

it appears plausible to consider the integral

$$S_n = \int_0^{\sqrt{n}} \left(1 - \frac{t^2}{n}\right)^n dt. \tag{4}$$

Heuristically it would seem from (3) that

$$\lim_{n\to\infty} S_n = \int_0^{\infty} e^{-t^2} dt. \tag{5}$$

This will, in fact, be proved.

Upon making the substitution $t = \sqrt{n} \sin x$ in (4) we have

$$S_n = \sqrt{n} \int_0^{\pi/2} \cos^{2n+1} x dx = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots 2n - 1} \frac{\sqrt{n}}{2n + 1} \tag{6}$$

by elementary integration. An application of Wallis' Product (see [2] for an elementary treatment) as in [1] shows that

$$\lim_{n\to\infty} S_n = \frac{\sqrt{\pi}}{2}. \tag{7}$$

Hence if we can prove (5) it will follow that

$$\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \tag{8}$$

or

$$\int_0^{\infty} e^{-u^2/2} du = \sqrt{\frac{\pi}{2}} \tag{9}$$

as required.

Consider

$$S - S_n = \int_0^{\sqrt{n}} \left\{e^{-t^2} - \left(1 - \frac{t^2}{n}\right)^n\right\} dt + \int_{\sqrt{n}}^{\infty} e^{-t^2} dt. \tag{10}$$

Since the second integral on the right converges it follows that

$$\lim_{n\to\infty} \int_{\sqrt{n}}^{\infty} e^{-t^2} dt = 0. \tag{11}$$

Thus to show that $\lim_{n\to\infty} S_n = S$ we need only prove that

$$\lim_{n\to\infty} \int_0^{\sqrt{n}} \left\{e^{-t^2} - \left(1 - \frac{t^2}{n}\right)^n\right\} dt = 0. \tag{12}$$

In order to do this we employ techniques familiar from the theory of the gamma function. It is easily shown in [3] that

$$0 \leqq e^{-u} - \left(1 - \frac{u}{n}\right)^n \leqq \frac{u^2}{n} e^{-u}. \tag{13}$$

Writing t^2 in place of u we have

$$0 \leqq \int_0^{\sqrt{n}} \left\{e^{-t^2} - \left(1 - \frac{t^2}{n}\right)^n\right\} dt \leqq \frac{1}{n} \int_0^{\sqrt{n}} t^4 e^{-t^2} dt \tag{14}$$

and the result (12) follows immediately upon letting $n \to \infty$ since

$$\int_0^{\infty} t^4 e^{-t^2} dt \tag{15}$$

converges. This completes the proof.

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2. Knopp, K., Theory and Application of Infinite Series, Hafner Publishing Co., New York, 1928, p. 384.
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APPROXIMATE DEPRESSED EQUATIONS

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The following method of solving a cubic equation with three real roots is found in a Freshman Mathematics text.* Given the cubic equation $x^3+px+q=0$ where $q^2/4+p^3/27<0$, find one root to any desired degree of approximation. Denote this approximate root by r and divide the cubic polynomial by $x-r$ to obtain

$$\frac{x^3 + px + q}{x - r} = x^2 + rx + p + r^2 + \frac{r^3 + pr + q}{x - r}.$$

Set $x^2+rx+p+r^2=0$ and solve for the remaining two roots. This note examines the approximate error involved in finding the second and third roots by this method.

We may call $x^2+rx+p+r^2=0$ an approximate depressed equation. Its roots are

$$x = -\frac{r}{2} \pm \frac{\sqrt{-4p - 3r^2}}{2}.$$

Let r_1 and r_2 denote these two roots with the upper and lower signs respectively. If r is subject to an error dr, then we have

* W. E. Milne and D. R. Davis, Introductory College Mathematics, Boston, 1941.

$$dx = -1/2\left[1 \pm \frac{3r}{\sqrt{-4p - 3r^2}}\right]dr$$

as the approximate error in the second and third roots due to an error dr in the first root.

The approximate errors in r_1 and r_2 due to an error dr in r are given in Table 1 for various values of r.

TABLE 1

r	Approximate error in r_1	Approximate error in r_2
0	$-.5\ dr$	$-.5\ dr$
$.1\sqrt{-p}$	$-.58\ dr$	$-.42\ dr$
$.2\sqrt{-p}$	$-.65\ dr$	$-.35\ dr$
$.4\sqrt{-p}$	$-.82\ dr$	$-.18\ dr$
$\sqrt{-3p}/3 \cong .58\sqrt{-p}$	$-1\ dr$	0
$.8\sqrt{-p}$	$-1.33\ dr$	$.33\ dr$
$\sqrt{-p}$	$-2\ dr$	$1\ dr$
$1.08\sqrt{-p}$	$-2.8\ dr$	$1.8\ dr$
$1.1\sqrt{-p}$	$-3.2\ dr$	$2.2\ dr$
$1.15\sqrt{-p}$	$-10.07\ dr$	$9.07\ dr$
$2\sqrt{-3p}/3 \cong 1.155\sqrt{-p}$	undefined	
$>2\sqrt{-3p}/3$	r_1, r_2 imaginary	

The approximate errors in r_1 and r_2 for negative r may be obtained from columns three and two respectively.

As an example, suppose we have found $r = 1.532$ as one of the real roots of $x^3 - 3x + 1 = 0$ correct to three decimal places whereas the correct value of this root is 1.53209 to five decimal places. By the method described above the approximate depressed equation is

$$x^2 + 1.532x - .652976 = 0,$$

from which we obtain to five decimal places

$$r_1 = .34743, \quad r_2 = -1.87943.$$

Using $dr = .00009$ and $r = 1.532 \cong .88\sqrt{-p}$, we infer from Table 1 that the approximate error in r_1 is equal to $-1.60\ dr = -.00014$. This compares with an actual error in r_1 of $-.00013$. We likewise infer that the approximate error in r_2 is $.60\ dr = .00005$. This compares with an actual error in r_2 of .00004.

It is seen from Table 1 that the errors in r, r_1, and r_2 are of the same order of magnitude unless $|r| > 1.1\sqrt{-p}$. In particular it should be noticed that if $|r| > 2\sqrt{-p/3} \cong 1.155\sqrt{-p}$, r_1 and r_2 will be imaginary although it is known that all roots are real. The reader may use $x^3 - 7x - 7 = 0$ with a known real root of 3.04892 to five decimal places as an example wherein the method of an approximate depressed equation leads to large errors in the second and third roots. Its other two roots are -1.35689 and -1.69202 to five decimal places.

can neglect $L=24$ and $L=27$. Thus Case II reduces to Case I for the remainder of the solution.

Case III. Smith's ranch is *not necessarily* in the corner of the county. Smith knows its location, but Green does not. Does this fact affect the solution?

An equivalent solution, but with much less detailed analysis, was offered by Frank Kocher. Different solutions, either containing oversights or not containing sufficient indication of method of reasoning, were offered by E. C. Coolidge and C. S. Ogilvy (jointly), Octave Levenspeil, J. V. Pennington, Azriel Rosenfeld, and the Proposer.

Chain of Circles in a Segment

E 1166 [1955, 364]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Let DE be a variable chord perpendicular to diameter AB of a given circle (O). The maximum circle (ω_0) inscribed in the smaller segment, DEB, touches chord DE in C. The circle (ω_1) is tangent to (ω_0), (O), and DC and another circle (ω_2) is tangent to (ω_1), (O), and DC. Find the ratio BC/CA for which the radius of circle (ω_2) is a maximum.

Solution by Hüseyin Demir, Zonguldak, Turkey. Denote the radii of (O) and (ω_i) by R and r_i respectively. Let (ω_1), (ω_2) touch CD in C_1, C_2. Then we easily get

$$CC_1 = 2\sqrt{r_0 r_1}, \qquad C_1C_2 = 2\sqrt{r_1 r_2}.$$

From right triangles having hypotenuses $O\omega_1 = R - r_1$, $O\omega_2 = R - r_2$ we get

$$(R - 2r_0 + r_1)^2 + 4r_0 r_1 = (R - r_1)^2, \tag{1}$$

$$(R - 2r_0 + r_2)^2 + 4(\sqrt{r_0 r_1} + \sqrt{r_1 r_2})^2 = (R - r_2)^2. \tag{2}$$

The value

$$r_1 = r_0(R - r_0)/R,$$

obtained from (1), when substituted in (2) yields

$$r_2 = (R - r_0)^2 r_0/(R + r_0)^2.$$

Now, introducing $k = BC/CA = r_0/(R - r_0)$ and applying the derivative test for a maximum, we get

$$k = (\sqrt{5} - 1)/4.$$

Also solved by G. B. Charlesworth, Walter Guber, A. R. Hyde, R. B. Plymale, and the Proposer. Some of these solutions were based upon a misinterpretation of the figure of the problem.

The Proposer remarked that the problem was suggested by an attempt to display circle (ω_2) to best advantage in a diagram. The following interesting allied facts were pointed out by the Proposer:

1. Circles (ω_0) and (ω_1) are maximum when C coincides with O, but (ω_2) is a maximum when $BC/CA=(\sqrt{5}-1)/4$, with the unexpected consequence that *CB is the side of a regular decagon inscribed in the circle on AC as diameter.*

2. $r_{2(\max)}=r_1/2$.

3. $r_n=2AB\cos^2 u/[\tan^n(u/2)+\cot^n(u/2)]^2$, u being the angle ABD, (communicated to the Proposer by Victor Thébault).

4. r_n is rational if AC and CB are rational.

An Inequality by Mathematical Induction

E 1167 [1955, 364]. *Proposed by C. D. Olds, San Jose State College*

Prove by mathematical induction that if a is a real number $\neq 1$, n a positive integer, then

$$\alpha_n=\frac{a^{2n+2}-1}{a(a^{2n}-1)}>\frac{n+1}{n}.$$

Solution by O. E. Stanaitis, St. Olaf College. It is apparent that we must further restrict a to be positive. Then the inequality is true for $n=1$, since

$$\alpha_1=a+1/a>2.$$

Assume the inequality holds for a particular value of n, say $n=k$. Then

$$\alpha_{k+1}=\alpha_1-1/\alpha_k>2-k/(k+1)=(k+2)/(k+1),$$

which is the form of the inequality when $n=k+1$. This completes the proof.

Also solved by J. R. Byrne, Hüseyin Demir, I. A. Dodes, H. M. Feldman, N. J. Fine, W. Y. Gamzon, Virginia Hanly, A. R. Hyde, Marvin Marcus, D. C. B. Marsh, L. A. Ringenberg, Azriel Rosenfeld, Enrique Rubini, M. R. Spiegel, C. W. Topp, Chih-yi Wang, David Zeitlin, and the Proposer. Late solutions by M. S. Klamkin and J. W. Ross.

Several solvers also showed that, for fixed a, $\alpha_{k+1}<\alpha_k$. Ringenberg pointed out that if we multiply both sides of the inequality by a, then we need not impose the further restriction that a be positive. Wang showed that the (corrected) problem is equivalent to showing that the area under the curve $y=x^n$, above the x-axis, and between $x=1$ and $x=a^2$, is always greater than the area under the curve $y=ax^{n-1}$, above the x-axis, and between $x=1$ and $x=a^2$.

Perfect Triangles

E 1168 [1955, 365]. *Proposed by R. R. Phelps, U.C.L.A.*

In an analogy with perfect numbers, let us define a *perfect triangle* as one whose integer valued sides add up to twice its area. An example is the (3, 4, 5) triangle. Find all perfect triangles.

Solution by N. J. Fine, University of Pennsylvania. We have

$$A^2=s(s-a)(s-b)(s-c),$$

where $2s=a+b+c=2A$. Hence

$$A = (A - a)(A - b)(A - c) = xyz,$$

say, where $x \geqq y \geqq z > 0$. Also,

$$x + y + z = 3A - (a + b + c) = A.$$

Hence $x+y+z=xyz$, so $y(yz-1) \leqq x(yz-1) = y+z \leqq 2y$, $yz \leqq 3$, $z^2 \leqq yz \leqq 3$, and $z=1$. Then $x(y-1)=y+1$, and $y=2$, $x=3$. Thus the (3, 4, 5) triangle is the only solution.

Also solved by M. Aissen and D. Reiner and D. Warner (jointly), Leon Bankoff, H. W. Becker, Arne Benson, G. B. Charlesworth, Hüseyin Demir, I. A. Dodes, Michael Goldberg, Sam Kravitz, D. C. B. Marsh, L. A. Ringenberg, Azriel Rosenfeld, Alan Wayne, R. E. Wild, and the Proposer. Late solution by M. S. Klamkin.

Editorial Note. As a corollary to the above result we have that the only triangle with integer valued sides and with inradius $r=1$ is the (3, 4, 5) triangle. For, in any triangle, $rs=A$. If $r=1$ then $s=A$, and the triangle is perfect.

The only triangles whose integer valued sides add up to exactly the area of the triangle are (5, 12, 13), (6, 8, 10), (6, 25, 29), (7, 15, 20), (9, 10, 17). This was proved in 1904 by W. A. Whitworth and D. Biddle. See Dickson, *History of the Theory of Numbers*, vol. II, p. 199.

Positive Increasing Sequences

E 1169 [1955, 365]. *Proposed by Joseph Lehner, Los Alamos Laboratory*

Let $0<x_1<x_2<\cdots$ and let $\sum_{n=1}^{\infty} x_n^{-h}$ converge. Then, if $0<h \leqq 1$ and $\alpha>0$, we have $x_{n+1}-x_n>\alpha$ for infinitely many indices n. If $h>1$, there are sequences $\{x_n\}$ with $x_{n+1}-x_n \to 0$.

Solution by O. E. Stanaitis, St. Olaf College. In the first part, suppose the contrary. Then, for some positive integer N, $x_{N+k}-x_{N+k-1} \leqq \alpha$ for $k=1, 2, \cdots$, and hence $x_{N+m} \leqq x_N + m\alpha$ for $m=1, 2, \cdots$. If we note that

$$\sum_{n=1}^{\infty} x_n^{-h} > \sum_{m=1}^{\infty} x_{N+m}^{-h} \geqq \sum_{m=1}^{\infty} (x_N + m\alpha)^{-h},$$

it follows that the first series is divergent, which contradicts the hypothesis.

The second part follows at once if we put $x_n=n^p$, where $1/h<p<1$.

Also solved by G. U. Brauer and Chih-yi Wang (jointly), I. A. Dodes, H. M. Feldman, N. J. Fine, D. M. Friedlen, D. S. Greenstein, M. S. Klamkin, H. D. Lipsich, and the Proposer.

Fine showed that for the second part one can find a *fixed* sequence that works for all $h>1$, namely $x_n=n/(\ln n)$, $n \geqq 2$. Brauer and Wang pointed out that in the second part there are also sequences $\{x_n\}$ with $x_{n+1}-x_n$ bounded or with $x_{n+1}-x_n$ unbounded.

Symmetric Hexagon Inscribed in a Closed Convex Curve

E 1170 [1955, 365]. *Proposed by Viktors Linis, University of Ottawa*

Show that there exists a centrally-symmetric hexagon inscribed in any closed convex curve such that the ratio of the respective areas is at least 2/3.

Solution by Michael Goldberg, Washington, D. C. The following demonstration paraphrases one given by L. Fejes Tóth in "Lagerungen in der Ebene, auf der Kugel und im Raum" (1953) pp. 102–103 and which is based on one given by Fáry [*Bull. Soc. Math. France*, 78, 152–161, (1950)].

Let the closed convex region be designated by G. Consider the maximum chord AB parallel to a given line g (see Fig. 1). Select a length s so that $0<s<AB$. Then take chords P_1P_2 and P_4P_5, each of length s, on each side of AB and parallel to AB. Consider the centrally-symmetric hexagon $P(s) \equiv P_1P_2P_3P_4P_5P_6$ obtained by drawing sides parallel to the diagonals P_2P_5 and

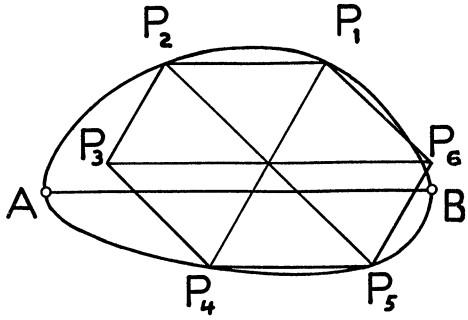


FIG. 1

P_1P_4. When s is small, the points P_3 and P_6 lie inside G. When s is close to AB, the points P_3 and P_6 are outside G. From considerations of continuity, as s is varied from zero to AB, there will be a value for which one of the points P_3 and P_6 will be on the boundary of the region G while the other is outside G. The hexagon determined in this manner will vary continuously with the direction of g. If we begin with P_3 on the boundary, and we turn g through 180°, then P_3 and P_6 are interchanged. Therefore, at some position the point P_6 must be on the boundary of G. At this position all the vertices of the centrally-symmetric hexagon lie on the boundary of G.

To show that the areas satisfy the relation $P/G \geqq 2/3$, consider Figure 2. Designate the segments of G outside the hexagon by s_1, s_2, s_3, s_4, s_5, s_6. Extend the sides of the hexagon to form the triangles $A \equiv A_1A_2A_3$ and $B \equiv B_1B_2B_3$. If we reflect s_1 through P_1, we obtain the region s_1' inside of the triangle $A_1P_1P_6$ but outside the segment s_6. This is a consequence of the convexity of G and the fact that $A_1P_1P_6$ is a reflection of $B_1P_1P_2$. It follows that $s_6+s_1' = s_6+s_1$ is smaller, or at most equal to, the area of $A_1P_1P_6$. If the same procedure is carried out at P_3 and P_5, we obtain $G \leqq A = 3P/2$. Equality is obtained if, and only if, G is

identical with one of the triangles A and B.

Also solved by the Proposer.

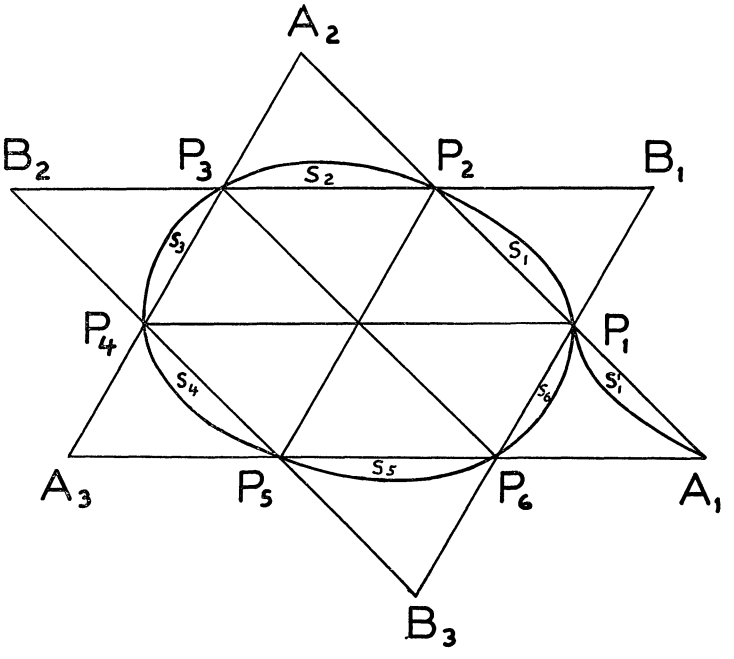


FIG. 2

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4668. *Proposed by D. C. Russell, University of London, England*

Construct the matrix A of a regular method of summation which sums $\sum_{n=0}^{\infty} z^n$ to $(1-z)^{-1}$ at each of the points $z=-1, -2, -3, \cdots$, but which sums the series at no other point in $|z|>1$.

4669. *Proposed by Michael Goldberg, Washington, D. C.*

1. What is the relation connecting the lengths of the sides of a rectangular skew (spatial) pentagon?

2. Among all possible rectangular skew pentagons, what are the lengths of the sides of the pentagon in which the ratio of the longest to the shortest is a minimum?

4670. *Proposed by K. L. Chung, Syracuse University*

If $f(x)$ is continuous and non-negative in $(0, \infty)$, and $\int_0^\infty f(x)dx = \infty$, then there exists an $h>0$ such that $\sum_{n=1}^{\infty} f(nh) = \infty$.

4671. *Proposed by J. P. Ballantine, University of Washington, Seattle, Wash.*

Show that, if $f(x)$ is any function possessing $2n+1$ continuous derivatives,

$$\sum_{r=0}^{n} \binom{2n-r}{n} \frac{a^r}{r!} [(-D)^r f(a) - D^r f(0)] = (-1)^n \frac{D^{2n+1} f(\xi)}{(2n+1)!},$$

where the first parenthesis under the summation is a binomial coefficient, and ξ lies between 0 and a.

4672. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Find a sequence of positive integers such that almost all (in the usual sense of limit density) positive integers are uniquely expressible as a sum of two of them.

SOLUTIONS

Distinct Boundaries of n Countries

4606 [1954, 572]. *Proposed by H. R. Smith, McCoy College, Baltimore, Maryland*

Given a set of n countries, $a_1, a_2, \cdots, a_n$, prove that there exist no more than $3n-6$ distinct boundaries of the form mn between the countries a_m and a_n. By "distinct" is meant that if rs and mn are two boundaries and a_r is the same as a_m then a_s is not the same as a_n.

I. *Solution by B. H. McCandless, Rutgers University.* Clearly the maximum number N of distinct boundaries will be obtained for n countries if one draws a map of $1, 2, \cdots, n$ countries successively, at each stage drawing the next country so that a maximum number of new boundaries will be formed.

The cases $n=1, 2, 3$ are trivial. Consider the case $n=4$. Here $N=3(4-2)=6$. If each of the four countries has a common boundary with each of the others, then at least one of the four must be completely surrounded. Otherwise we could draw a fifth country bounding each of the other four, so that we would have five countries each bounding all the others. But this is impossible, since we would then need five colors to color such a map.

Therefore, since at each stage one new country will be surrounded, we can add at most three new boundaries by drawing a new country. Thus the maximum number of boundaries for n countries is $N=3(n-2)$. (In all this, of course, boundaries consisting of one point are excluded.)

II. *Solution by V. F. Ivanoff. Santa Clara, California.* The proposed problem is dually equivalent to no. 3432. See solution [1931, 118].

Also solved by Michael Goldberg.

A Quotient of Infinite Series

4607 [1954, 572]. *Proposed by H. S. M. Coxeter, University of Toronto.*

Evaluate

$$\frac{1 - 2^{-2} + 4^{-2} - 5^{-2} + 7^{-2} - 8^{-2} + \cdots}{1 + 2^{-2} - 4^{-2} - 5^{-2} + 7^{-2} + 8^{-2} - \cdots}.$$

Solution by L. A. Ringenberg, Eastern Illinois State College. Subtracting term by term the denominator series D from the numerator series N (both unconditionally convergent) we find $N-D=-\frac{1}{2}N$. Hence $N/D=2/3$.

Also solved by Arlo Anderson, C. S. Ballantine, W. J. Blundon, J. D. Campbell, L. Carlitz, G. B. Charlesworth, R. V. Esperti, G. B. Findley, H. J. Fletcher, L. R. Ford, Jr., Emil Grosswald, V. S. Hanly, B. K. Harrison, A. R. Hyde, Edgar Karst, G. J. Kleinhesselink, Frank Kocher, D. C. B. Marsh, P. J. Owens, R. B. Plymale, G. E. Raynor, David Rosen, Peter Scherk, M. R. Spiegel, R. H. Sprague, O. E. Stanaitis, F. Underwood, and Chih-yi Wang.

Editorial Note. Expressions for N and D separately were given by several solvers. Fletcher and Spiegel find

$$N = \frac{-2}{\sqrt{3}} \int_0^{2\pi/3} \log\,(2 \sin \tfrac{1}{2}t)\,dt,$$

$$D = \frac{-2}{\sqrt{3}} \int_0^{\pi/3} \log\,(2 \sin \tfrac{1}{2}t)\,dt,$$

and obtain the desired result through elementary operations on the definite integrals. Wang evaluates N and D in terms of Pairman's trigamma function

$$T(x) = D^2 \log \Gamma(x+1) = \sum_{n=1}^{\infty} (x+n)^{-2},$$

and proceeds by use of the formula

$$T(mx) = \frac{1}{m^2} \sum_{n=0}^{m-1} T\left(x - \frac{n}{m}\right).$$

See C. Jordan, *Calculus of Finite Differences*, 2nd ed., pp. 60–61.

An Inequality

4608 [1954, 645]. *Proposed by José L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let a_i, b_i, $(i=0, 1, \cdots, 2n)$ be real numbers satisfying the following assumptions:

(a) $a_i + a_{i+1} \geqq 0$ for $i = 0, 1, \cdots, 2n-1$;

(b) $a_{2i+1} \leqq 0$ for $i = 0, 1, \cdots, n-1$;

(c) $\sum_{h=2p}^{2q} b_h > 0$ for $0 \leqq p \leqq q \leqq n$.

Prove that

$$\sum_{i=0}^{2n} (-1)^i a_i b_i \geqq 0$$

and that the equality sign holds only if $a_i=0$ for all i.

Solution by D. S. Greenstein, University of Pennsylvania. From (a) and (b) it follows that

(d) $a_{2i} \geqq |a_{2i+1}|, \quad a_{2i+2} \geqq |a_{2i+1}| \qquad (i = 0, 1, \cdots, n-1)$

and the sum in question may be written as $S = \sum_{i=0}^{2n} |a_i| b_i$. Let $c_1 > c_2 > \cdots > c_k \geqq 0$ $(i \leqq k \leqq n-1)$ be the set of absolute values assumed by the a_{2i+1}. (Thus $c_j = a_{2i+1}$ for some j.) Then, since $b_{2i} > 0 (i=0, 1, \cdots, n)$,

$$S = \sum_{|a_i| \geqq c_1} |a_i| b_i + \sum_{|a_i| < c_1} |a_i| b_i \geqq c_1 \sum_{|a_i| \geqq c_1} b_i + \sum_{|a_i| < c_1} |a_i| b_i$$

with equality only if no $|a_{2i}| > c_1$. The sum which c_1 multiplies is positive as a consequence of (c) and (d). Hence

$$S > c_2 \sum_{|a_i| \geqq c_1} b_i + \sum_{c_1 > |a_i| \geqq c_2} |a_i| b_i + \sum_{|a_i| < c_2} |a_i| b_i \geqq c_2 \sum_{|a_i| \geqq c_2} b_i + \sum_{|a_i| < c_2} |a_i| b_i.$$

Continuing in this manner, we finally get

$$S \geqq c_k \sum_{i=0}^{2n} b_i \geqq 0.$$

Clearly S will equal zero if and only if all the a_i are zero.

Also solved by W. J. Blundon, A. R. Hyde, P. J. Owens, Michael Skalskyj, and the Proposer.

Convergence in the Unit Circle

4610 [1954, 646]. *Proposed by Joseph Lehner, Los Alamos Scientific Laboratory*

Show that for almost all real, irrational θ, the series

$$\sum_{n=1}^{\infty} \log \left| \sin \pi n\theta \right| \cdot x^n$$

converges in the unit circle.

Solution by P. T. Bateman, University of Illinois. It is enough to show convergence for almost all irrational θ in the interval $[0, 1]$. Let ϵ be an arbitrarily given positive number. For each pair of integers h, k such that $k \geqq 1$ and $0 \leqq h \leqq k$ delete from the interval $[0, 1]$ those real numbers θ such that $|\theta - h/k| < \epsilon/k^3$. Let $\{\xi\}$ denote the distance from the real number ξ to the nearest integer. Using the inequality $|\sin \pi\xi| \geqq 2|\xi|$, we see that if θ is in $[0, 1]$ but not in one of the deleted intervals, then $|\sin \pi n\theta| \geqq 2\{n\theta\} \geqq 2\epsilon/n^2$ and thus $0 \geqq \log |\sin \pi n\theta| \geqq \log (2\epsilon) - 2 \log n$. Since $\sum_{n=1}^{\infty} (\log n)x^n$ converges for x inside the unit circle, the given series converges inside the unit circle for any θ in $[0, 1]$ which is outside the deleted intervals. But the sum of the lengths of the deleted intervals does not exceed $\sum_{k=1}^{\infty} 2k\epsilon/k^3 < 4\epsilon$. Since ϵ is arbitrary, the assertion of the problem follows.

Remarks. For any particular irrational number θ the series diverges at every point of the unit circle, since $\{n\theta\}$ can be arbitrarily small for suitable n (Hardy and Wright, *Theory of Numbers*, Theorem 201) and thus $\log |\sin \pi n\theta|$ is an unbounded function of n. On the other hand there are irrational numbers θ for which the given series has radius of convergence zero. For example, let $\theta = \sum_{k=1}^{\infty} 1/a_k$, where $a_1 = 3$ and $a_{k+1} = 9a_k \cdot 3^{a_k!}$ for $k = 1, 2, \cdots$. Using the inequality $|\sin \pi\xi| \leqq \pi\{\xi\}$ and the fact that $a_k | a_{k+1}$, we have

$$|\sin \pi a_k\theta| \leqq \pi\{a_k\theta\} = \pi(a_k/a_{k+1} + a_k/a_{k+2} + \cdots) < 9a_k/a_{k+1} = 3^{-a_k!}$$

and hence $\log |\sin \pi a_k\theta| < -a_k!$. But $a_k! x^{a_k}$ is an unbounded function of k for any non-zero x and hence the series $\sum_{n=1}^{\infty} \log |\sin \pi n\theta| \cdot x^n$ has radius of convergence zero for the value of θ we have chosen.

Also solved by Leonard Carlitz and the Proposer.

Five Mutually Orthogonal Spheres

4611 [1954, 646]. Corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Editorial Note. This is a well known proposition. See [1955, 50] for references.

A Limit Defined by a Recurrence Relation

4612 [1954, 646]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let a_0, a_1 be arbitrary and $a_n = a_{n-1} + a_{n-2}/n(n-1)$ for $n > 1$. Find

$$\lim_{n \to \infty} a_n.$$

I. *Solution by N. J. Fine, University of Pennsylvania.* For the initial conditions $(a_0, a_1) = (0, 1)$ and $(1, 1)$, the sequences a_n are monotone and bounded. Hence the required limit, L, exists for all initial conditions by linearity. If we write $y = \sum_{n=0}^{\infty} a_n x^n (|x| < 1)$, Abel's theorem shows that $L = \lim (1-x)y$ as $x \to 1^-$. From the recurrence formula we see that $u = (1-x)y$ satisfies the equation $(1-x)u'' = u$. One solution is given by

$$v(x) = \sum_{n=1}^{\infty} \frac{n(1-x)^n}{(n!)^2} \qquad \text{(all } x\text{)}.$$

To find a second solution, we put $u = vw$. The resulting equation in w is easily integrated, and we find

$$u = c_1 v \int_0^x \frac{dx}{v^2} + c_2 v.$$

The constants are determined by

$$c_2 v(0) = u(0) = y(0) = a_0,$$

$$\frac{c_1}{v(0)} + c_2 v'(0) = u'(0) = y'(0) - y(0) = a_1 - a_0.$$

So, finally, we have

$$L = \lim_{x \to 1} u(x) = c_1 = v(0) \cdot (a_1 - a_0) - v'(0) \cdot a_0$$

$$= a_0 \left(1 - \sum_{n=2}^{\infty} \frac{(n-1)}{(n!)^2}\right) + a_1 \sum_{n=1}^{\infty} \frac{n}{(n!)^2}.$$

II. *Solution by B. S. Popov, University of Skopje, Yugoslavia.* The recursion formula for a_n may be written

$$a_n = a_1 D_n(2, 3, \cdots, n-1) + \frac{a_0}{1 \cdot 2} D_n(3, 4, \cdots, n-1),$$

where $D_n(2, 3, \cdots, n-1)$ and $D_n(3, 4, \cdots, n-1)$ are the determinants

$$\begin{vmatrix} 1 & 1/2 \cdot 3 & 0 & \cdots & 0 \\ -1 & 1 & 1/3 \cdot 4 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 1/3 \cdot 4 & 0 & \cdots & 0 \\ -1 & 1 & 1/4 \cdot 5 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix},$$

respectively. It is known (O. Perron, *Die Lehre von der Kettenbrüchen*, 1929, s.313) that

$$\lim_{n\to\infty} D_n(2, 3, \cdots, n-1) = \psi(2, 1), \qquad \lim_{n\to\infty} D_n(3, 4, \cdots, n-1) = \psi(3, 1),$$

where

$$\psi(\gamma, x) = \sum_{n=0}^{\infty} \frac{x^n}{(\gamma)_n (1)_n}, \qquad (\gamma)_0 = 1, \qquad (\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1).$$

Thus we obtain

$$\lim_{n\to\infty} a_n = a_1\psi(2, 1) + \frac{a_0}{2\cdot 1}\psi(3, 1).$$

Similar second order difference equations occur in the theory of convergence of continued fractions. See also Chrystal, *Text Book of Algebra*, Ch. 34. In the same way it follows that if $a_n = a_{n-1} + x a_{n-2}/\gamma(\gamma+1)$, then

$$\lim_{n\to\infty} a_n = a_1\psi(\gamma, x) + \frac{a_0 x}{2\cdot 1}\psi(\gamma+1, x).$$

Also solved by L. Carlitz, F. H. Northover, Chih-yi Wang, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

The Theory of Numbers. By B. W. Jones. New York, Rinehart and Company, Inc. 1955. xi+143 pages. $3.75.

This book covers that phase of number theory which is usually described as "Elementary Number Theory." The book presupposes no knowledge of analysis or concepts in the realm of modern algebra. The first several chapters are intended primarily for the secondary school teacher.

The method of presentation is, as the author states in the preface, inductive. Although postulates and definitions are given and the majority of the results are stated as theorems, the reader is usually urged to anticipate these results from the exercises and numerous examples. The exercises are fairly standard but seem well chosen.

The subject matter covered is classical although there is more emphasis on Fibonacci sequences than is usually found in books of a similar character.

Chapter I, The Development of the Number System.

The usual postulates for the integers are first introduced. These in turn lead to the extension of the number system and the familiar techniques for introducing the real and complex numbers follow. The so called Fundamental Theorem of Arithmetic is given as well as the Euclidean Algorithm.

Chapter II, Repeating Decimals and Congruences.

Having introduced the decimal notation in the first chapter, the author first considers repeating decimals and very naturally is led to the notation of congruence. The classical theorems are presented with proofs and several examples of arithmetic functions are given. A discussion of Mersenne and Fermat numbers concludes the chapter but it is to be noted that the list of composite Fermat numbers is incomplete (*c.f.* Mersenne and Fermat Numbers. R. M. Robinson, *Proc. Amer. Math. Soc.*, vol. 5, October 1954, pp. 842–846).

Chapter III, Diophantine Equations.

The introduction to the solution of equations in integers is by way of "puzzle problems." Linear and then quadratic diophantine equations are treated.

Chapter IV, Continued Fractions.

The Fibonacci sequence is first introduced and then generalized. The treatment here is considerably detailed. Simple continued fractions are considered next and periodic continued fractions lead to the development of the Pell equation.

Chapter V, Nonlinear Congruences.

Polynomials with respect to a modulus are considered first and specialization leads to the concept of a primitive root and index. The classical theorems are then established.

Chapter VI, Quadratic Residues.

The Legendre symbol is defined and a proof of the law of quadratic reciprocity is given via the Gauss lemma. The generalization to the Jacobi symbol follows and the remainder of the chapter deals with applications to the representation of a positive integer as a sum of two squares.

C. A. NICOL
University of Texas

College Algebra and Plane Trigonometry. By A. Spitzbart and R. H. Bardell. Cambridge, Massachusetts, Addison-Wesley Publishing Company, 1955. xiii+408. $4.50.

Unified Algebra and Trigonometry. By E. P. Vance. Cambridge, Massachusetts, Addison-Wesley Publishing Company, 1955. ix+354. $4.50.

College Algebra and Plane Trigonometry. By J. H. Zant. Boston, Massachusetts, Ginn and Company, 1953. x+337+82. $4.00.

One pleasing aspect of these three books is the economy of space they exemplify. Each book is about as long as the usual algebra text alone, yet all the standard topics of both algebra and trigonometry are included. Furthermore, each book contains unusual features not often found in Freshman texts. For one looking for a text for a unified algebra and trigonometry course, perhaps to precede a unified analytic geometry and calculus course, each of these three books would be well worth considering.

The books by Spitzbart-Bardell and Vance exhibit especially original and unhackneyed approaches to the traditional first two courses of Freshman mathematics. They are written from a mature standpoint, and the authors have taken special care to avoid making statements and creating impressions which the student might have to "unlearn" later. They would be a challenge to the student and the teacher, but the rewards would be worth the extra effort required.

Each of the three books begins with a discussion of the real and complex number systems—most extensive in Spitzbart-Bardell—followed by an intensive review of elementary algebra. Then functions, graphs, angles and the trigonometric functions are taken up. Radian and degree measure are introduced at the same time and used throughout in each text. Each book concludes with a chapter on permutations, combinations and probability. In the middle sections the order of topics varies from book to book. However, the development in each text seems logical and justified. It is interesting to note that the general trigonometric reduction formulas, the proofs of which are so often glossed over in Freshman texts, are proved in complete detail in each of the three books, different methods of proof being used in each book. Spitzbart-Bardell and Vance have four-place tables while Zant includes extensive five-place tables. In each book the tables of logarithms of the trigonometric functions contain the characteristics.

Perhaps the most unified of the three texts is the one by Spitzbart and Bardell. Where direct integration of trigonometry into the sections on algebra would have been artificial, the authors have given problems involving the trigonometric functions, for example, in the sections devoted to arithmetic and geometric progressions, determinants, systems of equations, mathematical induction, linear and quadratic equations and variation. The concepts of inverse function and reflection of graph unify the inverse trigonometric, logarithmic and exponential functions.

Especially noteworthy is the chapter on the graphs of the trigonometric functions. After the usual line representation, an excellent discussion of periodicity—in which an example of a non-trigonometric periodic function is given—and point-by-point plotting of the graphs, contractions, expansions, translations in both the x and y directions and reflections of graphs are carefully discussed, and the topic of symmetry is developed as in analytic geometry. With this background, the rapid sketching of graphs becomes easy and more meaningful. Graphing by addition of ordinates is also thoroughly described.

Among other good points one might mention are some unusual graphs of composite functions, an ingenious proof of the addition formulas for general angles and a proof of the equivalence of the set of rational numbers and the set of repeating decimals.

There are quite a few statements of the type "It is left to the reader to verify" which would ordinarily demand amplification by the teacher. For example, the student is expected to verify that the unit of radian measure is independent of the length of the radius of the circle and that the ratios defining the trigonometric functions are independent of the choice of the point on the terminal side of the angle. One might also mention that the proof of the theorem on the conjugate non-real zeros of a polynomial with real coefficients is inadequate—a detailed proof along the lines indicated is given in Zant's book—and in the formula for the sum of a geometric progression the condition $r \neq 1$ is omitted.

The book by Vance has many excellent and unusual features. A careful discussion of one and two-dimensional rectangular coordinate systems is given, including the explicit assumption of a one-to-one correspondence between the set of points on a line and the set of real numbers. The formula for the distance between two points is derived as well as the general equation of the circle. With this background, the author is able to derive the addition formulas, the law of sines and the law of cosines both easily and—more important—accurately.

The author also gives careful consideration to several points often omitted in other texts, such as the prime factors of integers, polygonal approximations and length of arc of a circle and the definition of a^x for irrational x. Throughout, the distinction between equation and identity is maintained by the use of the symbols $=$ and $\equiv$ respectively. Care is taken in verifying inverse circular function equations to make sure the angles involved are in the right quadrants.

Some unusual sections are devoted to the exact values of the circular functions of $\pi/5$, approximations of functional values for small angles, *e.g.*, $\theta\sqrt{1-\theta^2} < \sin\theta < \theta$, and simple harmonic motion and harmonic analysis and synthesis.

The chapter on mathematical induction is well done. Three parts are given for an induction proof, the third part being a conclusion. The chapter on determinants is excellent. The term matrix is first defined, then the determinant of a square matrix. Complete proofs of the properties and expansion of determinants are then given. Complex numbers are introduced as the algebra of ordered pairs and developed along that line until the trigonometric form is reached. In

the chapter on progressions, sequences are defined by giving successive positive integer values to the independent variable of a function. Thus a function is associated with each arithmetic and geometric progression.

The problems throughout the book show careful selection and include many unusual and challenging ones. Some integral parts of the development are relegated to starred problems. With regard to this, the discussion of the ambiguous case in the solution of oblique triangles and vector addition of complex numbers are contained in unstarred problems.

A couple minor errors noted were the reference to an unstarred problem on page 92 and the postponement of the definition of the symbol 0! until some time after it is first implicitly used.

The sections on trigonometry in this book have appeared essentially unchanged as a separate text (this MONTHLY, vol. 62, pp. 458–459).

On the whole Zant's book is written on a more elementary level than the other two texts reviewed here, a pedagogical advantage in some cases. The book has several noteworthy features. There are good discussions of the classification of functions into algebraic and transcendental functions; equivalent, redundant and defective equations; the distinction between equation and identity and dependent and consistent systems of equations in two and three unknowns. The introduction to complex numbers is good, and the author emphasizes that complex numbers obey the laws of algebra stated earlier for real numbers. In the chapter on logarithms there are several sets of exercises on expressing numbers as powers of 10 and carrying out simple operations with numbers in that form first. Mathematical induction is well done, the order of steps in the proof being the reverse of the usual order. While interest and annuities are discussed, there is no treatment of continuous conversion, growth and the number e as in the other two books. The chapter on progressions includes harmonic progressions and a good discussion of limits and infinite sequences. In the chapter on equations of higher degree a method of finding non-real roots of equations of degree four or less is included as well as Horner's method for approximating real roots. In many places methods of solving problems are given as a rule, each step of the rule being illustrated by an example solved in a parallel column.

The treatment of graphs of trigonometric functions, inverse trigonometric functions and the properties of determinants is very brief. In solving triangles there is no discussion of significant figures or rounding off results. Answers to the examples are given to four or five significant figures even when the given sides of the triangle have only two or three significant figures and angles are given to the nearest degree. The formula for sin $(A+B)$ is proved only for all positive angles, not all angles as is assumed.

D. V. V. WEND
Walkerton, Indiana

Editorial Note. Since Professor Vance is the editor of this section, the editor-in-chief has assumed responsibility for the editing of this review. C.B.A.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

CONFERENCE ON HIGH SPEED COMPUTERS

A Conference on High-Speed Computers will be held at Louisiana State University, Baton Rouge, Louisiana, from February 15 through February 17, 1956. This conference is open to businessmen, office managers, accountants, engineers, mathematicians, chemists, physicists, economists, statisticians and other potential users from all sections of the country. Topics scheduled for discussion by nationally recognized speakers include office procedures, statistical operations and numerical methods designed for the adaptation of problems to machine solution. Several manufacturers of computing equipment will be represented through exhibits or demonstrations of computers in operation.

Inquiries concerning the conference may be directed to: Dr. J. W. Brouillette, Director, General Extension Division, Louisiana State University, Baton Rouge 3, Louisiana.

JOINT MEETING OF THE OPERATIONS RESEARCH SOCIETY OF AMERICA AND THE INSTITUTE OF MANAGEMENT SCIENCES

The Operations Research Society of America and the Institute of Management Sciences will hold a joint spring meeting on March 30 and 31, 1956, at the University of California at Los Angeles. Information concerning the program can be obtained from either Milton Franks, Engineering Department, University of California, Los Angeles 24, California, or Paul Stillson, The Ramo-Wooldridge Corp., 5740 Arbor Vitae, Los Angeles 45, California.

THE FRENCH BIBLIOGRAPHICAL DIGEST

The sixth issue of the second series of the French Bibliographical Digest is devoted to Mathematics. Part I, now available, deals with Pure Mathematics. Part II, which is to be published shortly, will deal with Applied Mathematics. Part I, which was prepared by Professor Jean Dieudonné of the University of Nancy, contains abstracts and reviews of the most significant articles and books contributed by French mathematicians between 1949 and 1954.

The Digest may be obtained free of charge from the Cultural Division of the French Embassy, 972 Fifth Avenue, New York 21, New York.

PERSONAL ITEMS

Dean W. L. Duren, Jr., of the University of Virginia represented the Association at the annual meeting of the American Council on Education which was held in Washington, D. C., on October 6–7, 1955.

Professor R. L. Wilder of the University of Michigan was awarded the honorary degree of Doctor of Science by Bucknell University at the dedication of

the F. W. Olin Science Building on September 28, 1955.

Agricultural and Mechanical College of Texas announces that Mr. H. D. Perry and Mr. Mack Williams have been appointed to instructorships.

Boston College reports the following: Assistant Professor Margaret Conroy of Washington University and Dr. Jacqueline L. Penez of Barnard College, Columbia University, have been appointed to assistant professorships; Dr. Evelyn M. Bender of Cornell University has been appointed to an instructorship.

Boston University announces: Assistant Professor G. E. Noether has been promoted to an associate professorship; Mr. Gustave Solomon, formerly a part-time instructor at Massachusetts Institute of Technology, and Mr. H. E. Perkins have been appointed to instructorships.

Bowling Green State University reports: Assistant Professor H. E. Tinnappel has been promoted to an associate professorship; Professor F. C. Ogg, chairman of the Department of Mathematics, served as a mathematician at the Aberdeen Proving Ground, Maryland, during the summer of 1955.

Brooklyn College announces the following: Dr. May H. Maria has been promoted to an assistant professorship; Associate Professor Moses Richardson has been promoted to a professorship; Dr. Melvin Hausner has returned from military service; Mr. Thomas Nicholson has retired.

Bucknell University reports that Mr. S. F. Dice of the University of Detroit and Mr. C. B. Sensenig, previously a graduate student at New York University, have been appointed to instructorships.

At California Institute of Technology: Dr. L. S. Shapley, formerly an associate mathematician at the Rand Corporation, Santa Monica, has been appointed Senior Research Fellow; Dr. R. A. Restrepo, Dr. R. C. Thorne, and Dr. D. R. Truax have been appointed Research Fellows; Dr. C. H. Wilcox, previously a fellow at Harvard University, has been appointed to an instructorship; Dr. F. B. Fuller and Dr. F. L. Spitzer have been promoted to assistant professorships.

Carnegie Institute of Technology announces: Dr. L. E. Dubins, Dr. G. H. F. Gardner of Cornell University, and Dr. A. F. Strehler, head of Lovelace Research Foundation, Albuquerque, New Mexico, have been appointed to assistant professorships; Mr. B. D. Aggarwala, previously a lecturer at the Indian Institute of Technology, Mr. Gus DiAntonie, formerly a lecturer at the University of Pittsburgh, Mr. J. T. Morse, recently a National Science Foundation Fellow at Massachusetts Institute of Technology, Mr. J. S. Rustagi, previously a research assistant at Stanford University, and Mr. S. H. Vosko, formerly a fellow at the Institute, have been appointed to instructorships.

Colorado Agricultural and Mechanical College reports the following: Dr. T. A. Newton of the University of Nebraska and Dr. E. E. Remmenga have been appointed to assistant professorships; Professor A. G. Clark, Dean of the College, is Acting Head of the Department of Mathematics.

Hampton Institute announces that Mrs. Eleanor Dawley and Mr. Andrew Hsu have been appointed to instructorships.

At Illinois Institute of Technology: Dr. Pasqualle Porcelli has been promoted to an assistant professorship; Mr. Frank Kozin, previously a graduate assistant at the Institute, has been appointed to an instructorship.

Iowa State College makes the following announcement: Assistant Professor George Seifert of the University of Nebraska has been appointed to an assistant professorship; Dr. F. W. Stallard, formerly a fellow at Oak Ridge Institute of Nuclear Studies, and Mr. D. A. Storvick of the University of Michigan have been appointed to instructorships; Associate Professors C. G. Maple and Bernard Vinograde have been promoted to professorships; Assistant Professors O. C. Kreider and C. H. Lindahl have been promoted to associate professorships; Mr. M. R. Bryson of Drake University has been appointed to an instructorship in the statistical laboratory.

Laval University reports: Mr. C. Roy has been appointed Professor of Mechanics; Dr. J. Lavallée of McGill University and Professor Fritz Rothberger of the University of New Brunswick have been appointed to professorships.

Marquette University announces the following: Miss Patricia A. Gogins, formerly a graduate assistant at the University, and Mr. J. E. Simpson have been appointed to instructorships; Mr. R. W. Reichhardt has been appointed to an assistant instructorship; Mr. Kenneth Blake, Miss Shih S. Liu, and Mr. D. F. Rauscher have been appointed to graduate assistantships; Mr. R. R. Rice of the Milwaukee School of Engineering has been appointed Lecturer; Assistant Professor Leo Branovan has been promoted to an associate professorship; Dr. C. B. Hanneken has been promoted to an assistant professorship.

Massachusetts Institute of Technology reports the following: Dr. E. A. Robinson, previously with the Gulf Research and Development Company, Pittsburgh, Pennsylvania, has been appointed to an instructorship; Dr. R. C. T. Smith, on leave from the University of New England in Australia, has been appointed Visiting Lecturer; Dr. Leonard Roberts has been appointed Research Associate.

McMaster University announces: Dr. Bernard Banaschewski of the University of Hamburg, Germany, has been appointed to an assistant professorship; Professor A. E. Johns has retired.

At Miami University: Professor Ernst Snapper of the University of Southern California has been appointed Research Professor; Assistant Professor T. C. Holyoke of Northwestern University has been appointed to an assistant professorship.

Mississippi State College announces the following appointments: Mr. T. W. Daniel to an instructorship; Mrs. Mildred L. Cozart, Mrs. Cornelia S. Barrentine, and Miss Marie Armstrong to acting instructorships.

Newark College of Engineering reports that Assistant Professors Carl Konove and Edgar Virene have been promoted to associate professorships.

At Northwestern University: Professor L. Virginia Carlton of Wesleyan

College and Dr. I. B. Fleischer have been appointed Lecturers; Assistant Professor W. A. Pierce, who is on leave of absence from Syracuse University, and Assistant Professor Esther Seiden of Howard University have been appointed to visiting assistant professorships; Mr. R. B. Kellogg, previously a graduate student at the University of Chicago, and Mr. George Kolettis, Jr., formerly a teaching assistant at the University of Chicago, have been appointed to instructorships; Assistant Professor M. A. Rosenlicht has been promoted to an associate professorship; Dr. W. E. Jenner has been promoted to an assistant professorship; Associate Professor Daniel Zelinsky is on leave of absence for the year, 1955–1956, in Japan on a Fulbright Grant; Assistant Professor Alex Rosenberg, who is on leave of absence for the year 1955–1956, is at the Institute for Advanced Study on a National Science Foundation Grant; Assistant Professor Meyer Dwass is on leave of absence for the year 1955–1956 and is at Stanford University as a research associate; Assistant Professor J. C. E. Dekker is on leave of absence for the year 1955–1956 and is working on a project supported by the National Science Foundation at the University of Chicago.

Ohio State University announces the following: Dr. J. G. Lamadrio, previously an assistant at the University of Michigan, and Dr. Robert Seall of Illinois Institute of Technology have been appointed to instructorships; Dr. Donald Austin of Syracuse University and Dr. Martin Davis of the University of California at Davis have been appointed to assistant professorships; Assistant Professor L. H. Miller has been promoted to an associate professorship; Dr. W. E. Deskins, Dr. R. C. Fisher, and Dr. R. F. Reeves have been promoted to assistant professorships; Assistant Professor Hortense Rickard has retired with the title Assistant Professor Emeritus.

Ohio University reports the following: Dr. R. F. Spring of the University of Illinois has been appointed to an assistant professorship; Professors Carl Denbow and Victor Goedicke, who are on leave of absence for 1955–1956, have been awarded fellowships by the Ford Foundation and are studying at Harvard University.

At Oklahoma Agricultural and Mechanical College: Assistant Professor R. B. Deal, Jr., has been promoted to an associate professorship; Dr. J. W. Hamblen of Purdue University has been appointed to an assistant professorship; Dr. S. D. Liao, post-doctoral fellow at Princeton University, has been appointed Research Instructor; Mrs. Anne P. Shoemaker has been appointed to a temporary instructorship.

Oregon State College reports: Professor J. C. R. Li was given the 1955 Carter Award for inspirational teaching in the School of Science; Associate Professor H. E. Goheen of Iowa State College has been appointed to an associate professorship; Dr. R. F. Link, formerly a statistician with the Sandia Corporation, Albuquerque, New Mexico, and Dr. A. Teresa Tymieniecka, previously a teaching assistant in philosophy at the University of California, have been appointed to instructorships; Mrs. Nan N. Reynolds, recently an associate engineer in aerodynamics research at Douglas Aircraft, Santa Monica, California,

has been appointed to an acting instructorship; Professor A. T. Lonseth has been appointed Acting Chairman of the Department of Mathematics; Associate Professor J. C. R. Li has been promoted to a professorship; Mr. R. H. Godard has been promoted to an assistant professorship; Associate Professor B. H. Arnold has returned from his sabbatical leave at the University College of Wales; Associate Professor B. W. Brewer is on sabbatical leave for the academic year 1955–1956 and is at the University of Wisconsin as an Honorary Fellow; Mr. R. D. Aeder of International Business Machines Corporation was instructor at a one-week programming school for the IBM Type 650 computer held July 25–29, 1955.

Rutgers University announces: Mr. A. E. Babbitt, Jr., has been appointed to an instructorship; Mr. W. G. Kern, Jr., Mr. D. R. King, and Mr. T. L. Koehler, Jr., have been appointed teaching assistants.

St. Lawrence University announces the following: Mr. J. J. Kinney, formerly a graduate student at Harvard University, has been appointed to an instructorship; Assistant Professor F. C. Warner has been promoted to an associate professorship.

St. Louis University reports that Mr. E. G. Eigel, Jr., Miss Bernardine Law, Mr. John Groves, and Mr. G. R. Kuhn have been appointed Fellows.

State College of Washington reports the following: Professor M. S. Knebelman, chairman of the Department of Mathematics, is Acting Associate Dean of the Division of Sciences and Arts; Professor S. G. Hacker is serving as Acting Chairman of the Department; Mr. Ward Crowley has been appointed Acting Assistant Professor; Mr. R. S. DeZur, formerly a graduate student at the University of Oregon, has been appointed to an acting instructorship; Dr. D. W. Bushaw and Dr. M. T. Wechsler have been promoted to assistant professorships.

University of Akron announces: Assistant Professor E. A. Tabler has been placed in charge of the Number Communications Course, a new development in the General Education curriculum; Assistant Professor R. C. Davis received a grant from the National Science Foundation to attend the Institute for Teachers of Collegiate Mathematics at Stanford University, Summer 1955; Assistant Professor Louis Ross received a scholarship to attend the short course in operations research which was held at Case Institute of Technology, September 1955; Dr. Samuel Selby, head of the Department of Mathematics, attended the Canadian Mathematical Congress during August 1955 at the University of Manitoba.

At the University of Alabama: Professor F. A. Lewis has resigned as head of the Department of Mathematics; Professor J. D. Mancill has been appointed Head of the Department; Assistant Professor O. R. Ainsworth has been promoted to an associate professorship; Professor M. O. Gonzalez is on leave of absence at the University of Havana, Cuba.

University of Arkansas announces the following appointments to instructorships: Mr. L. L. Koontz, Mr. B. G. Mullen, and Miss Lillian Spikes.

University of California reports: Assistant Professor S. P. Diliberto has been promoted to an associate professorship; Visiting Associate Professor H. D. Huskey has been appointed Associate Professor of Mathematics and Electrical Engineering; Assistant Professor Henry Helson of Yale University has been appointed to an assistant professorship; Professor Bernard Friedman of New York University and Professor G. P. Hochschild of the University of Illinois have been appointed to visiting professorships; Dr. Ronald Macauley, Dr. T. J. McMinn, and Dr. D. E. Schroer have been appointed to instructorships; Dr. D. B. Lowdenslager of the University of Virginia holds a research appointment; Dr. Michael Dummett holds a Commonwealth Fellowship for 1955–1956 and is spending the year at the University; Professor A. P. Morse is on sabbatical leave for 1955–1956; Professor A. Tarski is on sabbatical leave for 1955–1956 on a Guggenheim Fellowship and will give a series of lectures in Paris, France; Professor F. Wolf is on sabbatical leave for 1955–1956 and will spend the year in London and Rome; Professor G. C. Evans is spending six months in Europe. In June, 1955, a two-week conference on partial differential equations was held at the University under the auspices of the Office of Naval Research, the University of Kansas, and the University of California.

University of Cincinnati announces: Professor Louis Brand, formerly head of the Department of Mathematics, has retired with the title Professor Emeritus; Professor G. M. Merriman has been appointed Head of the Department; Assistant Professor F. W. Rogers has been promoted to an associate professorship; Mr. W. R. Harlow has been promoted to an assistant professorship; Mr. Smbat Abian, Mr. Wesley Love, and Mr. H. M. Weitkamp have been appointed to instructorships; Mr. Roger Chalkley, Mr. T. K. Cook, Mr. Charles Lambert, and Mr. Bruce Raymond have been appointed to teaching assistantships; Visiting Associate Professor W. B. Jurkat is on leave of absence for 1955–1956 in Germany and has been replaced by Professor Alexander Peyerimhoff of the University of Giessen.

University of Florida reports the following: Dr. R. H. Ackerson, previously a teaching fellow at Alabama Polytechnic Institute, Dr. A. T. Butson of Michigan State University, and Dr. T. O. Moore of the University of Missouri have been appointed to assistant professorships; Assistant Professors R. G. Blake and Kenneth Lewis have been promoted to associate professorships.

University of Georgia announces that a meeting of the Mathematics Colloquium of the University was held on October 12, 1955. Dr. J. W. Green of the University of California at Los Angeles gave a lecture entitled, "Mean Values of Harmonic Functions on Homothetic Curves."

At the University of Idaho: Dr. A. R. Amir-Moez of Santa Barbara College, University of California, and Mr. William Smith have been appointed to instructorships; Dr. C. H. Cunkle of the University of Missouri and Dr. T. R. Jenkins of the University of Michigan have been appointed to assistant professorships; Associate Professor A. E. Halteman has returned from a year's leave of absence spent at the University of Oregon.

University of Maryland announces: Mr. Henry Beiman, recently a teaching assistant at the University of Wisconsin, Miss E. Correl, formerly a graduate assistant at Purdue University, Dr. B. S. Fadnis, previously an assistant lecturer at Indian Institute of Technology, Dr. H. T. Hsu, and Dr. J. G. MacCarthy, previously a graduate assistant at the University of Pittsburgh, have been appointed to instructorships; Associate Professor D. M. Young, Jr., is on leave of absence and is with the Ramo-Wooldridge Corporation, Los Angeles, California.

University of Minnesota reports the following: Dr. G. E. Baxter of Massachusetts Institute of Technology, Dr. I. S. Krule, previously a teaching fellow at Tulane University, and Dr. G. O. Sabidussi of the Institute for Advanced Study have been appointed to instructorships.

University of Mississippi announces the following: Mrs. Mary C. Kerr and Mr. A. E. Dean have been appointed to instructorships; Professor W. H. Spragens has returned after a year at the University of Chicago on an internship in general education; Associate Professor L. L. Scott is on leave of absence and and is at the University of California on a Ford Foundation Fellowship.

At the University of North Dakota: Mr. J. S. Rue and Mr. D. H. Wentworth have been appointed to instructorships.

University of Pennsylvania makes the following announcement: Dr. Saul Kravetz has been appointed Associate; Dr. O. H. Alisbah has been appointed Lecturer; Assistant Professors R. D. Anderson and Bernard Epstein have been promoted to associate professorships; Professor S. P. Shugert has retired with the title Professor Emeritus; Associate Professor R. D. Anderson is on leave of absence at the Institute for Advanced Study for the year 1955–1956.

University of Pittsburgh reports: Assistant Professor Louis Sacks has been promoted to an associate professorship; Mr. Joseph Kachun has been promoted to an assistant professorship; Miss Florence J. Krajnak, previously a lecturer at the University's Johnstown Center, has been promoted to an instructorship at the Center.

University of Utah announces: Professor Bodo Volkmann of the University of Mainz, Germany, has been appointed Visiting Assistant Professor; Assistant Professor Allan Davis of the University of Nevada has been appointed to an assistant professorship; Mr. D. S. Adorno, formerly a graduate student at Pennsylvania State University, has been appointed to an instructorship.

Virginia Polytechnic Institute reports the following: Mr. H. G. Campbell and Mr. P. F. Clemens of Rennselaer Polytechnic Institute have been appointed to assistant professorships; Miss Emily B. Ragsdale and Mr. J. G. Rieley have been appointed to instructorships.

Wayne University announces the following: Associate Professor Benjamin Epstein has been promoted to a professorship; Assistant Professor Samuel Kaplan has been promoted to an associate professorship; Associate Professor Harvey Cohn has returned from a leave of absence at Stanford University.

Western Michigan College reports: Associate Professor H. H. Hannon has

been promoted to a professorship; Assistant Professor J. R. Meagher has been promoted to an associate professorship. .

At Wisconsin State College, River Falls: Dr. Lillian Gough of New York State Teachers College, Oswego, has been appointed to an associate professorship; Mr. Gerald Bisbey has been appointed to an instructorship; Professor Glen Junkman has retired after 38 years of teaching at the College.

Yale University announces: Dr. R. J. Nunke of Northwestern University, Dr. M. F. Ruchte, formerly a research fellow at Yale University, and Dr. E. C. Schlesinger, previously a fellow at Harvard University, have been appointed to instructorships; Assistant Professor D. F. Votaw, Jr., of the Department of Mathematics has been promoted to an associate professorship in the Department of Industrial Administration.

Associate Professor R. V. Andree of the University of Oklahoma is on leave of absence for the academic year 1955–1956 on a Carnegie Foundation grant and is Visiting Associate Professor at Haverford College.

Dr. Louis Auslander of Yale University is at the Institute for Advanced Study as a National Science Foundation Postdoctoral Fellow.

Mr. Thomas Bauserman of West Virginia University has been appointed to an assistant professorship at Marshall College.

Mr. C. B. Baytop of Hampton Institute has been appointed to an instructorship at Morgan State College.

Mr. H. W. Becker, formerly a transmitter engineer at Station WOW, Omaha, Nebraska, has been appointed to an instructorship at the Radio Engineering Institute, Omaha.

Visiting Assistant Professor H. D. Block of the University of Minnesota has been appointed to an instructorship at Cornell University.

Mrs. Helen L. Brooks, previously a lecturer at the University of Toledo, has been appointed to an instructorship at the University.

Mr. H. H. Brown, formerly a research mathematician at Franklin Institute, has a position as a professional engineer at Radio Corporation of America, Aviation Systems Section, Moorestown, New Jersey.

Mr. C. N. Campopiano, recently a project engineer at Sperry Gyroscope Company, Great Neck, New York, has accepted a position as a mathematician with Combustion Engineering, New York City.

Mr. B. R. Cato, Jr., of the University of Maryland has been appointed to an assistant professorship at the College of William and Mary.

Associate Professor Abraham Charnes of Carnegie Institute of Technology has been appointed to a professorship at Purdue University.

Dr. N. H. Choksy, previously a research assistant at the University of Wisconsin, has been appointed to an assistant professorship in electrical engineering at Johns Hopkins University.

Assistant Professor H. S. Cohen of Carnegie Institute of Technology has been appointed to an associate professorship at Rensselaer Polytechnic Institute.

Mr. G. R. Costello, formerly an aerophysics project engineer at Chance-Vought Aircraft, Dallas, Texas, is now an assistant department head in the Functional Engineering Department, Arma Corporation, Garden City, New York.

Assistant Professor H. K. Crowder, who is on leave of absence from Case Institute of Technology, is at the Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Associate Professor F. E. Ehlers of Oregon State College is now with Boeing Airplane Company, Seattle, Washington.

Mr. M. C. Erwin, who has been teaching at Reynolds High School, Indiana, has retired.

Dr. Elvy L. Fredrickson of Oregon State College has been appointed to an associate professorship at Lewis and Clark College.

Assistant Professor C. M. Fulton of the University of California at Davis has been promoted to an associate professorship.

Professor S. H. Gould of Purdue University has been appointed to a professorship at Williams College.

Mr. E. A. Green, previously a student at Queen's University, is now teaching at Fort William Collegiate Institute, Ontario, Canada.

Associate Professor G. H. Handelman of Carnegie Institute of Technology has been appointed to a professorship at Rensselaer Polytechnic Institute.

Assistant Professor H. S. Heaps of Nova Scotia Technical College has been promoted to an associate professorship.

Assistant Professor John van Heijenoort of New York University has been promoted to an associate professorship.

Mrs. P. S. Helmick of Grandview College has been appointed to an instructorship at Drake University.

Dr. T. P. Higgins of the University of Texas has a position as a research engineer at Boeing Airplane Company, Seattle, Washington.

Mr. C. W. Hunter has a position as a mathematician at Radio Corporation of America, Aviation Systems Laboratory, Waltham, Massachusetts.

Mrs. Verba W. Iturralde, formerly a teacher at El Paso Technical Institute, has been appointed to an instructorship at Texas Western College of the University of Texas.

Mr. W. G. Johnson, recently a part-time instructor at Syracuse University, has been appointed to an instructorship at Southern Illinois University.

Dr. J. B. Johnston has been appointed to an instructorship at Cornell University.

Mr. T. W. Kampe, formerly a graduate assistant at the University of Southern California, has been appointed to an instructorship at Pomona College.

Associate Professor L. H. Kanter of Mississippi State College has been appointed to an assistant professorship at Clarkson College of Technology.

Associate Professor W. S. Kimball of the University of Vermont has been appointed to an associate professorship at the University of Alaska.

Mr. Jack Klugerman, previously a student at Brooklyn College, now has a position as a computing engineer with Republic Aviation Corporation, Farmingdale, New York.

Professor L. Kuipers of the University of Indonesia has been appointed to an assistant professorship at the University of Redlands.

Assistant Professor R. M. Lakness of the University of California has been appointed to an assistant professorship at San Francisco State College.

Dr. Gene Levy of the University of Oklahoma has been promoted to an assistant professorship.

Mrs. Anne T. Limpert of St. Lawrence University has a position as a teacher at Gouverneur High School, New York.

Mr. Lionel London, formerly with the Armour Research Foundation, Chicago, Illinois, as a research engineer, is now a senior engineer at Cook Research Laboratory, Skokie, Illinois.

Rev. H. J. Luger has been appointed to an instructorship at Gonzaga University.

Mr. R. L. Manchester, previously at Franklinville Central School, New York, has been appointed Head of the Department of Mathematics of Iroquois Central School, Elma, New York.

Dr. Lawrence Markus of Yale University has been appointed Lecturer at Princeton University.

Dr. P. J. McCarthy, recently a research assistant at the University of Notre Dame, has been appointed to an instructorship at the College of the Holy Cross.

Dr. E. B. McLeod, Jr., of the University of Colorado has been appointed to an assistant professorship at Oregon State College.

Dr. R. M. McLeod, previously an assistant at Rice Institute, has been appointed to an instructorship at Duke University.

Dr. J. T. Mohat of the University of Texas has been appointed to an instructorship at Duke University.

Professor E. F. Moore, Director of Research and Professor of Statistics at Baylor University, has been appointed a staff supervisor in the Quality Control Branch, Phillips Petroleum Company, Rocket Fuels Division, McGregor, Texas.

Mr. R. A. Moreland, Jr., formerly a mathematician with the Sandia Corporation, Albuquerque, New Mexico, has been appointed to an instructorship at Texas Technological College.

Visiting Professor F. S. Nowlan of the College of William and Mary has been appointed Visiting Professor at the University of Houston.

Mr. C. T. Oergel, previously an analytical engineer for General Electric Company, Bloomfield, New Jersey, has been appointed Head of the Mechanical Engineering Department of Polytechnic Institute of Brooklyn.

Mrs. Erna H. Pearson has been appointed to an instructorship at Emory University.

Mr. W. J. Pervin, recently a senior scientist for Westinghouse Atomic Power

Division, Pittsburgh, Pennsylvania, has been appointed to an assistant professorship at the University of Pittsburgh.

Mr. A. G. Poorman, formerly a graduate assistant at Iowa State College, has been appointed to an assistant professorship at Ashland College.

Assistant Professor J. F. Price of Oregon State College has a position with Boeing Airplane Company, Seattle, Washington.

Dr. R. T. Prosser, previously a teaching assistant at the University of California, has been appointed Research Instructor at Duke University.

Dr. L. E. Pursell of Ohio State University has been appointed to an assistant professorship at Grinnell College.

Mr. John Rausen, formerly a lecturer at Columbia University, has been appointed to an instructorship at New York University.

Mr. S. T. Rio has been appointed Instructor and Chairman of the Department of Mathematics at Pacific University.

Dr. J. P. Roth of the University of California is spending the year 1955–1956 at the Institute for Advanced Study.

Dr. W. C. Sangren, previously a senior mathematician at Oak Ridge National Laboratory, Tennessee, is now Chief of Mathematics and Computing Division, Curtiss-Wright, Clifton, New Jersey.

Mr. E. D. Schell, recently chief of Mathematics and Electronic Computing Branch, Air Force Headquarters, has a position as a mathematician with the Sperry Rand Corporation, New York City.

Miss Nancy M. Scribano of Ohio University is now a mathematics teacher at New Rochelle High School, New York.

Assistant Professor C. E. Shotwell of the University of the South has a position as an applied science representative with the I. B. M. Corporation, Chattanooga, Tennessee.

Associate Professor S. S. Shu of Illinois Institute of Technology has been appointed to a professorship at Purdue University.

Professor Jack Silber of Roosevelt University spent the summer of 1955 as consultant to the Operations Analysis Office at the United States Air Force Missile Test Center.

Mr. L. C. Smith, previously a student at Hofstra College, has a position as a dynamics engineer at Republic Aviation Corporation, Farmingdale, New York.

Assistant Professor George Springer of Northwestern University has been appointed to an associate professorship at the University of Kansas.

Dr. D. W. Wall, formerly a programmer in the National Security Agency, Washington, D. C., has been appointed to an assistant professorship at the University of North Carolina.

Assistant Professor E. C. Watters of the United States Naval Academy has a position as a senior mathematician at Bendix Research Laboratories, Detroit, Michigan.

Dr. William Wehlau has been appointed Research Fellow in Astronomy at the University of Western Ontario.

Miss Nancy J. White, recently a mathematics teacher at Edward Hand Junior High School, Lancaster, Pennsylvania, has a position as an engineering assistant at General Electric Company, Cincinnati, Ohio.

Dr. Adil Yaqub has been appointed to an instructorship at Purdue University.

Dr. Hiroshi Yamauchi of Nuclear Development Associates, White Plains, New York, has been appointed to an assistant professorship at the University of Hawaii.

Mr. A. J. Yaspan has been appointed to an instructorship at Western Reserve University.

Professor Emeritus I. S. Carroll of Syracuse University died on October 17, 1955. He was a member of the Association for thirty-two years.

Professor Emeritus R. W. Marriott of Swarthmore College died on October 19, 1955. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 79 persons have been elected to membership by the Board of Governors on applications duly certified.

ALICE E. ANDERSON, M.A. (East Tennessee S.C.) East Tennessee State College.

S. I. ASKOVITZ, M.A., M.D. (Pennsylvania) Res. Fellow in Ophthalmology, Albert Einstein Medical Center, Philadelphia, Pa.

MRS. BONNIE AVERBACH, M.A. (Temple) Instr., Temple University.

J. H. BARNHILL, B.S. (North Carolina) Mathematician, Air Weather Service, Data Control Division, Asheville, N. C.

R. H. BATES, M.A. (Syracuse) Teacher, Milford Central School, N. Y.

A. A. BENVENUTO, M.A. (Syracuse) Grad. Asst., University of Illinois.

J. W. BEST, M.S. (Oklahoma A.&M.C.) Captain, United States Air Force.

FRED BIGELEISEN, Student, University of Buffalo.

G. E. BLOCH, B.S. (Brown) Ensign, United States Navy.

R. C. BOLLINGER, B.S. (Pittsburgh) Asst. Engr., Westinghouse Electric Corp., Pittsburgh, Pa.

MEYER BRILL, M.S. (Tulane) Instr., Tulane University.

R. E. BRINEY, A.B. (Northwestern U.) Grad. Student, Massachusetts Institute of Technology.

J. K. BROWN, United States Army.

D. L. BURKHOLDER, Ph.D. (North Carolina) Asst. Professor, University of Illinois.

G. D. BYRNE, B.S. (Creighton) Mathematician, White Sands Proving Ground, Las Cruces, N. Mex.

J. W. CARR, III, Ph.D. (M.I.T.) Asst. Professor, University of Michigan.

R. R. CHRISTENSEN, M.S. (Northwestern U.) Member of the Technical Staff, Ramo-Wooldridge Corp., Los Angeles, Calif.

R. F. CHURCH, Student, University of New Hampshire.

FREDERIC CUNNINGHAM, JR., Ph.D. (Harvard) Asst. Professor, University of New Hampshire.

E. B. DUFFY, Ch.E., M.S. (Cincinnati) Engr., Engineer Research and Development Lab., Fort Belvoir, Va.

G. J. DUFFY, B.S. (Loyola) Jr. Res. Technician, Argonne National Lab., Chicago, Ill.

L. W. EHRLICH, B.S. (Maryland) Grad. Asst., University of Maryland.

MRS. ELIZABETH A. ESSON, Student, Rutgers University, College of South Jersey.

J. H. FISHEL, M.S. (Illinois) Res. Asst., University of Illinois.

ELVY L. FREDRICKSON, Ph.D. (Oregon S.C.) Asso. Professor, Lewis & Clark College.

J. L. FREIER, Student, Brooklyn College.

L. T. GARDNER, A.B. (Antioch) Actuarial Analyst, Travelers Insurance Co., Hartford, Conn.

MRS. ELIZABETH B. GILMORE, B.S. (St. Joseph C.) Norfolk, Va.

R. J. GRAHAM, B.S. (Florence S.T.C.) Mathematician, Eglin Air Force Base, Fla.

JUNE V. GRENIER, B.S. (St. Joseph C.) Hartford, Conn.

D. W. GRISSINGER, B.S. (Pennsylvania S.U.) Jr. Engr., Philco Corp., Philadelphia, Pa.

W. E. GROVE, M.A. (Michigan S.U.) Instr., General Motors Institute, Flint, Mich.

P. L. HAMM, M.A. (Maine) Instr., University of Maine.

G. B. HARE, M.S. (Colorado) United States Army.

DOROTHY L. HARRIS, B.S. in Ed. (Miami U.) Grad. Student, Ohio State University.

D. J. HART, Lab. Asst., Chemical Division of Borden Co., Peabody, Mass.

R. E. HAYMOND, B.S. (South Carolina) Grad. Asst., California Institute of Technology.

J. M. HEATH, M.A. (Yale) Head, Department of Mathematics, Hopkins Grammar School, New Haven, Conn.

R. B. HERNDAY, B.Ed. (Wisconsin S.C., Whitewater) Instr., Milwaukee School of Engineering.

REV. F. A. HOMANN, S.J., A.B. (St. Louis) Grad. Student, St. Louis University.

J. B. JOHNSTON, Ph.D. (Calif. I.T.) Instr., Cornell University.

MRS. CLARA A. JONES, M.S. (South Carolina S.C.) Asst. Professor, South Carolina State College.

IRWIN KABUS, B.S. (C.C.N.Y.) Mathematician, Reeves Instrument Corp., New York, N. Y.

J. L. KATZ, M.S. (Poly. Inst. of Brooklyn) Instr., Polytechnic Institute of Brooklyn.

P. R. KUEPKER, Student, University of Oklahoma.

PEDRO LABORDE-MONTANER, Ph.D. (N.Y.U.) Professor, University of Puerto Rico.

MUNISWAMYREDDY LAKSHMANAN, M.A. (Madras) Lecturer, Central College, University of Mysore, Bangalore, India; Lecturer, Maryland College, Harper, Liberia.

C. W. LARSON, M.A. (Wisconsin) Instr., Ripon College.

H. L. LEWIS, M.A. (Texas) Instr., Southwest Texas State Teachers College.

J. C. LONG, A.B. (Indiana) Industrial Engr., Richardson Co., Indianapolis, Ind.

MRS. FLORA E. LYNN, A.B. (Tennessee) (Mary Baldwin) Math. and Latin Teacher, Clifton Forge H.S., Va.

ROSEMARIE MACMILLAN, B.S. (Alberta) Exploration Trainee, Mobil Oil of Canada, Calgary, Alberta.

H. S. MAR, B.S. (U. of Washington) Grad. Student, University of Washington.

G. E. MARTIN, M.A. (Florida S.U.) Grad. Teaching Asst., Purdue University.

B. L. MCALLISTER, M.A. (Utah) Teaching Asst., University of Wisconsin.

R. A. MORRIS, B.S. (Rhode Island) Fort Bliss, Texas.

R. C. MORROW, M.A. (Washington & Jefferson) Asso. Professor, United States Naval Academy.

K. H. MURPHY, M.S. (W.Va.U.) Instr., University of Illinois, Navy Pier, Chicago.

C. A. MUSES, Ph.D. (Columbia) Editor-in-Chief, The Falcon's Wing Press, Indian Hills, Colo.

EIZO NISHIURA, Student, San Jose State College.

G. M. ORTNER, M.A. (Michigan) Instr., General Motors Institute, Flint, Mich.

J. B. O'TOOLE, Ph.D. (Pittsburgh) Projects Engr., Philco Corp., Philadelphia, Pa.

SAMUEL PUCCIARELLI, B.S. (Manhattan C.) Jr. Engr., Republic Aviation Corp., Farmingdale, N. Y.

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B. L. REINHART, M.A. (Princeton) Instr., Princeton University.
R. R. RICE, B.S. (Wisc. S.C., Eau Claire) Instr., Milwaukee School of Engineering.
D. G. ROSE, JR., Moorestown, N. J.
A. A. RUTOSKEY, Student, University of Southern California.
E. D. RYAN, Student, University of Denver.
W. W. SHIRLEY, M.S. (Oklahoma A.&M.C.) Petroleum Reservoir Engr., Richfield Oil Corp., Los Angeles, Calif.
R. D. SINKHORN, Student, Municipal University of Wichita.
N. E. STEENROD, Ph.D. (Princeton) Professor, Princeton University.
F. G. STOCKTON, A.B. (Adams S.C.) Jr. Engr., Shell Development Co., Emeryville, Calif.
F. A. VARRICHIO, B.S. in Ed. (Fordham) Teaching Asst., Brown University.
W. E. WALDEN, A.B. (New Mexico C. of A.&.M.A.) Res. Asst., Los Alamos Scientific Lab., N. Mex.
R. B. WALTZ, B.S. (Franklin and Marshall) Jr. Engr., Philco Corp., Philadelphia, Pa.
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CYNTHIA L. YOUNG, Student, Tennessee Polytechnic Institute.
D. E. ZILMER, Ph.D. (Wisconsin) Mathematician, U. S. Naval Ordnance Test Station, China Lake, Calif.

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, April 28, 1956.
ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.
INDIANA, Wabash College, Crawfordsville, May 5, 1956.
IOWA, Grinnell College, Grinnell, April 20–21, 1956.
KANSAS, University of Wichita, April 21, 1956.
KENTUCKY, University of Kentucky, Lexington, April 28, 1956.
LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.
MICHIGAN, University of Michigan, Ann Arbor, March, 1956.
MINNESOTA, Augsburg College, Minneapolis, April 28, 1956.
MISSOURI, Fontbonne College, St. Louis, April 21, 1956.
NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.
NORTHEASTERN
NORTHERN CALIFORNIA, Stanford University, Stanford, January 14, 1956.
OHIO, Oberlin College, April 14, 1956.
OKLAHOMA
PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.
PHILADELPHIA
ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.
SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.
SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.
SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.
TEXAS, Southwest Texas State Teachers College, San Marcos, April, 1956.
UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.
WISCONSIN, Marquette University, Milwaukee, May, 1956.

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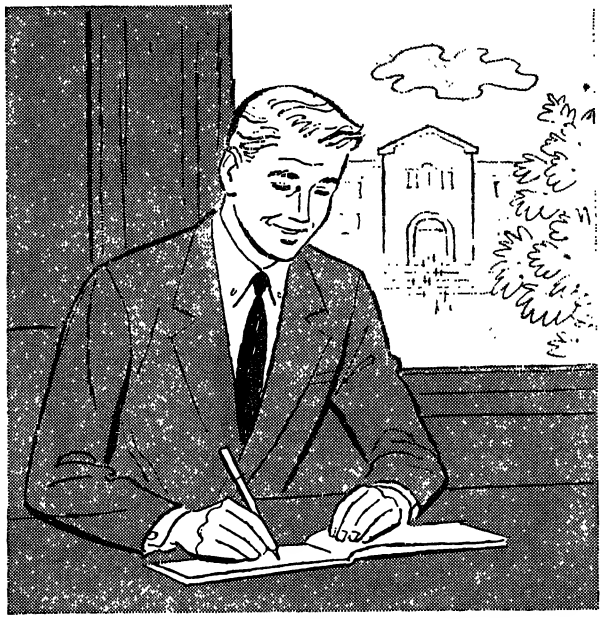
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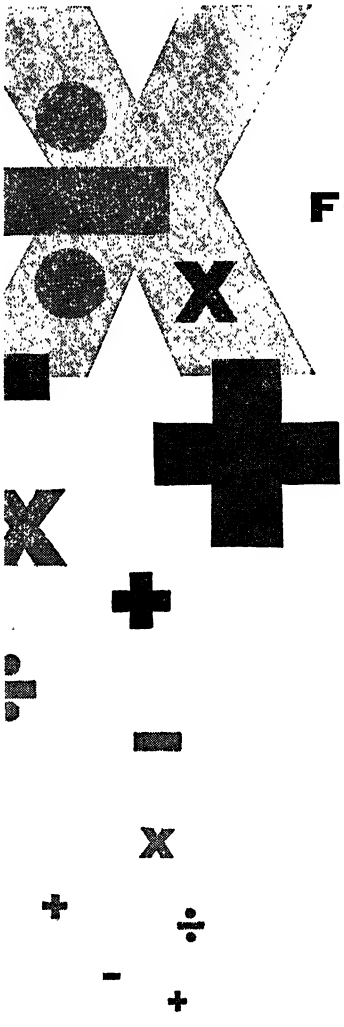
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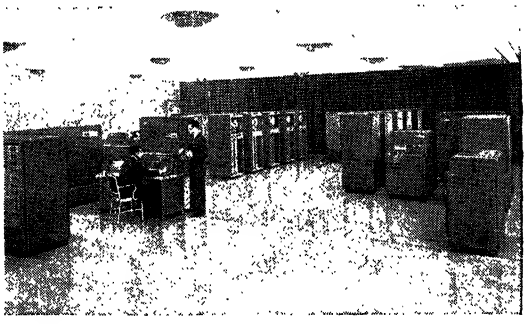
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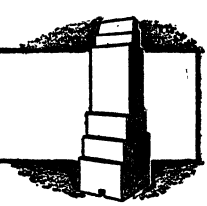
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MATHEMATICS AS A PROFESSION TODAY IN INDUSTRY*

THORNTON C. FRY, Bell Telephone Laboratories

INTRODUCTION

The word "Today" in the subject assigned to me implies that there has been a change in the status of mathematics in industry. This is true.

On the score of numbers alone, the change is impressive. At the beginning of this century the number of men in industry who could be described as mathematicians was negligible. Steinmetz was with General Electric—he had a Ph.D. in mathematics—and Campbell was with the Bell Telephone System. Perhaps there were others, though not many. Even in 1940, when I made a serious attempt to estimate the number, the figure I arrived at was only 150 in industry and government combined.[1] Last year, in a conference in New York on Training in Applied Mathematics, the present need, if not the actual number employed, was estimated to be around 1500.[2]

This rapid growth has brought new problems to the universities. As long as only an occasional mathematician entered industry, the curriculum could properly be planned without regard for him. Now that a significant percentage of graduates are industry bound, this is no longer true.

My object in this address will be to examine the place of the mathematician in industry, and bring out some of the basic requirements for his proper education.

We could make some progress toward these ends by taking a cross-section of the professional group as it exists today, noting the sources from which it was derived, the training of its members, and the variety of its activities; and drawing such conclusions as seem appropriate. This is what I attempted to do in 1940, with, I believe, useful results.[1] But in seeking light on educational needs it has a serious drawback. We do not educate *today's* mathematicians today, we educate *tomorrow's*, and our curriculum should be adapted to tomorrow's needs. On the present occasion, therefore, it will be more profitable to focus attention on the changing scene. We will thus understand better why the demand for industrial mathematicians is now so heavy whereas formerly it was not; why it developed when it did; and what the character of the future demand will be.

THE CHANGING SCENE

The young man who enters college next month at age 18 will, if present

* An address delivered at the request of the program committee at a meeting of the Mathematical Association of America, Ann Arbor, Michigan, August 29, 1955.

[1] "Industrial Mathematics," a report for the National Resources Planning Board. Published as part of "Research—A National Resource—II," a House Document, 77th Congress.

[2] "Applied Mathematics as a Responsibility of the Mathematical Profession." Published as part of "Proceedings of a Conference on Training in Applied Mathematics", sponsored by the American Mathematical Society and by the National Research Council under Contract NSF-C7 with the National Science Foundation.

industrial practices continue, retire 47 years from now at age 65. We would like to know what these 47 years will be like, and what demands they will place on his education. But man is strangely endowed; in three dimensions he can see what is ahead, but in the fourth, which is the dimension of time, only what is behind. So the best I can do is to look backward 47 years, and compare the world of 1908 with that in which we live today. By a process of extrapolation we can then form a judgment of what lies ahead.

Progress in science. In physics, in 1908, a lively subject of conversation was the special theory of relativity, which Einstein had formulated in 1905. Another was the quantum hypothesis, which Planck had published in 1901. But neither subject had yet had much impact on science. In fact, Einstein's general theory did not appear until 1914, and Bohr did not propose his model of the hydrogen atom until 1913. Millikan measured the charge on the electron in 1912; until then the existence of a unique and indivisible electric particle was in the realm of speculation.

These were the first stirrings of the revolution in physics which led to our present conceptions of atomic structure. Nuclear physics was to come even later. Isotope, energy level, nuclear spin, forbidden state—such words were not in the language when our student entered college in 1908. What physicist had yet heard of an eigenvalue? Or how many mathematicians either, for that matter?

In chemistry the change has been almost as marked. The combining weights of the elements had, of course, been measured, and when arranged in order of magnitude they looked too much like an arithmetical sequence to be ascribed to chance. But the deviations, too, were solidly established, and irritating because they were not understood. Moseley's paper on Atomic Numbers would not appear until 1914; and Aston's invention of the mass spectrograph, which gave the final confirmation, was five years later. The elements had also been arranged in the periodic table, in terms of their valences. The pattern of chemical behavior which this revealed was consistent in too many ways to be attributed to chance. But here also there was an irritating mystery, multiple valences. The Braggs had just seen those great spots-before-the-eyes of science, the scattering of X-rays by crystals. Eventually these would make the geometry of chemical bonds real and precise, so that the structural formulas of chemistry would no longer be limited to rows of letters. But in 1908 chemical valences and chemical bonds were still empirical facts with almost no systematic theory to tie them together. Except for a few dyestuffs, synthetic chemicals were unknown.

In communication, there were automobiles, telegraph and telephone; and the Wright brothers had made their historic flight in 1903. But flight was in its infancy, and no one had yet talked across the American continent, much less across the oceans. De Forest had just patented his vacuum tube in 1907, so of course there was no electronics.

In biology, Mendel's law was known. But there was as yet no talk of genes

or chromosomes; so of course the mechanism of inheritance and mutation was not understood. And the viruses and vitamins, which are newspaper words today, were then unknown.

Biochemistry was not even born. We can date its beginnings from the discovery of the protein building blocks by Fischer in 1913. Even the final identification of chlorophyl as the carrier between light and life dates only from 1913.

Who, in mathematics, had ever heard of Markoff processes, or the theory of games; of information theory or the theory of network synthesis?

The fact is, that in the last 47 years the whole fabric of physical science has changed. As one of the students who entered college in 1908, I have seen these changes. This is the exciting world, at that time still unknown, for which my college and university were asked to prepare me.

Progress in industry: New things. The technology which is founded on the physical sciences has shown equally radical progress. Its most obvious tokens are the new things we make: cellophane and nylon hose, TV sets and deep freeze units, aeroplanes, supersonic drills and atom bombs; transistors, electron microscopes and tracer isotopes. We cannot fail to marvel at the rate at which such new things have appeared, and the changes they have brought about in our standard of living.

These new things are ubiquitous, but they do not of themselves account for the growing demand for industrial mathematicians. In 1908 the makers of automobiles, locomotives and ocean liners felt no need to hire mathematicians; for the most part, in fact, they still do not. Why then are the makers of airplanes clamoring? Naval rifles and machine guns were designed by engineers; why is the mathematician needed to design a guided missile? Your home radio and TV sets are fine examples of what the engineer can do; why then was it necessary to develop a new branch of mathematics before a coaxial repeater could be built to amplify the signals of TV?

Better things. A second change in technology is the rising standard of performance which we demand of the products of industrial research. A few weeks ago I visited the oldest maker of submarine cables in the world, and watched the manufacture of the cable which will shortly carry 36 telephone conversations across the Atlantic. The precision required in this process was so great that a new factory, designed from the ground up to meet the stated demands, had been necessary. As one small example, instruments were provided to monitor continuously both the thickness and the dielectric constant of the plastic insulation as it came from the extruding press; and a servomechanism to adjust the speed of extrusion automatically so that the electrical capacity of the cable would be everywhere the same. Why was this precision necessary? Why must the dimensions of the copper wires and tapes be held to closer limits than ever before, and even the molecular weight of the plastic be controlled? If these things were not done, a submarine telephone cable might still be possible; but it would carry 30 or 25, or maybe even less messages; and the cost of the service rendered would be greater.

This example is unusual in degree perhaps, but not in kind. Servo control of machines is common in manufacturing today; so is the careful specification of primary materials. But in 1908 no one could have controlled the molecular weight of a polymer—in fact no one could even have measured it. Nor could the servo system have been designed—it was 25 years later before an adequate mathematical theory for that purpose was worked out.

It is natural that, as science moves forward and provides new knowledge and new techniques, industry finds profitable ways to use them. But only to the extent that the men of industry are alert to the new knowledge and competent to apply it. So the rising standard of performance puts requirements not only on materials and machines, but also on the men who design the products and processes. It requires them to bring to bear on each problem the entire wealth of modern science and technology, so far as it is pertinent, and to do this in a way which is imaginative without being impractical.

Frequently this is more than any man can do. This brings us to one of the most fundamental reasons why industry needs mathematicians, but we shall pass the point for the moment. It will arise again soon in an even more challenging form.

The system. The third important change in technology is the trend to develop systems to meet total situations, rather than separate functional components.

The cable, to which we referred a few moments ago, is actually part of such a system, the total purpose of which is to provide reliable transatlantic telephone service in adequate volume and at reasonable cost. The cable alone does not provide such service. There must be repeaters to amplify the signals at intervals and restore their proper wave form; terminal apparatus to generate the signals at the sending end and distribute them to the proper destinations at the receiving end; adequate and reliable power plants; equipment for testing, monitoring and adjusting the working apparatus, *etc.* All these functional units are interdependent; the shortcomings of one may to some extent be compensated by imposing stronger requirements on another. And all are influenced by the form in which the speech signals are encoded and many similar matters; for example, the use of pulse coded transmission would make it easier to restore the proper wave form of the signals, but would require a more extravagant frequency bandwidth; a vocoder system would conserve bandwidth at the expense of reduced clarity and more complicated terminal equipment.

The problem then, is to get the combination of parts which will give the best answer to the over-all problem. This may require a cable with requirements more stringent than the cable manufacturer likes, and less stringent than the repeater engineers would prefer. But if the system is wisely planned, no requirement will be so severe as to be technologically unreasonable; and those which are difficult to meet will assure advantages to the over-all performance of the system which are commensurate to their cost.

An even better example of system design is NIKE, the antiaircraft defense

system which is now being installed around all major cities. Here the problem was to protect strategic land locations against high-flying bombers carrying modern weapons of destruction. Before any part of this system was designed—even before design of any part was authorized—a careful study was carried out to formulate the problem clearly, and then determine whether a satisfactory solution was possible in the existing state of technology, and if so, by what means. Such a study had to take account of probable future developments in the enemy's aircraft and bombs; it had to look into the relative merits, for the case at hand, of homing and guided missiles; what could be expected in propulsion, in air frame design, in radar accuracies, in jamming techniques, and a multitude of other matters. Only when this study was completed was the development of the NIKE system undertaken. As it happened, not a single functional part of the system was taken from what the market provided. Air frame, warhead, propulsion, radars, computers, even the vehicles in which the equipment is housed, were all newly designed to contribute their proper share to the over-all performance of the system.

The team. Let us look for a moment at the breadth and depth of technical competence which was necessary to carry out this study. Among the more obvious things are all the skills that go into the design of radars, computers, jet engines, air frames, and so on. Even at this point one can say that no one man can be expertly informed about all. There are also less obvious matters: the theory of games for example. It is clear that the enemy will not cooperate by flying a desired course or even passively follow his own original plan in the presence of unanticipated danger. What strategy should the NIKE missile then follow to win, taking full account of its own limitations on maneuverability, fuel capacity, *etc.*, as well as the probable limitations of the enemy?

I could enlarge this list further, but I think the point has been made. The development of a system—and even more importantly the planning of such a development—requires detailed knowledge of too many scientific fields to be the work of any one man. It requires a *team of experts*.

Research teams are not unusual today, either in the universities or in industry. In 1908 it was different. They were not necessary then; today they are. And the reasons why they are now necessary have, I hope, emerged from what I have said so far. Essentially there are three: the scope of science has greatly increased; we demand the higher standard of performance which modern science makes possible; and we are increasingly developing systems to meet a total situation, rather than machines to perform some basic function.

I can put this more briefly. We know more, therefore we can do more. But it is *collectively* that we know most, and hence *collectively* we must act to accomplish most.

Why mathematicians? Mathematicians are useful members of such teams. Why? The answer which trips off our tongue most glibly is, "Because there are problems to be solved."

This answer is wrong. Not merely inadequate, mind you. It completely

misses the essential point. Mathematicians do sometimes solve problems, in industry as elsewhere. But they are usually not good at it. Certainly not as good as machines. And certainly the solving of problems is not often of such central importance as to require special representation on a research team.

The indispensable function of the mathematician is formulating problems, not solving them. This is even true of the mathematician who specializes in computation, as one of the outstanding members of our mathematical group does. As scientists from all parts of our organization bring problems to him for calculation, the first discussion almost invariably concerns the origin of the problem and its formulation; and it is not unusual for the scientist to go away convinced that he had brought a different problem than he had intended.

It is because the mathematician is expert in analyzing relations, in distinguishing what is essential from what is superficial in the statement of these relations, and in formulating broad and meaningful problems, that he has come to be an important figure in industrial research teams. Obviously, these traits are most needed at the early stage where a situation is being studied and plans are being laid. He plays much less part in the execution of these plans. NIKE is a case at point: the study and planning of this project required a team of experts from many engineering areas—propulsion, aerodynamics, radar, digital circuitry, *etc.*—as well as from the military. But the central members of the team were the mathematicians. This was not merely because the theory of games was involved. It was principally because they, better than other scientists, are adept in detecting the essential thread that lies obscured by the irrelevant details and divergent languages of the other sciences. Later, as the development progressed, they played a less prominent part in its activities.

Not every development project separates itself as neatly into a study and an execution phase as did NIKE; and the role of the mathematician is not always so central. More often than not he is a consultant to other members of the team, rather than their leader. But the service which he performs is generally that of critical analyst and expert formulator of situations and concepts.

LOOKING AHEAD

We may now try to extrapolate into the future, and thus appraise the educational job that should be done today.

We do not know what the future will be in detail. We do know it will be different. Hence, our job is surely to educate men as best we can to meet that *different* future, whatever it may be.

We know that science will progress and probably with great rapidity. This is inevitable, because tools now available permit studies that in the past would have been impossible. To mention only a few, there are such familiar things as electron microscopes, tracer elements and supersonics, the possibilities of which have not yet been fully exploited. The study of materials at low temperatures, where the voice of the nucleus can be heard above the thermal noise, has only just begun. And, especially in chemistry and biochemistry, we have as yet

caught only a glimpse of the importance of pure chemicals.

We can be reasonably sure of some of the directions that scientific progress will take. The nature of chemical bonds will some day be understood completely enough and precisely enough to predict untried reactions with assurance. The relation between chemical structure and physical properties will also be understood, so that elasticity, tensile strength, electrical resistivity, and the like, can be computed for a substance which has not yet been synthesized. It may even happen, some day, that this process will be operated in reverse; the customer having specified the physical characteristics which he needs, a new molecule will be computed which will have them.

We can be reasonably sure that life processes will soon be better understood. We already know that the fundamental chemical changes in cells are profoundly influenced by minute quantities of chemicals in their environment. These chemical processes cannot be studied in the test tube because sufficiently pure chemicals are not available. In the last few years a few substances—germanium, silicon and indium—have been purified to the point where less than one foreign molecule in a billion remains; but these are not of great importance to biochemical research. When ways are found to purify those which are more significant, progress will be almost explosive.

Industry too will progress. The pushbutton factory is already here. I know a large one which operates round the clock; the total number of so-called productive employes is six—two for each shift! In the jargon of the day this is "automation." What its future will be no one can now foretell. But it has arrived in conjunction with two other events of major historical importance which make its rapid growth inevitable. One is the control of nuclear reactions, which can provide limitless quantities of easily transportable power. Like every major increase of available power in the past, it will both reduce the demand for muscular power and increase the products of labor. The other is the creation of solid state devices such as ferrites, ferroelectrics, and transistors. The reliability and low wastage of power inherent in such devices will make it practicable to construct communication systems of capacity hitherto unknown, and control devices of almost fantastic complexity. One consequence will surely be decentralization in ways now scarcely dreamed of. For example, we may expect the parts of a single push-button factory to be widely dispersed, and yet operate automatically under a central system of control.

But all this means that the industrial problem of developing a product, and designing a factory to produce it, will become even more complex. It will clearly be, even more often than today, a problem of integral system design. Only thus can full advantage be taken of the possibilities opened up by the advances in science. And only those who are masters of science in all its avenues can fully exploit its possibilities. So quite clearly the job will be even more beyond the capacities of any one man than now, and the team will be an even greater necessity.

Again, we may put this briefly: In the future we shall know more, therefore

we shall be able to do more. But since we shall only know *collectively*, we shall have to work *collectively*.

What sort of mathematician? Let us attempt to visualize this team of the future, which we must educate today, and especially the mathematician on it.

There is certainly nothing to suggest that he will be less wanted than today. It is not an accident that the great increase in industrial demand for mathematicians in the last 15 years coincided with the rapid development of team research. It will continue to grow because the industrial exploitation of advancing science will demand ever more competent teams, and the teams will not be fully effective without the mathematician. There may even be relatively more situations like NIKE, in which he will play a central role; though I believe his position will be advisory more often than controlling.

The team will need to understand more things, and its individual members will need to understand them more thoroughly. To put it more briefly, they will have to be better educated. If anyone is inclined to feel that this is a speculative judgment, I need only point out that industry is hiring today a larger proportion of Ph.D.'s than ever before. Even on its engineering staffs, such men are needed in numbers that are not available. I know of companies, both here and abroad, which are hiring hundreds of engineering graduates and giving graduate education while on the company payroll.

Better educated teams, however, mean faster company for the mathematicians on them. If there is any doubt that industrial mathematicians are playing in the big league today, there can be none about tomorrow. So in addition to the increased breadth that will be necessitated by the rapid development of science, they will need increased depth and versatility to keep pace with their associates.

This is a challenging situation. Our present patterns of education for Ph.D.'s do not produce industrial mathematicians of the breadth of scientific outlook needed today. They are even less adequate for the tomorrow in which our graduates will live. Clearly the Ph.D.—today's Ph.D.—is not enough.

It is not a unique problem. The same outlook faces the other physical sciences as well. But except for engineering—which has failed even to provide Ph.D. training for more than a negligible proportion of its students—the other sciences are alert to the need. In mathematics there is still a widespread misunderstanding that the industrial mathematician is just a problem solver.

Let us not make that mistake. We are talking about problem formulators, very superior consultants who can play in big league company. That is what industry needs. It cannot, without serious penalty, settle for less.

What sort of man should he be? He must be experienced and adept in the basic thought processes of mathematics, and of course he must also have a working knowledge of its various branches. These are the usual requirements of a mathematician anywhere.

Beyond this is the requirement that he be able to work effectively as a member of a team. To do so, he must have broad scientific interests and enough

knowledge of other sciences to enable him to deal intelligently with his associates; and he must have a personality suitable for joint effort.

I expanded on these ideas in the previous paper to which I have referred,[1] and need not go into them more particularly today. There is, however, one statement from that paper which I would like to quote verbatim:

> "He must be a man of outstanding ability. No one wants the advice of mediocrity. Among industrial mathematicians there is no place for the average man."*

CONCLUSION: THE EDUCATOR'S JOB

The educator's job today is nowise different from that of 1908. It is to fit the student for the life he will lead; life in a future as yet unknown, and probably as different from the present as was the case in 1908.

If we are to train men to meet this new and challenging future, we must train them in those things which are most lasting in our science—its habits of thought, its fundamental disciplines, its peculiar relation to philosophy and to the other sciences. Such training would not be meaningful if it did not deal with specific theorems and specific applications; these can only be chosen from the problems of today and yesterday; but their purpose should not be mistaken; it is to illustrate certain habits of thought and to train in certain disciplines; it is to illuminate the problems of tomorrow, not to solve the problems of today.

As I see it, just two things are essential to guarantee this result. The first is a clear appreciation of the need. Not merely that "mathematicians" are wanted, but what sort of service they will perform. Given this understanding of the need, I will trust our educators to build the curriculum to meet it.

The second is time. Even today our young men are likely to be in their late twenties by the time they get their already inadequate Ph.D. If we seek to make this adequate by adding more years to the curriculum, we will consume even more of their imaginative youth. We should seriously ask if there is not a better way.

I think there is. It is the atavistic way. We must compress what is essential from the past of our science into fewer years, so that its present can be reached at an acceptable age. This will not be easy. Not that it is inherently difficult, but it runs counter to other educational purposes which have been more widely

* Lest this be understood to mean that the world has no important place for the man of average, or less than average, training or ability, I may also quote from my address before the Conference on Training in Applied Mathematics:[2]

> "It is important to recognize that . . . industrial and government employers . . . need men with more than one level of mathematical training—from technicians who are skilled in performing certain mathematical operations, to creative scientists who can invent and improve their art."

I have no doubt that industry will need many computers, and will want them even better trained. They also will deserve the respect and dignity due all useful members of society. But my present address is concerned with scientists; not with technicians.

publicized and are generally—I think too generally—accepted. What I have in mind is a very radical reappraisal of the educational values of mathematical subjects, and their psychological adaptability to various ages. Why, for example, do we not teach algebra along with, or before arithmetic? Is it harder to understand that $a+a=2a$ than that $1+1=2$? At how early an age can imaginary numbers be introduced? Does not a child acquire the ability to appreciate (or is it accept?) such simple logical relationships at an earlier age than he acquires the experience and judgment needed for many problems of arithmetic? If these things were done, could we not teach calculus to ninth year students? And could we not teach it in the complex plane at the start? Finally, could we not do all these things without devoting more hours to mathematics than is now devoted by those who take it at all as a high school elective?

I can see no less radical way to get the time that is needed to educate mathematicians for industry, and still preserve their most valuable creative years for productive work. It will not be easy to bring about. But the stakes are worth it; for such an atavistic approach to mathematics is as badly needed by students who enter engineering or the physical sciences as by the mathematicians themselves.

A MEDIEVAL ITERATIVE ALGORISM

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A standard interpolation expedient of ancient astronomy consisted of putting $f(\theta)=k \sin \theta$ to obtain general values of a periodic, symmetrical function of which it was known only that $f(0°)=f(180°)=0$, and $f(90°)=k$. This device is at least as old as Ptolemy (*c.* 150 A.D.). The last column of his table for the computation of planetary latitudes* is in effect a table of cosines, to be used as indicated above. The table is a true interpolation scheme in the sense that it was set up without direct reference either to the underlying astronomical phenomenon or to the complicated mathematical idealization to which it yields an approximation.

Of course Ptolemy had no knowledge of the sine or cosine functions as such. His interpolation function was obtained from a table of lunar latitudes,† itself computed by use of a table of chords.‡

* The Almagest, Book 13 (Syntaxis mathematica, ed. J. L. Heiberg, 2 vols., Leipzig, 1898–1903; German translation by K. Manitius, 2 vols., Leipzig, 1912–13). The edition and French translation hereafter referred to is by Halma, Composition mathématique . . . , Paris, 1813–1816.

† *Ibid.*, Halma, vol. i, p. 316.

‡ *Ibid.*, Halma, vol. i, pp. 38–45.

The Hindus, who invented the ancestor of the modern sine function, made extensive practical application of this style of interpolation. The Hindu sine (written here Sin with a capital S to distinguish it from the modern function) is definable as $\text{Sin}\,\theta \equiv R \sin\theta$, where R is the radius of the defining circle.

Various values for R were current. Where numbers were customarily represented in a place-value system with base sixty, it was convenient, following the lead of Ptolemy, to put $R=60$. With this arrangement, standard with the Arabic-writing Islamic astronomers, Sin θ has the same sexagesimal digits as $\sin\theta$, but having the "sexagesimal point" moved one place over.

An R ascribed§ to the Hindu Āryabhaṭa is 3438, obtained by putting the linear unit equal to one minute of arc length along the circumference of the defining circle. If now θ is in minutes, $\text{Sin}\,\theta \approx \theta$ for small θ, a property resembling the corresponding one for radians.

A third value of R is 150, probably so set because with this parameter Sin ϵ very nearly equals the sexagesimal base, ϵ being the inclination of the ecliptic.‖ If θ is the longitude of a point on the ecliptic, its declination δ is given by the expression $\sin\delta = \sin\epsilon \sin\theta = k \sin\theta$. Thus a table of declinations can be used as the basis for an interpolation scheme which resembles, but is not identical with that described in the first paragraph.¶

For the representation of smooth periodic phenomena with the maximum not midway between the zeros, say at $\theta = 90° - m$ for $0° < m < 180°/\pi$, a natural modification of the standard procedure would be to effect some continuous transformation of the θ's leaving the endpoints ($\theta = 0°$ and $\theta = 180°$) undisturbed and displacing the midpoint ($\theta = 90°$) an amount m to the left (see the figure). Such a transformation is defined by the expression

$$t = \theta - m \sin\theta, \tag{1}$$

and the function

$$\Phi(t) = k \sin\theta(t) \tag{2}$$

has the desired property. Tables of this $\Phi(t)$ exist in two Islamic astronomical handbooks** in connection with the determination of lunar parallax components. Neither author motivates the method, and one of them clearly indicates that it

§ *Cf.* Rasā'il ul-Bīrūnī (in Arabic), Osmania Oriental Publications Bureau, Hyderabad-Dn., 1948, p. 178; also Burgess, E., Translation of the Sūrya-Siddhānta, a Textbook of Hindu Astronomy, . . . , reprinted from the edition of 1860, Univ. of Calcutta, 1935, pp. 59–64.

‖ Tables containing this function appeared in medieval Europe; *cf.* for example, Millás-Vallicrosa, Estudios sobre Azarquiel, Madrid, 1943–1950, p. 44; and Neugebauer and Schmidt, Hindu astronomy at Newminster in 1428, Annals of Science, vol. 8, 1952, pp. 221–228.

¶ The tables of solar and lunar equations in Die astronomischen Tafeln des . . . al-Khwārizmī (edited by Björnbo, Besthorn, and Suter, Copenhagen, 1914, pp. 132–136) were computed by this method.

** *Ibid.*, pp. 191–192, and Bodleian (Oxford) Ms. Seld. A. 30, Az-Zīj al-Jadīd li Ibn ash-Shāṭir. For the astronomical setting of the problem, see Kennedy, E. S., "Parallax theory in Islamic Astronomy," to appear in *Isis*.

is not his invention. From the fact that most of the material in the other is known to be of Hindu origin it is conjectured that this technique also was worked out in India.

A third Arabic work, written in the ninth century by *Ḥabash al-Ḥāsib al-Marwazī*†† (*i.e.*, The Computer from Merv, in Turkestan) gives a neat recursion relation for determining Φ. He prescribes the formation of the sequence

$$\begin{aligned} &\theta_0(t) = t + m \sin t, \\ &\cdots\cdots\cdots\cdots \\ &\theta_n(t) = t + m \sin \theta_{n-1}(t), \\ &\cdots\cdots\cdots\cdots \end{aligned} \qquad (n = 1, 2, 3, \cdots) \tag{3}$$

the limit of which satisfies (1). Hence

$$\Phi(t) = k \sin \lim_{n\to\infty} \theta_n(t). \tag{4}$$

The symbolism is, of course, modern; in all of the sources the processes are written out as verbal statements.

Ḥabash gives no proof, contenting himself with the statement that the desired function is

$$k \sin \theta_3(t).$$

In fact the sequence does converge, and with sufficient rapidity that θ_3 yields a fair approximation to the tables.

We show briefly the convergence of (3), or more generally of the sequence

$$\begin{aligned} &\theta_0(t) \equiv 0, \\ &\cdots\cdots\cdots\cdots \\ &\theta_n(t) = t + mS[\theta_{n-1}(t)], \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{5}$$

where the function S satisfies a Lipschitz condition

$$|S(t_1) - S(t_2)| \leqq k|t_1 - t_2| \quad \text{with} \quad k < 1/m \quad \text{and} \quad S(0) = 0.‡‡$$

Since

$$|\theta_n(t) - \theta_{n-1}(t)| = m|S[\theta_{n-1}(t)] - S[\theta_{n-2}(t)]| \leqq mk|\theta_{n-1}(t) - \theta_{n-2}(t)|,$$

we have

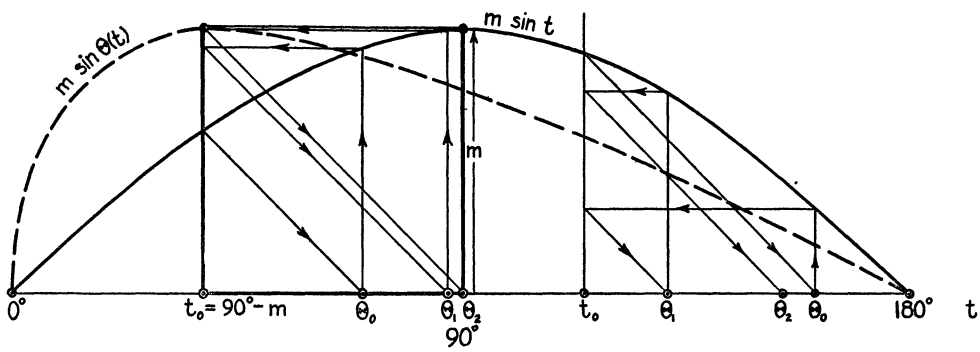
$$\left|\frac{\theta_n(t) - \theta_{n-1}(t)}{\theta_{n-1}(t) - \theta_{n-2}(t)}\right| \leqq mk < 1,$$

†† Manuscript 784,2° in the Yeni Jami library in Istanbul. Ḥabash was one of the scientists attracted to the Baghdad and Damascus courts of the Caliph al-Ma'mūn.

‡‡ This follows directly from, for example, Kantorovich, Vulikh and Pinsker, Functional Analysis in Partially Ordered Spaces, Moscow, 1950 (in Russian), p. 467, Cor. 2.24.

so that the ratio test shows the convergence of the series with general term $\theta_n(t)-\theta_{n-1}(t)$, and hence of the sequence $\theta_n(t)$.

From the point of view of the historian it is of interest to ask not only about the validity of the process, but also who invented it and how. Neither question can at present be satisfactorily answered. There is no particular reason for thinking that Ḥabash was the inventor. It is natural to suspect a Hindu origin for the algorism, by extrapolation from the fact that the techniques in which this section of Ḥabash's work is embedded are demonstrably Hindu. The nature of the operation provides no clue, for computations by successive approximation were used by Hellenistic, Hindu, and Muslem scientists alike. As for a reconstruction of the process of invention, it will be illuminating to consider a graphical method of generating the sequence $\{\theta_n\}$. In the figure below, and for any



fixed t_0, draw a line with slope -1 from point $(t_0, m \sin t_0)$ to intersect the horizontal axis at the point marked θ_0. From θ_0 erect a perpendicular meeting the curve in a point, and project the latter horizontally to P on the ordinate through t_0. In turn project P downward to the right as shown to obtain θ_1, and so continue the cycle of projections. It can be shown that for $t_0=90°-m$ the bounded sequence $\{\theta_n\}$ is monotonic increasing, converging to 90°. Hence $\sin\theta(t)$ has its maximum at $t=90°-m$. It is submitted that the unknown inventor, thoroughly familiar with the technique of trigonometric interpolation, but desiring a function of the nature of the curve shown dotted in the figure, set up the equivalent of expressions (1) and (2). This gives the desired type of function, but has the disadvantage that, given a t, one cannot directly obtain a corresponding Φ. A direct method being desirable, the formal similarity between expression (1) and

$$t = \theta_n(t) - m \sin \theta_{n-1}(t)$$

may well have been noticed, and the convergence of the implicit sequence concluded from considerations analogous to our graphical method.

SOME GENERALIZATIONS OF VANDERMONDE'S CONVOLUTION

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Among the elegant results implied by the binomial theorem, one of the most attractive and widely known is Vandermonde's convolution, which states that:

$$\sum_{i=0}^{k}\binom{r}{i}\binom{q}{k-i}=\binom{r+q}{k}. \tag{1}$$

This formula is valid for all integral values of r and q. Since each member is a polynomial in r and also in q of degree k, it follows that (1) is a polynomial identity in r and q.

It is with generalizations of this type relation that we shall be concerned in the present paper. We define the general expression:

$$A_n(\alpha,\beta)=\binom{\alpha+\beta n}{n}\frac{\alpha}{\alpha+\beta n}. \tag{2}$$

From relation (1) we easily see that

$$\sum_{k=0}^{n}\binom{\alpha+k}{k}\binom{\gamma+n-k}{n-k}=\binom{\alpha+\gamma+n+1}{n}, \tag{3}$$

and taking note of (2) it is clear that this and relation (1) are special cases of the following when $\beta=0$ and when $\beta=1$.

$$\sum_{k=0}^{n}A_k(\alpha,\beta)A_{n-k}(\gamma,\beta)=A_n(\alpha+\gamma,\beta). \tag{4}$$

In order to demonstrate (4) we may proceed as follows, developing certain preliminary results first.

Taking the nth difference of a polynomial of degree less than or equal to $n-1$ we must have zero for $n\geqq 1$. Hence we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\alpha+k\beta}{n}\frac{\alpha}{\alpha+k\beta} \tag{5}$$

$$=(-1)^nA_n(\alpha,\beta)+\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\binom{\alpha+k\beta}{n}\frac{\alpha}{\alpha+k\beta}=0,\ n\geqq 1.$$

Hence

$$A_k(\alpha,\beta)=\sum_{j=0}^{k-1}(-1)^{j+k+1}\binom{k}{j}\binom{\alpha+\beta j}{k}\frac{\alpha}{\alpha+\beta j},\qquad k\geqq 1. \tag{6}$$

This relation enables us to evaluate a certain infinite series as follows:

$$\sum_{k=0}^{\infty} A_k(\alpha, \beta)z^k = 1 + \sum_{k=1}^{\infty} A_k(\alpha, \beta)z^k$$

$$= 1 + \sum_{k=1}^{\infty} z^k \sum_{j=0}^{k-1} (-1)^{j+k+1}\binom{k}{j}\binom{\alpha + \beta j}{k}\frac{\alpha}{\alpha + \beta j}$$

$$= 1 + \sum_{j=0}^{\infty} (-1)^j \frac{\alpha}{\alpha + \beta j} \sum_{k=j}^{\infty} (-1)^k \binom{k+1}{j}\binom{\alpha + \beta j}{k+1} z^{k+1}$$

$$= 1 + \sum_{j=0}^{\infty} (-1)^j \binom{\alpha + \beta j}{j}\frac{\alpha}{\alpha + \beta j} \sum_{k=j}^{\infty} (-1)^k \binom{\alpha + \beta j - j}{k+1-j} z^{k+1}$$

$$= 1 - \sum_{j=0}^{\infty} \binom{\alpha + \beta j}{j}\frac{\alpha}{\alpha + \beta j} z^j \sum_{k=1}^{\infty} (-1)^k \binom{\alpha + \beta j - j}{k} z^k$$

$$= 1 - \sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j \sum_{k=0}^{\infty} (-1)^k \binom{\alpha + \beta j - j}{k} z^k + \sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j.$$

Therefore we find

$$\sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j \sum_{k=0}^{\infty} (-1)^k \binom{\alpha + \beta j - j}{k} z^k = 1,$$

and this becomes

$$\sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j(1 - z)^{\alpha+\beta j-j} = 1.$$

Making changes in the notation and examining the convergence we find

$$\sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j = x^{\alpha}, \tag{7}$$

where:

$$z = \frac{x - 1}{x^{\beta}} \quad \text{and} \quad |z| < \left|\frac{(\beta - 1)^{\beta-1}}{\beta^{\beta}}\right|.$$

Again, we also find the following:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{\alpha + \beta k}{n} = (-1)^n \beta^n, \qquad n \geqq 0, \tag{8}'$$

which follows easily by finite difference arguments, and is well-known. From (8) we find:

$$(-1)^n \binom{\alpha + \beta n}{n} = (-1)^n \beta^n - \sum_{k=0}^{n-1} (-1)^k \binom{n}{k}\binom{\alpha + \beta k}{n}, \qquad n \geqq 1.$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\binom{\alpha+\beta n}{n}z^n &= 1+\sum_{n=1}^{\infty}(-1)^n\binom{\alpha+\beta n}{n}z^n \\
&= 1+\sum_{n=1}^{\infty}(-\beta z)^n-\sum_{n=1}^{\infty}z^n\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\binom{\alpha+\beta k}{n} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\sum_{n=k}^{\infty}\binom{n+1}{k}\binom{\alpha+\beta k}{n+1}z^{n+1} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}\sum_{n=k}^{\infty}\binom{\alpha+\beta k-k}{n+1-k}z^{n+1} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k} \\
&\quad \left\{\sum_{n=0}^{\infty}\binom{\alpha+\beta k-k}{n}z^{n+k}-z^k\right\} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}z^k(z+1)^{\alpha+\beta k-k} \\
&\quad +\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}z^k.
\end{aligned}$$

Therefore we have

$$\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}z^k(1+z)^{\alpha+\beta k-k}=\frac{1}{1+\beta z}.$$

This may now be rewritten in the form

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta k}{k}z^k=\frac{x^{\alpha+1}}{(1-\beta)x+\beta}, \tag{9}$$

where

$$z=\frac{x-1}{x^\beta}, \qquad |z|<\left|\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\right|.$$

A discussion of the convergence of this last series may be found in Pólya and Szegö, *Aufgaben und Lehrsätze*, Springer Verlag, Berlin, 1925, first volume, problems 206 to 218; in particular problem 216, *zweiter Abschnitt*. These series may also be found as Pólya and Szegö indicate from the application of the Lagrange theorem, a statement of which is given in Whittaker and Watson, *A Course of Modern Analysis*, 4th Edition, 1927, page 133.

Now from relation (7) we have the following:

$$x^{\alpha}\cdot x^{\gamma} = x^{\alpha+\gamma} = \sum_{k=0}^{\infty} A_k(\alpha+\gamma,\beta)z^k$$

$$= \left(\sum_{i=0}^{\infty} A_i(\alpha,\beta)z^i\right)\left(\sum_{j=0}^{\infty} A_j(\gamma,\beta)z^j\right)$$

$$= \sum_{k=0}^{\infty} z^k \sum_{i=0}^{k} A_i(\alpha,\beta)A_{k-i}(\gamma,\beta).$$

Therefore we may compare coefficients and we find

$$\sum_{k=0}^{n} A_k(\alpha,\beta)A_{n-k}(\gamma,\beta) = A_n(\alpha+\gamma,\beta), \tag{10}$$

which is exactly relation (4), and since it is a polynomial expression on each side here of degree n in α, β, γ, it is an identity in them.

Just as we have derived (10) we may combine relations (7) and (9), and we find the following:

$$\frac{x^{\alpha+\gamma+1}}{(1-\beta)x+\beta} = \sum_{k=0}^{\infty}\binom{\alpha+\gamma+\beta k}{k}z^k = x^{\gamma}\cdot\frac{x^{\alpha+1}}{(1-\beta)x+\beta}$$

$$= \sum_{i=0}^{\infty}\binom{\gamma+\beta i}{i}\left(\frac{\gamma}{\gamma+\beta i}z^i\right)\sum_{j=0}^{\infty}\binom{\alpha+\beta j}{j}z^i$$

$$= \sum_{k=0}^{\infty} z^k \sum_{i=0}^{k}\binom{\gamma+\beta i}{i}\frac{\gamma}{\gamma+\beta i}\binom{\alpha+\beta(k-i)}{k-i}.$$

Comparing the coefficient of z^k in these expressions we find therefore that with appropriate changes in symbols,

$$\sum_{k=0}^{n}\binom{\gamma+\beta k}{k}\binom{\alpha+(n-k)\beta}{n-k}\frac{\gamma}{\gamma+\beta k} = \binom{\alpha+\gamma+\beta n}{n}. \tag{11}$$

This result is quite as valuable as relation (10) although the latter is more symmetrical. One more relation will be developed before some applications are made. Let

$$f(\alpha,x) = \frac{x^{\alpha+1}}{(1-\beta)x+\beta}.$$

Then,

$$f(\alpha,x)\cdot f(\gamma,x) = f(\alpha+\epsilon,x)\cdot f(\gamma-\epsilon,x),$$

so that by utilizing relation (9) we find at once

$$(12)\qquad \sum_{k=0}^{n}\binom{\alpha+k\beta}{k}\binom{\gamma+(n-k)\beta}{n-k}=\sum_{k=0}^{n}\binom{\alpha+\epsilon+k\beta}{k}\binom{\gamma-\epsilon+(n-k)\beta}{n-k},$$

which is again a symmetrical expression but not too useful.

Some Applications and Examples in the Literature

I. One of the unsolved problems in this MONTHLY revolves around this type of expansion being considered here. K. L. Chung proposed in problem no. 4211, (1946, page 397) the summation

$$(13)\qquad \binom{nd}{n}=d(d-1)\sum_{k=1}^{n}\frac{(dk-2)!}{(k-1)!(dk-k)!}\binom{nd-kd}{n-k},\qquad n\geqq 1,$$

which may be rewritten in the form

$$\binom{nd-1}{n-1}=\sum_{k=0}^{n-1}\binom{d-1+dk}{k}\frac{d-1}{d-1+dk}\binom{(n-1)d-kd}{n-1-k}$$

because (13) may first of all be rewritten as

$$\begin{aligned}\binom{nd}{n}\frac{1}{d-1}&=\sum_{k=1}^{n}\binom{dk}{k}\binom{nd-kd}{n-k}\frac{1}{dk-1}\\&=\sum_{k=1}^{n}\binom{dk-1}{k-1}\frac{d}{dk-1}\binom{nd-kd}{n-k}\\&=\sum_{k=0}^{n-1}\binom{dk+d-1}{k}\frac{d}{dk+d-1}\binom{nd-kd-d}{n-1-k}.\end{aligned}$$

But this is evaluated from relation (11) immediately if we first allow n there to be replaced by $n-1$, and let $d=\beta$, $\gamma=d-1$, and $\alpha=0$.

II. Another instance of this summation is in Torelli's sum. Torelli's result is quoted in the *Jahrbuch über die Fortschritte der Mathematik*, Volume 26, Jahrgang 1895, page 281. The original paper was entitled *Qualche formola relativa all' interpretazione fattoriale delle potenze* (Batt. G. XXXIII, pp. 179–182). We have here

$$x^{\overline{n}}=\prod_{j=0}^{n-1}(x+j)=n!\binom{x+n-1}{n}$$

$$(14)\qquad (x+y)^{\overline{n}}=\sum_{r=0}^{n}\binom{n}{r}y(y-rh+1)^{\overline{r-1}}(x+rh)^{\overline{n-r}}.$$

Or, therefore,

$$\binom{x+y+n-1}{n}$$

$$= (-1)^n \sum_{r=0}^{n} (-1)^r \binom{y-rh+r}{r}\binom{-x-rh}{n-r}\frac{y}{y-rh+r}$$

$$= \sum_{r=0}^{n}\binom{y-rh+r}{r}\binom{x-1+rh+n-r}{n-r}\frac{y}{y-rh+r}$$

$$= \sum_{k=0}^{n}\binom{y+(1-h)k}{k}\binom{x-1+nh+(n-k)(1-h)}{n-k}\frac{y}{y+(1-h)k},$$

which clearly follows from relation (11) when we put

$$\gamma = y, \qquad \alpha = x - 1 + nh, \qquad \beta = 1 - h.$$

III. One of the often quoted relations of a convolution type is formula number 17, pages 64–68, of the *Synopsis der höheren Mathematik*, Volume I, by Johann Georg Hagen, Berlin, 1891. This was in German, although Hagen was a professor at Georgetown University for some time. Netto, in his *Lehrbuch der Combinatorik*, Teubner, Leipzig, 1901, devotes Chapter 13 to Hagen's combinatorial identities, and points out a few errors in some of the relations. In each case, reference is made by Hagen and Netto to some work in this field by Heinrich August Rothe, whose paper *Formulae de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita*, Leipzig, 1793, would seem to be one of the earliest references to these general convolutions.

In any event, Hagen's formula 17 may be derived in the following manner. From relation (7) we have

$$\sum_{k=0}^{\infty} A_k(\alpha, \beta) w^{p+qk} = w^p x^\alpha,$$

where

$$w = \frac{(x-1)^{1/q}}{x^{\beta/q}}.$$

On the interval of absolute convergence we may differentiate and we find

$$\sum_{k=0}^{\infty} A_k'(\alpha, \beta)(p+qk) w^{p+qk-1}$$

$$= pw^{p-1}x^\alpha + \frac{\alpha w^p x^{\alpha-1} q x^{(\beta/q)+1}}{(x-1)^{(1/q)-1}\{(1-\beta)x+\beta\}}$$

$$= pw^{p-1}x^\alpha + \alpha q w^{p-1}\frac{x^{\alpha+1}}{(1-\beta)x+\beta} - \alpha q w^{p-1}\frac{x^\alpha}{(1-\beta)x+\beta}.$$

Therefore we have

$$(15)\quad \sum_{k=0}^{\infty} A_k(\alpha,\beta)(p+qk)w^{p+qk} = pw^p x^\alpha + \alpha q w^p \frac{x^{\alpha+1}}{(1-\beta)x+\beta} - \alpha q w^p \frac{x^\alpha}{(1-\beta)x+\beta}.$$

Now, we let

$$S = \left\{\sum_{k=0}^{\infty} A_k(\alpha,\beta)(p+qk)w^{p+qk}\right\}\cdot\left\{\sum_{j=0}^{\infty} A_j(\gamma,\beta)w^{p+qk}\right\}$$
$$= w^p \sum_{k=0}^{\infty} w^{p+qk} \sum_{i=0}^{k} A_i(\alpha,\beta)A_{k-i}(\gamma,\beta)(p+qi).$$

But, through the use of (15), this is equal to

$$S = w^p x^\gamma \left\{pw^p x^\alpha + \alpha q w^p \frac{x^{\alpha+1}}{(1-\beta)x+\beta} - \alpha q w^p \frac{x^\alpha}{(1-\beta)x+\beta}\right\}$$
$$= w^p \left\{p\sum_{k=0}^{\infty} A_k(\alpha+\gamma,\beta)w^{p+qk} + \alpha q \sum_{k=0}^{\infty}\binom{\alpha+\gamma+\beta k}{k} w^{p+qk} - \alpha q \sum_{k=0}^{\infty}\binom{\alpha+\gamma-1+\beta k}{k} w^{p+qk}\right\}$$
$$= w^p \sum_{k=0}^{\infty} w^{p+qk}\left\{\frac{p(\alpha+\gamma)}{\alpha+\gamma+\beta k} + \alpha q - \alpha q\frac{\alpha+\gamma+\beta k-k}{\alpha+\gamma+\beta k}\right\}\cdot\binom{\alpha+\gamma+\beta k}{k}$$
$$= w^p \sum_{k=0}^{\infty} w^{p+qk}\binom{\alpha+\gamma+\beta k}{k}\frac{p(\alpha+\gamma)+k\alpha q}{\alpha+\gamma+\beta k}.$$

But now we equate coefficients of these two expansions of S and the result is the relation

$$(16)\quad \sum_{k=0}^{n} A_k(\alpha,\beta)A_{n-k}(\gamma,\beta)(p+qk) = \frac{p(\alpha+\gamma)+n\alpha q}{\alpha+\gamma+\beta n}\binom{\alpha+\gamma+\beta n}{n},\quad n\geqq 0,$$

which is a polynomial identity in α, γ and β. Letting $\gamma+\beta n=y$ and $\alpha=x$, we have

$$(17)\quad \sum_{k=0}^{n}\binom{x+\beta k}{k}\binom{y-\beta k}{n-k}\frac{p+qk}{(x+\beta k)(y-\beta k)} = \frac{p(x+y-\beta n)+nxq}{(x+y)x(y-\beta n)}\binom{x+y}{n},$$

which is Hagen's relation 17 (and number 17 here also). Clearly, relations (10) and (11) are the special cases already treated.

It would be interesting to construct a proof of these convolutions using finite series, the method of finite differences, or otherwise, just as Vandermonde's convolution relation (2) was obtained. We remark that Vandermonde's convolution is the special case when we let $\beta=0$. The equivalent form of Vandermonde's result is that obtained when $\beta=1$. It would be rather interesting to try to prove relation (10) for example by induction on β or on n. However, the resulting expressions do not seem to reduce very manageably.

DISTANCE SUMS ON A SPHERE AND ANGLE SUMS IN A SIMPLEX

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1. Introduction. It was shown in [1] that if S is the sum of the dihedral angles in a tetrahedron and T the sum of the trihedral angles then,

(A) $$T = 2S - 4\pi$$

and

(B) $$2\pi \leqq S \leqq 3\pi.$$

Equation (1) was obtained from the well known formula relating the area A of a spherical triangle and the sum $\sum$ of its angles,

(C) $$A = \sum - \pi.$$

This paper is concerned with generalizing relations (B) and (C). The methods of [1] do not appear to permit a complete generalization of (B) although they give the best upper bound for S and a non-trivial lower bound. This paper gives best upper and lower bounds for S. Furthermore, (C) does not generalize to an equation, but to an inequality.

In [1] I remarked that one would expect those results to be known. It seems appropriate to state here that they were. (See [2].) It should also be mentioned that results related to those of this paper have been investigated by T. S. Motzkin in [3], using different methods.

2. Terminology. By the *dihedral* angle between two $(n-1)$-dimensional hyperplanes in E_n is meant the supplement of the angle between their directed normals. Given n directed planes in E_n whose intersection is a point p, by the *multihedral* angle they determine we mean the surface content they cut out on the unit sphere S_{n-1} whose center is p. We suppose throughout that $n>2$.

Given the unit sphere S_{n-1}, with center p, every $(n-1)$-dimensional hyperplane through p determines a unit hypersphere S_{n-2} on the S_{n-1}. By the angle between two hyperspheres on S_{n-1} we mean the dihedral angle between their determining hyperplanes.

We are now in a position to formulate the problem precisely. A simplex in E_n consists, for our purposes, of $n+1$ linearly independent points and the $n+1$ hyperplanes they determine. We consider each hyperplane directed toward the point not lying on it. Then to measure a multihedral angle at one vertex, we have simply to measure the surface content of a spherical simplex on the unit S_{n-1} with the vertex as center. One of our problems, then, is to find the surface content of a spherical simplex in terms of the $\frac{1}{2}n(n-1)$ angles between the n hyperplanes bounding it. This will give us a multihedral angle in terms of its surrounding dihedral angles.

To find the sum S of the dihedral angles in a simplex we choose a point O inside the simplex and drop perpendiculars to the faces. Calling the sum of the $\frac{1}{2}n(n+1)$ angles between these $n+1$ perpendiculars R, $S=\frac{1}{2}n(n+1)\pi-R$. But if we draw a unit sphere with center O, R is the sum of the (spherical) distances between the points $p_1, \cdots, p_{n+1}$ in which these perpendiculars intersect the surface of the sphere. Now, since O is inside the simplex there is no plane through O with all of the perpendiculars lying on one side of it, and hence the points $p_1, \cdots, p_{n+1}$ are not contained in any hemisphere. We call such a set of points *global*.

Our second problem then, discussing the sum of the dihedral angles in a simplex, has been reduced to that of discussing the sum of the distances between the points of a global $(n+1)$-tuple in S_{n-1}.

3. The spherical excess of a spherical simplex. We have to deal with the surface content A of a spherical simplex K in S_{n-1}. We can consider K as the common part of n hemispheres $K_1, \cdots, K_n$, whose boundaries (each an S_{n-2}) we denote by $K_1', \cdots, K_n'$. If we let α_{ij} denote the angle between K_i' and K_j', and A_{ij} denote the content of the spherical lune which is the common part of K_i and K_j, then $\alpha_{ij}=2\pi A_{ij}/C_n$, where $C_n=2\pi^{n/2}/\Gamma(n/2)$, the surface content of the S_{n-1}.

Now taking the origin at the center of the S_{n-1}, the coordinates of the points in K_i satisfy a linear homogeneous inequality $f_i(x)\geqq 0$, and K_i' is given by $f_i(x)=0$. The n forms $f_1, \cdots f_n$ divide the entire sphere into 2^n regions, each region being characterized by the signs of the n functions in it. We shall designate a typical region $R_{i_1,\ldots,i_m}$, meaning that $f_{i_1}, \cdots, f_{i_m}$ are positive in the interior of this region and the other functions negative. By $R^*_{i_1,\ldots,i_m}$ we mean the region diametral to $R_{i_1,\ldots,i_m}$, that is, the region in which $f_{i_1}, \cdots, f_{i_m}$ are negative and the others positive.

With this notation,

$$A_{ij} = A + \sum R_{ij\alpha_i\cdots\alpha_m}, \tag{1}$$

where $R_{ij\alpha_1\cdots\alpha_m}$ ranges over all the regions in which f_i and f_j are positive. Now suppose a region R makes p forms positive and $n-p$ negative. Then either R or R^* will occur in $\frac{1}{2}p(p-1)+\frac{1}{2}(n-p)(n-p-1)$ of the equations (1).

Now

$$\frac{p(p-1)}{2}+\frac{(n-p)(n-p-1)}{2}=p^2-np+\frac{n^2-n}{2}$$

and we must consider the maximum and minimum values of this function for integral values of p less than n. Computation shows that the maximum equals $\frac{1}{2}(n-1)(n-2)$ and is taken on for $p=n-1$, and the minimum is attained for $p=[n/2]$. When n is odd, the minimum is $(n-1)^2/4$ and for n even the minimum is $n(n-2)/4$.

Thus adding the $\frac{1}{2}n(n-1)$ equations (1) we will have

$$\sum A_{ij}\leqq\frac{n(n-1)}{2}A+\frac{(n-1)(n-2)}{2}\sum R_\beta,$$

where R_β takes on the value of the surface content of one out of each diametral pair of regions. Thus we have

$$\sum A_{ij}\leqq(n-1)A+\frac{(n-1)(n-2)}{2}\frac{C_n}{2}.$$

Multiplying by $2\pi/C_n$ gives

$$\sum\alpha_{ij}\leqq\frac{(n-1)(n-2)}{2}\pi+\frac{2(n-1)\pi}{C_n}A.$$

Similarly, for n odd,

$$\sum A_{ij}\geqq\frac{n(n-1)}{2}A+\frac{(n-1)^2}{4}\sum R_\beta,$$

and for n even,

$$\sum A_{ij}\geqq\frac{n(n-1)}{2}A+\frac{n(n-2)}{4}\sum R_\beta,$$

giving

$$\sum\alpha_{ij}\geqq\frac{(n-1)^2}{4}\pi+\frac{n^2-1}{2C_n}A,\ n\text{ odd},$$

$$\sum\alpha_{ij}\geqq\frac{n(n-2)}{4}\pi+\frac{n^2}{2C_n}A,\ n\text{ even}.$$

These inequalities constitute the extension of the formula connecting area and spherical excess. It is clear, further, that these bounds on $\sum A_{ij}$, and hence

on $\sum\alpha_{ij}$, are the best possible since any pair $R_{i_1},\cdots,_{i_m}$ and $R^*_{i_1},\cdots,_{i_m}$ can be made to assume a fraction of the spherical content arbitrarily close to 1 by properly choosing the simplex K.

4. Lower bound on the distance sum for global sets. We can now use the results of the last section to find a lower bound on the sum of the distances between points of a set which is global, *i.e.*, not contained in a hemisphere. We first confine our attention to global $(n-1)$-tuples in S_{n-1}.

Let $p_1, \cdots, p_{n+1}$ be a global set in S_{n-1}, let $R=\sum p_ip_j$, and let $H_1, \cdots, H_{n+1}$ be the hemispheres with centers at $p_1, \cdots, p_{n+1}$, with $H_1', \cdots, H_{n+1}'$ their bounding S_{n-2}. Then $p_ip_j=\pi-\alpha_{ij}$, where α_{ij} is the angle between H_i' and H_j' as before.

Letting R_i be the sum of the distances between the points $p_1, \cdots, p_{i-1}$, $p_{i+1}, \cdots, p_{n+1}$, we will have $R=(R_1+\cdots+R_{n+1})/(n-1)$. The common part of the hemispheres $H_1, \cdots, H_{i-1}, H_{i+1}, \cdots, H_{n+1}$ is a simplex K_i, whose surface content we may designate A_i and the sum of whose angles we may call $\sum_i$. Then

$$R_i=\frac{n(n-1)}{2}\pi-\sum_i, \quad \text{and} \quad \sum_i \leqq \frac{(n-1)(n-2)}{2}\pi+\frac{2(n-1)\pi}{C_n}A_i\cdot$$

Thus

$$R_i \geqq (n-1)\pi-\frac{2(n-1)\pi}{C_n}A_i$$

and

$$R \geqq (n+1)\pi-\frac{2\pi}{C_n}(A_1+\cdots+A_{n+1}).$$

Now, since the points p_i are global, the simplexes K_i and K_j have no points in common, since any such point would be the center of a hemisphere containing $p_1, \cdots, p_{n+1}$. Furthermore, diametral to K_i is a simplex K_i^* which does not intersect any K_j either, since the points of K_i^* are outside H_e and the points of K_j are inside H_e, $e\neq i, j$. Thus $K_1, \cdots, K_{n+1}, K_1^*, \cdots, K_{n+1}^*$ are mutually exclusive sets and $A_1+\cdots+A_{n+i}\leqq C_n/2$. Consequently $R\geqq n\pi$.

To find a bound on the distance sum for a global set of k points in S_{n-1} with $k>n+1$ we simply observe that we can imbed the S_{n-1} in an S_{k-2} and move the points slightly to make them a global subset of the S_{k-2}.

Thus we have the result: If $p_1, \cdots, p_k$ form a global set in S_{n-1}, $\sum p_ip_j\geqq(k-1)\pi$. That this bound is the best can be seen by taking a point p_1 and $k-1$ points clustered around p_1', its diametral point.

5. Upper bounds for the distance sum. To find an upper bound on the sum of the distances between points $p_1, \cdots, p_k$ of S_{n-1} we use methods very similar to those of Section 3. We take hemispheres $H_1, \cdots, H_k$ with centers at

$p_1, \cdots, p_k$ and let $H_1', \cdots H_k'$ be the bounding S_{n-2}. If A_{ij} is the content of the common part of H_i and H_j, α_{ij} the angle between H_i' and H_j', then as before, $p_ip_j=\pi-\alpha_{ij}$ and $\alpha_{ij}=2\pi A_{ij}/C_n$. Still following the preceding pattern, we associate a linear homogeneous function f_i with each H_i' and classify points of the sphere according to the signs of the functions f_i. Certain combinations of signs may fail to appear but there will be at most 2^k regions and they will come in pairs of diametral regions $R_{i_1,\cdots,i_m}$ and $R'_{i_1,\cdots,i_m}$.

Now $A_{ij}=\sum R_{ij\alpha_1,\cdots,\alpha_m}$ where $R_{ij\alpha_1,\cdots\alpha_m}$ ranges over all regions in which f_i and f_j are positive. If R is a region in which p of the functions are positive, $k-p$ then will be positive in R' and either R or R' will appear in

$$\frac{p(p-1)}{2}+\frac{(k-p)(k-p-1)}{2}=p^2-kp-\frac{k^2-k}{2}$$

equations for A_{ij}. The minimum of this function is attained for $p=[k/2]$ and is $(k-1)^2/4$ or $k(k-2)/4$ according as k is odd or even.

Thus

$$\sum A_{ij} \geqq \begin{cases} \dfrac{(k-1)^2}{4}\cdot\dfrac{C_n}{2}, & k \text{ odd}, \\ \dfrac{k(k-2)}{4}\cdot\dfrac{C_n}{2} & k \text{ even}, \end{cases}$$

and

$$\sum \alpha_{ij} \geqq \frac{(k-1)^2}{4}\pi \quad \text{for } k \text{ odd}$$

$$\geqq \frac{k(k-2)}{4}\pi \quad \text{for } k \text{ even}.$$

Letting $R=\sum p_ip_j$, $R=k(k-1)\pi/2-\sum\alpha_{ij}$, and thus when k is odd, $R\leqq(k^2-1)\pi/4$, while when k is even, $R\leqq k^2\pi/4$. This can be summarized by saying $R\leqq[k^2/4]\pi$. It will be noted that no use has been made of the dimension of the sphere and hence the maximum can be attained on any sphere by clustering the k points around two diametral points in groups as nearly equal as possible.

6. Summary. In conclusion we might summarize the results obtained, apply them to the angle sums in a simplex in E_n, and point out in what respect the results are incomplete.

First, if A is the surface content of a simplex in the unit sphere S_{n-1}, and $\sum$ the sum of its dihedral angles, we have

$$\text{I:} \qquad \frac{(n-1)(n-2)}{2}\pi+\frac{2(n-1)\pi}{C_n}A \geqq \sum \geqq \left[\frac{(n-1)^2}{4}\right]\pi+\left[\frac{n^2}{2}\right]\frac{A}{C_n}.$$

This expresses equally well the relation between a multihedral angle and its surrounding dihedral angles, and hence can be used in conjunction with the formulas below to obtain bounds on the sum of the multihedral angles in a simplex.

Second, if $p_1, \cdots, p_k$ is any set of points on the S_{n-1} and $R = \sum p_i p_j$, then

II: $$R \leqq \left[\frac{k^2}{4}\right]\pi.$$

If, in addition, the points $p_1, \cdots, p_k$ form a global set, then

III: $$R \geqq (k-1)\pi.$$

We can apply these formulas now to give bounds on S, the sum of the dihedral angles in a simplex in E_n.

If n is odd, we have

IV: $$\frac{n(n-1)}{2}\pi \geqq S \geqq \frac{n^2-1}{4}\pi$$

and if n is even,

IVa: $$\frac{n(n-1)}{2}\pi \geqq S \geqq \frac{n^2}{4}\pi.$$

Since each face of a simplex in E_n is itself a simplex in a lower dimensional space, we can find bounds on the sum of the angles of any dimension in a simplex in E_n. However, it is by no means clear that bounds obtained using the foregoing results would be the best possible. This question remains open for investigation.*

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* After this paper was written my attention was called to Coxeter's review of [4] in *Mathematical Reviews*, January, 1954, p. 55. Formulas IV and IVa are a special case of Höhn's inequalities with $k=2$. I have not seen Höhn's thesis but apparently the methods are quite different. Formulas I, II, and III are not connected with Höhn's work, so far as I can tell.

ON THE ACCURACY OF THE ADJOINT METHOD OF DIFFERENTIAL CORRECTIONS

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1. Introduction. Many problems arising in the applied sciences lead to systems of ordinary differential equations, and it is often desirable to determine not only solutions of such systems but also the effect of small perturbations upon these solutions. The calculation of such "differential effects" can be accomplished in a number of ways. There is the "direct" method in which the desired effects are obtained by the comparison of neighboring solutions. Among the "indirect" ways of predicting small effects the "method of adjoints" is much in use, especially in ballistics and related fields.

Now the system of adjoint differential equations involves the computation of certain partial derivatives, which, especially if empirical functions occur, are not always obtainable with great accuracy. The problem discussed here, seemingly not treated anywhere else, is concerned with just what degree of accuracy is required of the partial derivatives in order to guarantee prescribed exactness of the predicted differential effects.

While the cases taken up here are not of the greatest generality they are nevertheless thought to be sufficiently representative to give useful answers.

2. The fundamental formula. Let us consider the system of ordinary differential equations

$$\dot{z} = f(t, z) \qquad z(0) = z_0 \tag{2.1}$$

where

$$z = \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} f_1(t, z_1, \cdots, z_n) \\ \cdot \\ \cdot \\ \cdot \\ f_n(t, z_1, \cdots, z_n) \end{bmatrix}$$

represent column matrices of n elements.

Let, similarly,

$$\dot{v} = h(t, v) \qquad v(0) = v_0 \tag{2.2}$$

be a "neighboring" system in the sense that h differs little from f, and v_0 little from z_0. Assuming that unique solutions $z(t) = z(t, z_0)$, $v(t) = v(t, v_0)$ exist and are known, and putting

$$\Delta z(t) = v(t) - z(t)$$
$$\Delta f(t) = h(t, z(t)) - f(t, z(t)),$$

it follows that the $\Delta z(t)$ satisfy the system

$$(\Delta z)^{\cdot} = a\Delta z + \Delta f(t) + b\Delta z + O[(\Delta z)^2], \tag{2.3}$$

with

$$a = (a_{ij}), \qquad a_{ij} = \partial f_i(t, z(t))/\partial z_j$$
$$b = (b_{ij}), \qquad b_{ij} = \partial \Delta f_i(t)/\partial z_j.$$

Accordingly, the variational equations of (2.1) and (2.2) are

$$\dot{\eta} = B\eta + \Delta f, \qquad B = a + b, \qquad \eta(0) = \Delta z(0) \tag{2.4}$$

where

$$\eta = \begin{bmatrix} \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_n \end{bmatrix},$$

and η_i is the variation of z_i, $i = 1, 2, \cdots, n$.

Now it can be shown that if $z(t)$ satisfies (2.1) and $v(t) = z(t) + \Delta z(t)$ satisfies (2.2), then $\Delta z(t)$ is approximated by the solution of (2.4). In fact [1],

$$\Delta z(t) = \eta(t) + O(M^2),$$

where

$$M = \max [N_1, M(f_1), \cdots, M(f_n)],$$

with

$$N_1 = \max \left[\|\Delta z(0)\|, \max_t \|\Delta f(t)\| \right], \qquad \|\Delta z\| = \left[\sum_i (\Delta z_i)^2 \right]^{1/2}$$

and

$$M(f_k) = \max \left[\sum_j (\partial f_k(t, z)/\partial z_j)^2 \right]^{1/2}, \qquad k = 1, 2, \cdots, n.$$

The system adjoint to (2.4) is

$$\dot{\lambda} = -B'\lambda, \tag{2.5}$$

the prime denoting transposition of the matrix.

Since, by (2.4) and (2.5),

$$(\lambda'\eta)^{\cdot} = \lambda'\Delta f,$$

it follows that

$$\lambda'(T)\eta(T) = \lambda'(0)\eta(0) + \int_0^T \lambda'(\tau)\Delta f(\tau)d\tau, \tag{2.6}$$

for any $t = T$.

This is the fundamental formula; it permits the determination of the effect of small perturbations $\eta(0)$ upon quantities q whose differential changes dq at the

value T of the independent variable are expressible in the form $dq=\lambda'(T)\eta(T)$. There are many quantities capable of such representation. In fact, the following general relationship holds: Let u, w be any two variables usable as independent variables along the undisturbed solution near a point P, and let ξ be any disturbance. Let further the differential effect of ξ on any quantity q at equal values u_0 of u be denoted by $dq(\xi \mid u=u_0)$. Then the differential effect of ξ on q at equal values u_0 of u, and of equal values w_0 of w are related by

$$dq(\xi \mid w = w_0) = dq(\xi \mid u = u_0) - (dq/dw)_P dw(\xi \mid u = u_0). \tag{2.7}$$

A detailed proof of this theorem is given in [2]. This proof may be outlined as follows: Let P be the point on the undisturbed solution at which u, w, q assume, respectively, the values u_0, w_0, q_0. For sufficiently small M there exists on the disturbed solution a point Q at which $u=u_0$, and a point R at which $w=w_0$. Then at point Q

$$\begin{aligned} u_Q &= u_0 \\ w_Q &= w_0 + \Delta w(\xi \mid u = u_0) \\ q_Q &= q_0 + \Delta q(\xi \mid u = u_0). \end{aligned}$$

Similarly, at R:

$$\begin{aligned} u_R &= u_0 + \Delta u(\xi \mid w = w_0) \\ w_R &= w_0 \\ q_R &= q_0 + \Delta q(\xi \mid w = w_0). \end{aligned}$$

On the arc QR then the quantity q changes by

$$q_R - q_Q = \Delta q(\xi \mid w = w_0) - \Delta q(\xi \mid u = u_0),$$

the quantity w by

$$w_R - w_Q = -\Delta w(\xi \mid u = u_0).$$

By the mean value theorem there exists a value w^* between w_R and w_Q so that

$$(q_R - q_Q)/(w_R - w_Q) = (dq/dw)_{w^*}. \tag{2.8}$$

On the other hand,

$$(dq/dw)_{w^*} = (dq/dw)_P + O(M).$$

With the proper substitutions into equation (2.8), and using the further facts that

$$\begin{aligned} \Delta q(\xi \mid u) &= O(M) \\ \Delta q(\xi \mid u = u_0) &= dq(\xi \mid u = u_0) + O(M^2), \end{aligned} \tag{2.8a}$$

there is now readily obtained the equation (2.7), as claimed.

Applying (2.7) now to a set of quantities $q=x$, $w=y$, $u=t$ there results

$$dx(\xi \mid y) = dx(\xi \mid t) - (dx/dy)_P dy(\xi \mid t).$$

For the set $q=t$, $w=y$, $u=t$,

$$dt(\xi \mid y) = -(dt/dy)_P dy(\xi \mid t).$$

For $y=0$, $t=T$ the preceding equations become

$$dx(\xi \mid 0) \equiv dR = \eta_1(T) - [\dot{x}(T)/\dot{y}(T)]\eta_2(T) \tag{2.9}$$

$$dt(\xi \mid 0) \equiv dT = -[1/\dot{y}(T)]\eta_2(T). \tag{2.10}$$

It follows then from (2.9) and (2.10) that in general

$$dq = \lambda'(0)\eta(0),$$

where the integration of the adjoint equations (2.5) resulting in a desired effect dR of the variable x at $t=T$ must consequently be carried out starting with the terminal conditions

$$\lambda'(T) = [1, m, 0, \cdots, 0], \tag{2.11}$$

that of the equations (2.5) leading to a desired value dT of the variable t at $t=T$ with the terminal conditions

$$\lambda'(T) = [0, n, 0, \cdots, 0], \tag{2.12}$$

with $m = -[\dot{x}(T)/\dot{y}(T)]$, $n = -[1/\dot{y}(T)]$.

By (2.8) we have

$$\Delta q = \lambda'(0)\eta(0) + O(M^2).$$

Now it is clear that in computing the matrix B certain small errors ϵ are unavoidable, so that the system (2.5) will be replaced actually by one of the form

$$\dot{\Lambda} = -A' \odot \Lambda, \qquad \Lambda(T) = \lambda(T), \tag{2.13}$$

with the symbol $\odot$ denoting pseudo-multiplication.

This, in turn, will lead to a quantity

$$Dq = \Lambda'(0) \odot \eta(0),$$

and thence to a quantity

$$S(\eta) = Dq(\eta) - \Delta q(\eta) - \Delta q(\eta),$$

whose magnitude provides a measure not only of the second order terms but also of the effect of the errors ϵ upon the predicted value dq. Either S or

$$\sigma \equiv [dS/d(\Delta q)]_{\eta=0} = \sum_i [\partial S/\partial \eta_i(0)][\partial \eta_i(0)/\partial \Delta q]_{\eta=0}$$

$$= \sum_i [\Lambda_i(0) - \lambda_i(0)]\lambda_i^{-1}(0) \equiv \sum_i \sigma_i$$

may be used for such a measure, the latter being more useful for comparative purposes. It is clear that σ is the slope at the origin $\eta=0$ of the graph of S versus dq.

3. The case $n=1$. An interesting general result may be obtained easily in the case $n=1$. Then

$$\dot{Z} = F(Z, t), \qquad \dot{\Lambda} = -F_z(Z(t),t) \odot \Lambda.$$

Since $F_z \odot \Lambda = F_z\Lambda + \rho_1$, we have

$$\dot{\Lambda} = -F_z\Lambda - \rho_1,$$

with ρ_1 denoting a small function of t. Thus

$$\Lambda(t) = \exp\left(-\int_0^t F_z d\tau\right)\left[C - \int_0^t \rho_1(\tau) \exp\left(\int_0^t F_z d\tau'\right) d\tau\right].$$

With the terminal condition $\Lambda(T)=1$,

$$\Lambda(t) = \exp\left(-\int F_z d\tau\right)\left[\exp\left(\int_0^T F_z d\tau\right) + \int_0^T \rho_1 \exp\left(\int F_z d\tau'\right) d\tau\right].$$

For $t=0$, then,

$$\Lambda(0) = \exp\left(\int_0^T F_z d\tau\right) + \int_0^T \rho_1 \exp\left(\int F_z d\tau'\right) d\tau. \tag{3.1}$$

Consequently,

$$\Lambda(0) - \lambda(0) = \exp\left(\int_0^T F_z\right) - \exp\left(\int_0^T f_z\right) + \int_0^T \rho_1 \exp\left(\int F_z\right).$$

Let us define now

$$\epsilon(t) = F_z(Z) - f_z(z).$$

Then obviously

$$\Lambda(0) - \lambda(0) \approx \exp\left(\int_0^T f_z d\tau\right)[\exp I(T) - 1] + \int_0^T \rho_1 \exp\left(\int F_z\right),$$

where

$$I(T) = \exp \int_0^T \epsilon(\tau) d\tau. \tag{3.2}$$

Since

$$\begin{aligned} S &= \Lambda'(0) \odot \eta(0) - \lambda(0)\eta(0) \\ &= \rho_2(\eta) + [\Lambda(0) - \lambda(0)]\eta(0), \end{aligned}$$

it follows that

$$\sigma \approx (\exp(I(T)) - 1) + \exp\left(-\int_0^T f_z\right)\left[(\rho_2^*(\eta) + \int_0^T \rho_1 \exp\left(\int f_z\right)\right] \tag{3.3}$$

where ρ_1, ρ_2, and ρ_2^* denote small functions.

We are now especially interested in the effect of ϵ upon σ; putting ρ_1 and ρ_2 identically zero we find

$$\sigma \approx \exp(I(T)) - 1 \approx T\epsilon(t^*), \tag{3.4}$$

if we assume $|T\epsilon(t)| \ll 1$.

The slope σ is thus seen to be approximately proportional to the error ϵ in the partial derivative, as well as the total length T of integration, a result which seems reasonable. If, for example, $T=20$, $f_z=0.1$, $\epsilon \approx 0.001$, then $\sigma \approx 0.02$; errors in f_z of about 1% lead to values in the slope σ of about 2%. On the other hand, in order to expect the slope for a solution integrated to $t=200$ to be about 1%, the partial derivative must be correct to about 0.00005.

4. The analytical example. Let us next consider a special case involving a solution z having four components. It is well known that the plane motion of the center of gravity of a body subject to gravity and resistance may be described in a system x, y of rectangular coordinates by a set of equations of the form (2.1) with

$$z = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}, \quad f = \begin{bmatrix} \dot{x} \\ \dot{y} \\ -k\dot{x} \\ -k\dot{y} - g \end{bmatrix}, \quad z(0) = \begin{bmatrix} x_0 \\ y_0 \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix}. \tag{4.1}$$

Here g is the gravitational acceleration, and $k(y, \dot{x}, \dot{y}) = G(\dot{x}, \dot{y})H(y)C^{-1}$, where $H(y)$ is the air density ratio, and G, C are drag function and ballistic coefficient, respectively, functions that are usually determined by experiments. However, in order to obtain closed expressions for the solutions we shall limit ourselves here to the case where k is so slowly changing that it may be considered a positive constant. Then (4.1) is solved by

$$x(t, x_0, \dot{x}_0) = x_0 + \dot{x}_0 k^{-1}(1 - e^{-kt}) \tag{4.2}$$

$$y(t, y_0, \dot{y}_0) = y_0 + (y_0 + \gamma)k^{-1}(1 - e^{-kt}) - \gamma t \tag{4.3}$$

with $\gamma = gk^{-1}$. These expressions obviously tend to the well-known trajectory solutions in vacuum as k approaches zero. Solving (4.2) for $t=t(x, x_0, \dot{x}_0)$, and substituting into (4.3) leads to

$$y(x, x_0, y_0, \dot{x}_0, \dot{y}_0) = y_0 + (\dot{y}_0 + \gamma)(x - x_0)\dot{x}_0^{-1} - \gamma k^{-1} \log\{\dot{x}_0[\dot{x}_0 - k(x - x_0)]^{-1}\}. \tag{4.4}$$

The range R is then defined by means of

$$y(R, 0, 0, \dot{x}_0, \dot{y}_0) = 0, \tag{4.5}$$

and the time of flight T by means of

$$x(T, 0, \dot{x}_0) = R. \tag{4.6}$$

In doing this we associate the "normal" trajectory with

$$z(0) = [0, 0, \dot{x}_0, \dot{y}_0].$$

We note, from (4.2) and (4.3), that

$$\begin{aligned} R &= \dot{x}_0 k^{-1}(1 - e^{-kT}) \\ \dot{x}(T) &= \dot{x}_0 e^{-kT} \\ \dot{y}(T) &= \dot{x}_0 \dot{y}_0 - kR(\dot{y}_0 + \gamma)\dot{x}_0^{-1}. \end{aligned} \tag{4.7}$$

Let us next consider a "neighboring" set of solutions, defined as solutions $w(t)$ of (4.1) associated with $w(0) = z(0) + \eta(0)$, where the components $\eta_i(0)$ of $\eta(0)$ are small. Denoting by dR, dT the differential changes in range and time of flight, we find, by (4.1),

$$dR = - y_{\dot{x}}^{-1}[y_{x_0}\eta_1(0) + \cdots + y_{\dot{y}_0}\eta_4(0)] \equiv \alpha'\eta(0),$$

with

$$\alpha' = [\alpha_1, \alpha_2, \alpha_3, \alpha_4],$$

and the partial derivatives $y_x = \partial y/\partial x$, *etc.*, to be evaluated at R, 0, O, $\dot{x}_0$, $\dot{y}_0$. A short calculation shows that

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= - \dot{x}_0(\dot{x}_0 - ER)[\dot{x}_0\dot{y}_0 - ER(\dot{y}_0 + \gamma)]^{-1} = - \dot{x}(T)/\dot{y}(T) \\ \alpha_3 &= R\dot{x}_0^{-1} \\ \alpha_4 &= - Rx(T)[\dot{x}_0\dot{y}(T)]^{-1}. \end{aligned} \tag{4.8}$$

For the change dT, by (4.3)

$$\begin{aligned} dT &= y_t^{-1}[y_{y_0}\eta_2(0) + y_{\dot{y}_0}\eta_4(0)] \equiv \beta'\eta(0), \\ \beta' &= [0, \beta_2, 0, \beta_4], \end{aligned}$$

and the partial derivatives y_{y_0}, $y_{\dot{y}_0}$ to be computed at T, 0, $\dot{y}_0$. It is seen that

$$\begin{aligned} \beta_2 &= - [\dot{x}_0\dot{y}_0 - ER(\dot{y}_0 + \gamma)]\dot{x}_0^{-1} = - 1/\dot{y}(T) \\ \beta_4 &= - R[\dot{x}_0\dot{y}(T)]^{-1}. \end{aligned} \tag{4.9}$$

5. The solution of the adjoint system. Let us proceed next to the method of adjoints, as it applies to the system (4.1). It was already pointed out in section 2 that in practice (2.5) is replaced by a system of the form (2.13); in our case we wish to put

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -K & 0 \\ 0 & 0 & 0 & -K \end{bmatrix} \tag{5.1}$$

with $K(t) = k - \epsilon(t)$, and $\epsilon(t)$ assumed to be small in the sense that $\max_t |\epsilon(t)| \ll k$. The associated adjoint system is then, written out in detail,

$$\begin{aligned} \dot{\Lambda}_1 &= 0 \\ \dot{\Lambda}_2 &= 0 \\ \dot{\Lambda}_3 &= -\Lambda_1 + K\Lambda_3 \\ \dot{\Lambda}_4 &= \Lambda_2 + K\Lambda_4. \end{aligned} \tag{5.2}$$

The pertinent terminal conditions are of the form

$$\Lambda'(T) = [\lambda_{q1}, \lambda_{q2}, 0, 0], \tag{5.3}$$

with $\Lambda(T) = \lambda(T)$ as exhibited in (2.11) and (2.12), depending on whether $q = R$ or $q = T$.

The solutions of (5.2), (5.3) are

$$\begin{aligned} \Lambda_1 &= \lambda_{q1} \\ \Lambda_2 &= \lambda_{q2} \\ \Lambda_i(t) &= e^{I(t)}\left[C_i - \lambda_{q,i-2}\int_0^t e^{-I(\tau)}d\tau\right], \qquad i = 3, 4, \end{aligned} \tag{5.4}$$

with

$$I(\tau) = k\tau - \int_0^\tau \epsilon(\tau')d\tau'$$

and the C_i denoting constants of integration. Use of (5.3) shows that

$$C_i = \lambda_{q,i-2}\int_0^T e^{-I(\tau)}d\tau,$$

whence

$$\Lambda_i(t) = e^{I(t)}\lambda_{q,i-2}\int_t^T e^{-I(\tau)}d\tau, \qquad i = 3, 4.$$

We need $\Lambda_i(0)$; obviously

$$\Lambda_i(0) = \lambda_{q,i-2}\int_0^T e^{-I(\tau)}d\tau, \qquad i = 3, 4. \tag{5.5}$$

Let us now consider the case $\epsilon(t) \equiv 0$, assuming, then, that the partial derivatives a_{ij} could be calculated exactly. In this case (5.4) becomes

$$\Lambda_i(0) = \lambda_i(0) = \lambda_{q,i-2}k^{-1}(1 - e^{-kT}), \qquad i = 3, 4. \tag{5.6}$$

i. The case $q=R$. Here $\lambda_{q1}=1$, $\lambda_{q2}=m$ so that, by (5.4), (5.5), and (4.7)

$$\lambda'(0) = [1, m, R\dot{x}_0^{-1}, R\dot{x}_0^{-1}m].$$

Thus $\lambda(0)=\alpha$, as defined in (4.8), and the accuracy of $dR=\lambda'(0)\eta(0)$ is illustrated.

ii. The case $q=T$. Here $\lambda_{q1}=0$, $\lambda_{q2}=n$, so that now

$$\lambda'(0) = [0, n, 0, R\dot{x}_0^{-1}n] = \beta',$$

as seen from (4.9). Thus also $dT=\lambda'(0)\eta(0)$. Since now $\Lambda_i(0)-\lambda_i(0)=0$ for all i, it follows that

$$\sigma = \sum_i [\Lambda_i(0) - \lambda_i(0)]\lambda_i^{-1}(0) = 0.$$

In general, however, $\epsilon(t)\not\equiv 0$, whence $\sigma\neq 0$. Let us now assume that a constant value t^* in $0\leqq t\leqq T$ exists such that $\epsilon(t)\approx\epsilon(t^*)\equiv\epsilon^*$. In that case clearly

$$\int_0^T e^{-I(\tau)}d\tau \approx (k - \epsilon^*)^{-1}[1 - e^{-(k-\epsilon^*)T}],$$

and

$$\Lambda_i(0)/\lambda_i(0) \approx [k/(k - \epsilon^*)][(1 - e^{-(k-\epsilon^*)T}/(1 - e^{-kT})].$$

Therefore,

$$\sigma_i \approx (\epsilon^*/k) + [(1 - e^{\epsilon^*T})/(e^{kT} - 1)].$$

Further if we suppose again that $\max_t |\epsilon(t)|\, T\ll 1$ then

$$\sigma_i \approx (\epsilon^*/k)[1 - kT/(e^{kT} - 1)],$$

provided

$$\max |\epsilon| \ll \min (k, T^{-1}). \tag{5.7}$$

It follows that

$$\sigma \approx 2^j(\epsilon^*/k)[1 - kT/(e^{kT} - 1)] \tag{5.8}$$

with $j=1$ for $q=R$, and $j=0$ for $q=T$.

The slope is thus again seen to be approximately proportional to both ϵ and T. For example, for $T=20$, $k=0.1$, $\epsilon\approx 0.001$ it is found that $\sigma\approx 0.007$; an error of 1% in the value of k leads to an error of about 0.7% in the slope.

References

1. E. J. McShane, J. L. Kelley, F. V. Reno, Exterior Ballistics, University of Denver Press, 1953, p. 477.

2. See *l.c.* 1, p. 433.

MATHEMATICAL NOTES

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Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

SOLUTION OF A PROBLEM OF E. M. WRIGHT ON CONVEX FUNCTIONS*

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With R denoting the real number field and f a function on R to R, consider the following two statements about f:

(A) $f(x+\delta)-f(x) \geqq f(y+\delta)-f(y)$ for all $\delta>0$ and $x>y$.

(B) $f(\frac{1}{2}x+\frac{1}{2}y) \leqq \frac{1}{2}f(x)+\frac{1}{2}f(y)$ for all x, y.

It is well known that (A) implies (B), and that for continuous f the two are equivalent. (See [2], for example.) In a recent note [3], Professor E. M. Wright showed that (A) is equivalent to certain other interesting inequalities, and raised a question as to the existence of a function f for which (B) is true but (A) is false. The purpose of this note is to describe such a function†. Needless to say, the argument given leans heavily on the Axiom of Choice.

THEOREM. *There is a function f on R to R such that for all x, $y \in R$ it is true that*

(i) $f(\lambda x+(1-\lambda)y) \leqq \lambda f(x)+(1-\lambda)f(y)$ *for all rational* λ *with* $0 \leqq \lambda \leqq 1$;

(ii) *if* $x \neq y$, *there is a* $\delta>0$ *such that* $f(x+\mu\delta)-f(x)<f(y+\mu\delta)-f(y)$ *for all rational* $\mu>0$.

Proof. Recall first that the line R and the plane R^2 are both of dimension $2^{\aleph_0}$ as vector spaces over the rational field.‡ (This follows, for example, from the lemma on p. 20 of [1].) Thus there is an (additive, rationally homogeneous) isomorphism τ of R onto R^2. For each $x \in R$ let $f(x)=|\tau(x)|^2$, where $|\ |$ is the Euclidean norm in the plane R^2. That (i) is true follows readily from the corresponding property of $|\ |^2$ and the fact that τ is an isomorphism. It remains to establish (ii).

Consider an arbitrary pair x and y of distinct points of R and let $x'=\tau(x)$, $y'=\tau(y)$. Let U be the non-empty open set of all $p \in R^2$ such that $(y', p)>(x', p)$, where $(\ ,\ )$ denotes the inner-product in R^2. Since for all z, $q \in R^2$ it is true that $d|z+tq|^2/dt=2(z, q)+2t|q|^2$, it then follows that $|x'+tp|^2-|x'|^2<|y'+tp|^2-|y'|^2$ for all $p \in U$ and $t>0$. Thus with $\delta=\tau^{-1}(p)$ and μ a positive rational number, it is true that $f(x+\mu\delta)-f(x)<f(y+\mu\delta)-f(y)$. It remains only to show that $\tau^{-1}U$ includes at least one positive number. Suppose not. Then τ^{-1} is an additive real function on R^2 which is bounded above on the non-empty open set

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† Since Wright works with functions on $[0, \infty[$ to R, his question is answered by the restriction to $[0, \infty[$ of the function described in the Theorem.

‡ The Axiom of Choice enters in the proof of this fact.

U, and hence is bounded above on a translate V of U which includes the origin. But then τ^{-1} is bounded on the set $V \cap (-V)$, which is a neighborhood of the origin. Since τ^{-1} is rationally homogeneous, it follows that τ^{-1} is continuous at the origin, and hence by additivity at each point of R^2. Since, however, τ^{-1} maps R^2 biuniquely onto R, it cannot be continuous, and the contradiction completes the proof.

References

1. Reinhold Baer, Linear Algebra and Projective Geometry, New York, 1952.
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934.
3. E. M. Wright, An inequality for convex functions, this MONTHLY, vol. 61, 1954, pp. 620–622.

NOTE ON CONVEX FUNCTIONS

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E. M. Wright in an interesting note [1] on a convex inequality states that it is unknown whether functions exist which satisfy condition (i) below and not condition (ii). It is not difficult to construct such a function, making use of a Hamel basis. (This makes use of the axiom of choice. See [2] or [3].) The conditions are as follows:

(i) $$f\left(\frac{a+b}{2}\right) \leqq \frac{f(a)+f(b)}{2} \qquad \text{for real } a \text{ and } b.$$

(ii) If $a \leqq b$ and $\delta > 0$ then $f(a+\delta) - f(a) \leqq f(b+\delta) - f(b)$.

Let H be a Hamel basis for the real numbers over the rationals. Suppose without loss of generality that 1 and π belong to H. Then each real number x has the unique representation $x = \sum_{h \in H} r_{x,h} \cdot h$, where the $r_{x,h}$ are rational numbers, only a finite number of which are not zero. Let

$$f(x) = \sum_{h \in H} r_{x,h}^2 \qquad \text{for each real } x.$$

It is easy to check that (i) holds. To see that (ii) does not hold, let $a=1$, $b=\pi$, and $\delta=1$. Then $a<b$, $\delta>0$, and $f(a+\delta)-f(a)=4-1>2-1=f(b+\delta)-f(b)$.

The function f may be modified so that (i) is still satisfied; and so that (ii) is satisfied for any preassigned set of values of $\delta>0$ of power less than the continuum, but not for all $\delta>0$.

References

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ON AN INEQUALITY CONCERNING AN INDEFINITE FORM

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1. Introduction. Let us consider the indefinite form

$$\phi(x) = (x_1^p - x_2^p - \cdots - x_n^p)^{1/p}, \qquad p \geqq 1, \tag{1}$$

for values of the x_1 in the region

$$R: \quad \text{a. } x_i \geqq 0, \qquad \text{b. } x_1 > (x_2^p + x_3^p + \cdots + x_n^p)^{1/p}. \tag{2}$$

Observe that $x+y$ belongs to R whenever x and y separately do. This follows from Minkowski's inequality.

The inequality we shall establish is the reverse triangle inequality,

$$\phi(x + y) \geqq \phi(x) + \phi(y). \tag{3}$$

for x and y both in R. The proof will depend upon a representation for $\phi(x)$ which we shall establish in the next section.

2. Representation for $\phi(x)$. Let us now demonstrate that

$$\phi(x) = \operatorname*{Min}_{S(z)} \sum_{i=1}^{n} x_i z_i, \tag{1}$$

where $S(z)$ is the region defined by the inequalities

$$\text{a. } z_1 \geqq 1, \qquad z_k \geqq 0, \qquad k = 2, \cdots, n.$$
$$\text{b. } (z_2^q + \cdots + z_n^q) \leqq z_1^q - 1, \qquad q = p/(p-1).$$

Proof. Using Holder's inequality on the terms $x_2z_2 + \cdots + x_nz_n$ we see that

$$x_2z_2 + \cdots + x_nz_n \geqq -\left(\sum_{k=2}^{n} x_k^p\right)^{1/p}(z_1^q - 1)^{1/q}, \tag{3}$$

and accordingly that the lower bound of $\sum_{i=1}^{n} x_iz_i$ is attained in $S(z)$. Hence

$$\phi(x) = \operatorname*{Min}_{z_1 \geqq 1}\left[x_1z_1 - \left(\sum_{k=2}^{n} x_k^p\right)^{1/p}(z_1^q - 1)^{1/q}\right]. \tag{4}$$

There is a unique minimum at the point where the derivative with respect to z_1 is set equal to zero. At this point we have

$$\frac{x_1}{\left(\sum_{k=2}^{n} x_k^p\right)^{1/p}} = \frac{z_1^{q-1}}{(z_1^q - 1)^{1-1/q}}. \tag{5}$$

Solving for z_1 and performing the required algebraic substitutions, we obtain the stated expression for $\phi(x)$.

3. Proof of the inequality. Using the foregoing lemma for x and y both in R, we have

$$(1)\qquad \phi(x+y) = \operatorname*{Min}_{S(z)} \sum_{i=1}^{n} (x_i + y_i)z_i = \operatorname*{Min}_{S(z)} \left(\sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i z_i \right)$$

$$\geqq \operatorname*{Min}_{S(z)} \left(\sum_{i=1}^{n} x_i z_i \right) + \operatorname*{Min}_{S(z)} \left(\sum_{i=1}^{n} y_i z_i \right)$$

$$\geqq \phi(x) + \phi(y),$$

which is the desired result.

The case $n=4$, $p=2$ seems of particular interest.

UNIFORM CIRCULAR MOTION IS SINGULAR

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What can be said of the motion of a particle P if

$$\begin{cases} P \text{ lies on a circle,} \\ |\text{acceleration of } P| = \text{constant} \neq 0? \end{cases}$$

The partial answer that comes at once to mind is the familiar one involving constant speed. We shall show, however, that *this answer is the singular solution to the problem* and that *the general solution is a type of oscillation along an arbitrary quadrant of the circle.*

Let C be the circle, r its radius, and a the constant numerical value of the acceleration. From the decomposition of the acceleration vector into its tangential and normal components we have

$$(1)\qquad (dv/dt)^2 + v^4/r^2 = a^2.$$

Regarded as an equation in dv/dt, (1) yields as locus of double-points precisely $v^2/r=a$. This locus satisfies (1), is not a particular case of the general solution (9) (below), and corresponds to motion with constant speed. We have thus—pending derivation of (9)—justified the title of this note.

To find the general solution of (1) we substitute vdv/ds for dv/dt and then put $w=v^2$, obtaining

$$(2)\qquad (dw/ds)^2 + 4w^2/r^2 = 4a^2.$$

The general solution of (2) gives

$$(3)\qquad v^2 = ar \cos (2s/r),$$

provided that s is measured from an appropriate point.* If we now introduce a central angle θ so that $s=r\theta$, (3) may be written as

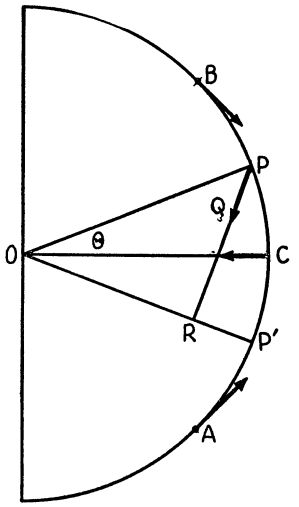
$$(4)\qquad v^2/r = a \cos 2\theta,$$

* The constant of integration here merely corresponds to an arbitrary shifting along C of the path of motion.

which shows that *the motion is confined to the quadrant*

$$-\pi/4 \leqq \theta \leqq \pi/4. \tag{5}$$

Before continuing with the solution, we exploit the fact that (4) suggests a simple method of constructing the acceleration vector geometrically (since v^2/r is precisely the normal component, and a is the length, of that vector). Placing the origin at the center of C and identifying the polar angle with θ, we



draw OP and its reflection (OP') in the polar axis, locate R on OP' (or its extension) so that $OR=PR$, and locate Q on PR (or its extension) in the direction of R from P so that $PQ=a$. Then $\overline{PQ}$ is the required acceleration vector for the particle P, for: the normal component of $\overline{PQ}$ is $a \cos 2\theta$ (as required by (4)), since $\angle OPQ=\angle POR=2\theta$ by our construction; $\angle OPQ \leqq \pi/2$ (as required of acceleration vectors in general) by (5); and the tangential component of $\overline{PQ}$ is directed toward the polar axis (as required by the implication of (4) that $|v|$ decreases as $|\theta|$ increases). (The acceleration vector is shown in the figure also at the ends A, B and the midpoint C of the path of motion.)

Returning to (4) and noting that $v=rd\theta/dt$, we obtain

$$dt = \sqrt{r/a}\,|\,d\theta\,|\,/\sqrt{1-2\sin^2\theta} \quad (t \text{ assumed always increasing}). \tag{6}$$

Next we put

$$\sqrt{2}\sin\theta = \sin\phi, \qquad \theta = \phi = 0 \text{ when } t = 0,\dagger \tag{7}'$$

with ϕ increasing monotonely as θ oscillates between $-\pi/4$ and $\pi/4$. Then from (6) we have‡

† The constant of integration which we anticipate simply determines when we shall take $t=0$.

‡ Standard notations for elliptic integrals and functions may be found in Madelung, Die mathematischen Hilfsmittel des Physikers, Dover, 1943, pp. 76–77.

$$t = \sqrt{r/(2a)} \int_0^{\phi} du/\sqrt{1 - \sin^2 u/2} \equiv \sqrt{r/(2a)}F(1/\sqrt{2}, \phi),$$

whence a period T of the oscillation is given by

$$T = 4\sqrt{r/(2a)}K(1/\sqrt{2})$$

and, by (7) (here we restore the arbitrary constants),

$$s - s_0 = r \arcsin \{(1/\sqrt{2}\ \mathrm{sn}\ [\sqrt{2a/r}(t - t_0)]\}. \tag{8}$$

Finally, differentiation of (8) gives

$$v = \sqrt{ar}\ \mathrm{cn}\ [\sqrt{2a/r}(t - t_0)]. \tag{9}$$

ON DIVIDING AN OBJECT EFFICIENTLY

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In what follows E denotes a plane circular disk; "curve" denotes either an arc (homeomorphic image of the closed unit interval) or a simple closed curve (homeomorphic image of the circle). Restricting ourselves to rectifiable curves, we consider some problems closely related to several classical results of the calculus of variations, especially to Dido's problem [1, pp. 3, 446, 465, 488, 528].

Question 1. Let P and Q be two points (not necessarily distinct) on the boundary of a plane circular disk E. What curve in E, joining P to Q and partitioning E into two portions of equal area, has least length?

Solution. If P and Q are diametrically opposite, then the solution is clearly the diameter through P and Q, since the shortest curve joining two points is a straight line and a diameter does in fact divide the disk equally.

Now assume that P and Q are not diametrically opposite one another. Then, by continuity arguments, there is a unique circle C through P and Q such that the area of the portion of the interior of this circle within E is precisely half the area of E. (If P coincides with Q, demand that C be tangent to the boundary of E.)

Now of all curves enclosing a fixed area, it is well known that the circle has minimum length. In particular if we apply this fact to the area contained within C we deduce that the portion of C within E is the curve of minimal length dividing E equally.

Question 2. What curve in E, partitioning E into two connected portions of equal area has least length?

Solution. Any arc in question must join two distinct points P and Q on the boundary of E. Any simple closed curve in question may be translated till it touches the boundary of E in at least one point P. If this arc or curve is to be of minimal length, then it must, by the answer to question 1, be either a circular arc joining P to Q, or a circle tangent at P. Finally, among these arcs and curves,

it is easy to see that the diameter has minimal length.

Question 3. Let R be a rectangle of height h and base $\pi h/2$. Let P be the midpoint of its base. What simple closed curve in E through P, partitioning R into two connected portions of equal area has least length?

Solution. The method which answers question 1 breaks down. The circle given by that method is tangent to the upper base of the rectangle and divides R into three connected portions. There is no simple closed curve of minimal length satisfying the demands of this question.

Further Questions. What are the answers to questions 1 and 2 if E is an ellipse? If a simple closed curve is given on the surface of the sphere, what is the surface of minimal area through this curve dividing the sphere equally (compare to the Plateau Problem)? What if E is to be divided into three equal portions?

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CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ROUND ROBIN MATHEMATICS*

J. E. FREUND, Virginia Polytechnic Institute

It is quite common that members of a mathematics department are approached by someone in the athletic department requesting help in the construction of a round robin schedule for intramural sports. The purpose of this note is to show a very simple way of constructing round robin schedules for any number of teams.

The requirements for a round robin schedule are that each team plays each other team once and that if there is an even number of teams, each team will play on every day of the "season." If there is an odd number of teams only one team has a "bye" on each day of the season and no team has more than one "bye."

If there are n teams and *n is an odd number* let us number these teams 0,1, 2, $\cdots$, $n-1$. Team i will play against team j on the kth day of the season ($k=0, 1, 2, \cdots, n-1$) if

$$i + j \equiv k \pmod{n}$$

* Work done in connection with Contract No. DA-36-034-ORD-1527RD, Office of Ordnance, U. S. Army.

with the exception that the team for which

$$2i \equiv k \pmod{n}$$

draws a bye. Using the theory of congruences it can easily be shown that this formula meets all of the conditions, *i.e.*, each team plays each other team once, one team gets a "bye" in each round, and each team gets only one "bye." Using this formula we can construct the following table for $n=13$ teams:

k							bye
0	1–12	2–11	3–10	4 – 9	5 – 8	6 – 7	0
1	0 – 1	2–12	3–11	4–10	5 – 9	6 – 8	7
2	0 – 2	3–12	4–11	5–10	6 – 9	7 – 8	1
3	0 – 3	1 – 2	4–12	5–11	6–10	7 – 9	8
4	0 – 4	1 – 3	5–12	6–11	7–10	8 – 9	2
5	0 – 5	1 – 4	2 – 3	6–12	7–11	8–10	9
6	0 – 6	1 – 5	2 – 4	7–12	8–11	9–10	3
7	0 – 7	1 – 6	2 – 5	3 – 4	8–12	9–11	10
8	0 – 8	1 – 7	2 – 6	3 – 5	9–12	10–11	4
9	0 – 9	1 – 8	2 – 7	3 – 6	4 – 5	10–12	11
10	0–10	1 – 9	2 – 8	3 – 7	4 – 6	11–12	5
11	0–11	1–10	2 – 9	3 – 8	4 – 7	5 – 6	12
12	0–12	1–11	2–10	3 – 9	4 – 8	5 – 7	6

If there is *an even number of teams*, say, $n+1$ where n is again an odd number, these teams will be labeled 0, 1, 2, $\cdots$, n. Team i will then play against team j on the kth day of the season ($i, j, k=0, 1, 2, \cdots, n-1$) if

$$i + j \equiv k \pmod{n}$$

with the exception that the team for which

$$2i \equiv k \pmod{n}$$

will play against team n on the kth day of the season. In other words, the team which draws the "bye" in the schedule for $n-1$ teams is matched against team n.

In order to construct a round robin schedule for 14 teams we simply take the

round robin schedule for the 13 teams 0, 1, 2, $\cdots$, 12 and match on each day the team having the bye against team 13.

In some sports it is desirable to designate in each game one team as the "home team." If there is an odd number of teams each team will play an even number of games and it will be the "home team" half of the time if we use the following scheme:

(i) if $i+j$ is *odd*, the home team is the team having the *smaller* number, (ordering the numbers from 0 to $n-1$ in the usual sense).

(ii) if $i+j$ is *even*, the home team is the team having the *larger* number.

For example, in the round robin schedule for 13 teams, team 1 is the home team when $k=0$, 3, 5, 7, 9, and 11 while team 8 is the home team when $k=1$, 4, 6, 8, 10, and 12. It can be shown that this produces the desired results. Of course, if there is an even number of teams, each team plays an odd number of games and no such arrangement is possible.

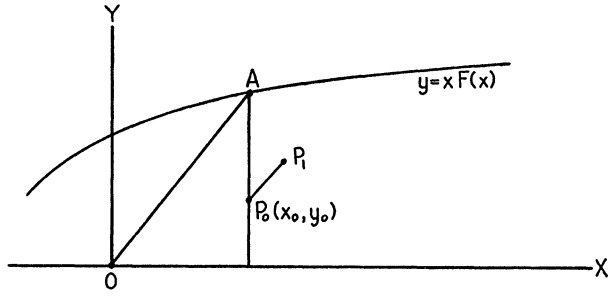
A GRAPHICAL INTEGRATION

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

The following is believed to be a new simple graphical method of integrating. Let the differential equation be given by

$$\frac{dy}{dx} = F(x) \qquad \text{(where } x = x_0,\ y = y_0\text{)}. \tag{1}$$

One first plots the curve $y=xF(x)$ (assumed to be continuous). From the point $P_0(x_0, y_0)$, a vertical line is drawn intersecting the curve $y=xF(x)$ in the point A. Then the tangent at P_0 to the integral curve of equation (1) must be parallel to OA. A small segment P_0P_1 of this tangent is drawn with a parallel ruler. From point P_1, the procedure is repeated. The proximity of the successive points P_0, P_1, *etc.* will determine the accuracy of the construction.



This method leads to a particularly simple construction for $y=\log x$.

Remarks: The above method can also be adapted to graphically solving the D.E., $dy/dx=F(x)/G(y)$, $(x=x_0, y=y_0)$. We let $dz/dx=F(x)$, $(x=x_0, z=z_0)$, and $dz/dy=G(y)$, $(y=y_0, z=z_0)$. Both of these latter equations are solved by the above

method yielding the graphs of $z-z_0=\int_{x_0}^{z}F(x)dx$, and $z-z_0=\int_{y_0}^{y}G(y)dy$. It is now a simple matter to obtain the plot of y vs. x. Another method would be to specialize the functions in the graphical solution of

$$\frac{dy}{dx}=\frac{\phi(x)-G(y)}{\psi(x)-F(y)},$$

by letting $G(y)=\psi(x)=0$. (See author's note, this MONTHLY, vol. 61, 1954, pp. 565–567).

ON A GRAPHICAL SOLUTION

M. E. LEVENSON, Brooklyn College

The present paper is concerned with a graphical solution of the equation

$$\frac{dy}{dx}+p(x)y=q(x), \qquad p(x)\not\equiv 0.$$

This construction is based on the following property of this equation. If the family of integral curves of the equation is cut by the line $x=x_1$, the tangents at the points of intersection are concurrent and meet at the point

$$\left(x_1+\frac{1}{p(x_1)},\ \frac{q(x_1)}{p(x_1)}\right).$$

This property of the differential equation recently appeared as a problem on the William Lowell Putnam Examination (March, 1954) and also appears as an exercise in Langer's excellent textbook *A First Course in Ordinary Differential Equations* (p. 67).

The construction is as follows: Draw the curves (see Figure).

$$C_1:\ y=\frac{q(x)}{p(x)}$$

and

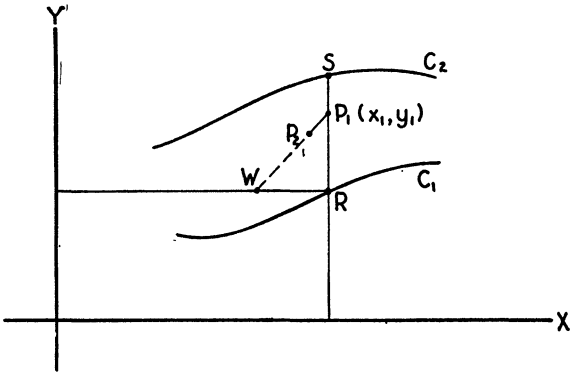
$$C_2:\ y=x+\frac{1}{p(x)}.$$

Consider a point $P_1(x_1, y_1)$. Draw the vertical line through this point meeting C_1 in R and C_2 in S. Draw the horizontal line through R. On the horizontal line, locate the point W whose abscissa is equal to the ordinate of the point S. The point W so constructed has the coordinates

$$\left(x_1+\frac{1}{p(x_1)},\ \frac{q(x_1)}{p(x_1)}\right)$$

and therefore the line joining W to $P_1(x_1, y_1)$ is the tangent line through the

point $P_1(x_1, y_1)$. Now taking a point P_2 on WP_1 close to P_1, we repeat the construction for P_2, *etc.*



It is obvious that the roles of x and y may be interchanged and that the construction can be modified rather simply to give the orthogonal trajectories. It should be mentioned that the graphical procedure given by M. S. Klamkin (this MONTHLY, vol. 61, 1954, p. 565–567) appears to be limited to the cases where his functions F and G have unique inverses.

THE FORMAL SOLUTION OF A DIFFERENTIAL EQUATION

R. K. RITT, University of Michigan

We consider the ordinary differential equation with constant coefficients

$$V_m\left(\frac{d}{dt}\right)x = F(t), \qquad F(t + 2\pi) = F(t) \tag{1}$$

where

$$V_m(p) = \sum_{r=0}^{m} a_{m-r}p^r = a_0\prod_{j=1}^{m}(p - b_j),\ b_i \neq b_j \text{ if } i \neq j,\ b_j \neq \text{an integer.} \tag{2}$$

If

$$[V_m(p)]^{-1} = \sum_{j=1}^{m} A_j(p - b_j)^{-1}, \tag{3}$$

then the formal application of the finite Fourier transform gives

$$\begin{cases} x(t) = \sum_{j=1}^{m} A_j u_j(t) \\ u_j(t) = (1 - \exp 2\pi b_j)^{-1}e^{b_j t} * F(t) \end{cases} \tag{4}$$

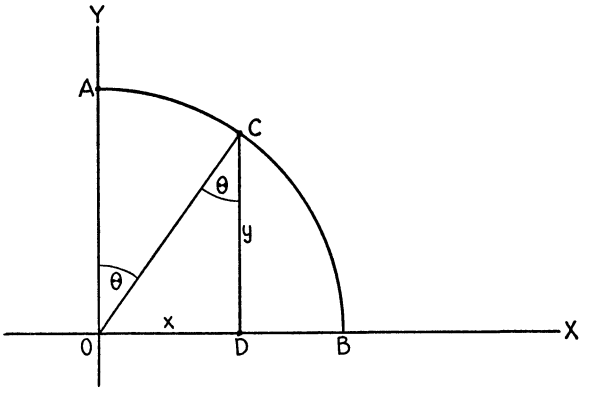
where $f(t) * g(t) = \int_0^t f(t-s)g(s)ds + \int_t^{2\pi} f(t+2\pi-s)g(s)ds$. In this note we prove that if $F(t)$ is piecewise continuous, (4) represents a function which has $(m-1)$ con-

ON THE DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

In the usual course in calculus it is the custom to introduce the trigonometric functions and their derivatives before considering the derivatives of the inverse trigonometric functions. In the so-called unified courses in calculus where some of the elementary concepts of integration are taken up early, there seems to be some advantage in introducing these derivatives in the reverse manner.

To illustrate the possibilities in this direction we show how to find the derivative of arcsin x. Referring to the figure, ACB represents one quarter of



the circumference of a circle with center at the origin and having unit radius. The area of $OACD$ given by

$$\int_0^x \sqrt{1-x^2}\,dx$$

is also equal to the sum of the areas of OCD and OAC. This sum is given by

$$\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x$$

from elementary formulas for the area of a triangle and sector of a circle. Hence

$$\int_0^x \sqrt{1-x^2}\,dx=\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x. \tag{1}$$

By differentiation it follows that

$$\sqrt{1-x^2}=\frac{d}{dx}\left(\frac{x}{2}\sqrt{1-x^2}\right)+\frac{1}{2}\frac{d}{dx}(\arcsin x)$$

which yields easily enough

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \tag{2}$$

or, if one likes,

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{3}$$

where u depends on x.

From this result it is easy to obtain the derivatives of the other inverse trigonometric functions. Thus

$$\frac{d}{dx}(\arccos x) = \frac{d}{dx}\left(\frac{\pi}{2} - \arcsin x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{d}{dx}\left(\arcsin \frac{x}{\sqrt{1-x^2}}\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arc\,cot} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \arctan x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arc\,csc} x) = \frac{d}{dx}\left(\arcsin \frac{1}{x}\right) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arc\,sec} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \operatorname{arc\,csc} x\right) = \frac{1}{x\sqrt{x^2-1}}.$$

We may now obtain the derivatives of the trigonometric functions. The general procedure is illustrated by determining $d(\sin x)/dx$. Let

$$y = \sin x$$

so that

$$x = \arcsin y.$$

Then it follows at once that

$$\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$$

or,

$$\frac{dy}{dx} = \sqrt{1-y^2} = \cos x. \tag{4}$$

Derivatives of the other trigonometric functions can be obtained in a similar way.

The writer believes that the mode of presentation of this paper is more easily

understood by the student than the conventional procedure for several reasons. First, there is no need for the investigation of $\lim_{\theta \to 0} (\sin \theta/\theta)$ which is involved in the conventional procedure. Second, there is no need to manipulate a difference quotient into a form suitable for limiting processes. Third, it is tied in with an interesting geometric interpretation. Incidentally the integration formula (1) is a useful by-product which normally would involve for its evaluation the techniques of integration by parts or trigonometric substitution.

It should be remarked that in obtaining all of the above results, angles were restricted to those lying in the first quadrant. It is not at all difficult to consider other angles, the procedure being entirely analogous to that used in the conventional approach.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1201. *Proposed by C. S. Ogilvy, Hamilton College*

What is the area of the maximum cross section of the unit cube?

E 1202. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let O be an arbitrary point on an arbitrary line λ passing through the centroid G of a tetrahedron $ABCD$. If λ cuts the planes BCD, CDA, DAB, ABC in A', B', C', D', show that

$$A'O/A'G + B'O/B'G + C'O/C'G + D'O/D'G = 4.$$

E 1203. *Proposed by S. I. Birnbaum, Polytechnic Institute of Brooklyn*

Prove that for given sets X, Y, T we have $X = Y = 0$ if and only if

$$T = (X \cap T') \cup (Y \cap T') \cup (X' \cap T \cap Y').$$

E 1204. *Proposed by A. J. Goldman, Princeton University*

Let C be the circle $x^2+y^2=1$, $f(x, y)$ a continuous real-valued function defined on C, and A an angle such that $0 < A \leqq \pi$. Show that there are two radii of C which form an angle A and have endpoints at which $f(x, y)$ has the same value.

E 1205. *Proposed by H. J. Cohen, City College of New York*

If $f''(x)$ is continuous on $[0, 1]$ and if $f(0)=f(1)=0$, show that

$$\int_0^1 |f''(x)/f(x)|\,dx > 4.$$

SOLUTIONS

Mersenne Numbers

E 1171 [1955, 442]. *Proposed by N. D. Brenner, University of Pennsylvania*

(1) Is a number of the form 2^p-1, where p is a prime, ever divisible by another number of the same form?

(2) Are two such numbers necessarily relatively prime?

Solution by W. E. Briggs, University of Colorado. Let h be the smallest positive integer such that $2^h\equiv 1 \pmod r$, r a prime. Then, if $2^t\equiv 1 \pmod r$, $h \mid t$, since otherwise $2^t=2^{kh+n}\equiv 2^n\equiv 1 \pmod r$, where $0<n<h$. Hence $2^p-1\equiv 2^q-1\equiv 0 \pmod r$, r a prime, implies that h divides both p and q.

Also solved by H. L. Alder, G. E. Bardwell, W. J. Blundon, I. A. Dodes, D. C. Duncan, A. D. Freedman, A. J. Goldman, Virginia S. Hanly, Vern Hoggatt, Douglas Holdridge, A. R. Hyde, D. C. B. Marsh, C. S. Ogilvy, G. B. Robison, Azriel Rosenfeld, R. R. Seeber, Jr., R. E. Shafer, Allen Simon, Donald Solitar, E. P. Starke, R. P. Tapscott, and the Proposer. Late solutions by Hüseyin Demir and R. K. Meany.

Editorial Note. We may show more generally that if n is any positive integer greater than 1, then $(n^a-1, n^b-1)=1$ if and only if $(a, b)=1$.

A Partitioning of the Binomial Coefficients

E 1172 [1955, 442]. *Proposed by D. R. Morrison, Tulane University*

Show, for every positive integer n, that of the following sums of binomial coefficients two are equal and the third differs from them by 1; that is, they partition their sum, 2^n, into three parts which are as nearly equal as three integers whose sum is 2^n can be.

$$S_1 = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots$$

$$S_2 = \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots$$

$$S_3 = \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots$$

I. *Solution by Patricia Ann Brown, Trinity University, Texas.* Let ω be a root of $x^2+x+1=0$. Then

$$\begin{aligned} 1 &= (1+\omega)(1+\omega^2) = (1+\omega)^n(1+\omega^2)^n \\ &= (S_1+\omega S_2+\omega^2 S_3)(S_1+\omega^2 S_2+\omega S_3) \\ &= [(S_1-S_2)^2+(S_2-S_3)^2+(S_3-S_1)^2]/2. \end{aligned}$$

II. *Solution by Virginia S. Hanly, Ohio State University.* Using the identity

$$\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$$

we readily find that

$$\begin{aligned} S_1(n)+S_2(n) &= S_2(n+1), \\ S_1(n)+S_3(n) &= S_1(n+1), \\ S_2(n)+S_3(n) &= S_3(n+1). \end{aligned}$$

Clearly, if the desired result is true for any number n, it is also true for the succeeding number $n+1$. It is easily verified to be true for $n=2$. Hence the desired result is true for all n by mathematical induction.

III. *Note by E. P. Starke, Rutgers University.* This is exactly problem E 300 [1938, 320], proposed by Daniel Finkel and solved by Emma Lehmer. In the *National Mathematics Magazine*, March 1939, pp. 292–293, the following results are given:

If the binomial coefficients are distributed analogously into four sets, two of the sums are always equal and the other two are equal for odd n but unequal for even n; the differences between the sums are always powers of 2.

If the binomial coefficients are distributed analogously into five sets, the five sums will have just three distinct values; the three possible differences of distinct sums being three consecutive terms of the Fibonacci series,

$$1, 1, 2, 3, 5, 8, 13, \cdots, a_{n+1} = a_n + a_{n-1}.$$

The property of equality of certain sums holds for any number of sets, but the facts about the differences of sums do not continue to work out so neatly.

Also solved by G. E. Bardwell, W. J. Blundon, Louis Brickman, J. R. Byrne, Leonard Carlitz, G. B. Charlesworth, Richard Courter, V. D. Gokhale, B. K. Gold, A. J. Goldman, C. D. Gorman, Douglas Holdridge, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, C. D. Olds, Paul Payette, Walter Penney, L. A. Ringenberg, H. A. Robinson, Azriel Rosenfeld, R. R. Seeber, Jr., R. E. Shafer, R. P. Tapscott, and Chih-yi Wang. Late solutions by Hüseyin Demir and W. S. Lawton.

Klamkin called attention to the similar problem No. 117, *Mathematics Magazine*, Nov.–Dec. 1952, p. 104.

Computation of an Arc Length

E 1173 [1955, 442]. *Proposed by R. A. Laird, New Orleans, La.*

The sum of the lengths of five equal contiguous chords inscribed in a given circular arc is 5280 feet. The length of the long chord of the given arc is 5208 feet. Find, to the nearest inch, the length of the given arc.

Solution by A. R. Hyde, West Hartford, Conn. Let R, S, $10A$ denote the radius, arc length, and central angle respectively. Then

$$\sin A = 528/R, \qquad \sin 5A = 2604/R = (217/44)\sin A,$$

$$S = 10RA = 5280(A/\sin A) = 5280[1 + (\sin^2 A)/6 + 3(\sin^4 A)/40 + \cdots].$$

But

$$\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A,$$

whence

$$16\sin^4 A - 20\sin^2 A + 3/44 = 0$$

and

$$\sin^2 A = 5/8 - (1/2)\sqrt{17/11} = 0.00342.$$

Two terms of the series suffice, giving

$$S = 5280(1.00057) \text{ feet} = 5283 \text{ feet to the nearest inch.}$$

Also solved by W. V. Gamzon, Raymond Huck, Edgar Karst, Sam Kravitz, D. C. B. Marsh, Morris Morduchow, C. S. Ogilvy, Walter Penney, N. C. Perry, L. A. Ringenberg, Azriel Rosenfeld, R. R. Seeber, Jr., Chih-yi Wang, R. H. Wilson, Jr., and the Proposer. Several incorrect solutions were also received. Late solution by L. V. Mead and Don Cenis (jointly).

The Proposer has developed the approximation formula

$$S = P + (P - C)/(n^2 - 1), \quad \text{with error} \quad E < S^5/1920n^2R^4,$$

where

S = required arc,
C = long chord of the arc,
n = number of equal contiguous chords inscribed in the arc,
P = sum of the equal contiguous chords,
R = radius of the arc.

The case $n=2$ of this approximation formula is the well known Huygens' formula for the approximation to the length of a circular arc.

Hermitian Matrices

E 1174 [1955, 442]. *Proposed by M. D. Marcus and B. N. Moyls, University of British Columbia*

Let A be an n-square Hermitian matrix whose characteristic roots are the diagonal elements A_{ii}, $i=1, \cdots, n$. Prove that A is diagonal.

Solution by W. V. Parker, Alabama Polytechnic Institute. If the characteristic roots of $A=(A_{ij})$ are $A_{11}, A_{22}, \cdots, A_{nn}$, then

$$\sum_{i<j} A_{ii}A_{jj} = \sum_{i<j} (A_{ii}A_{jj} - A_{ij}A_{ji}).$$

This implies

$$\sum_{i<j} A_{ij}A_{ji} = 0.$$

In case A is Hermitian, this becomes

$$\sum_{i<j} A_{ij}\overline{A}_{ij} = 0,$$

which is possible only if $A_{ij}=0$, $i\neq j$. For a generalization, see this MONTHLY [1954, 330].

Also solved by G. E. Bardwell, Leonard Carlitz, C. D. Gorman, D. S. Greenstein, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, and the Proposers.

Mutually Tangent Spheres

E 1175 [1955, 442]. *Proposed by G. A. Yanosik, New York University*

(1) Three mutually tangent spheres, with radii $r_1<r_2<r_3$, rest upon a horizontal plane. Find the radius R of the largest sphere which will slip through the space between the three given spheres.

(2) Given four mutually tangent spheres of slightly different radii $r_1<r_2<r_3<r_4$, find the radius R of the largest fifth sphere which will fit in the space which is more or less bounded by the four given spheres.

Notes by Leon Bankoff, Los Angeles, Calif. (1) When the fourth sphere is in the critical position for passage, the centers of the four spheres are coplanar. The problem is thus reduced to its two-dimensional analogue, having the solution (see *School Science and Mathematics*, Jan. 1953, p. 75, problem 2293).

$$R = r_1r_2r_3/(\sqrt{2r_1r_2r_3(r_1+r_2+r_3)} + r_1r_2 + r_2r_3 + r_3r_1).$$

The initial condition of tangency of the three given spheres to a horizontal plane merely implies that $r_1 \geqq r_2r_3/(\sqrt{r_2}+\sqrt{r_3})^2$, and has no bearing on the general solution. See this MONTHLY [1941, 267], problem E 432.

(2) A fifth sphere will fit in the space "bounded" by the four given spheres

only if such space exists. The condition $r_1 < r_2 < r_3 < r_4$ is not sufficient to preclude the possibility that the centers of the four given mutually tangent spheres may be coplanar. Clearly, from the preceding solution, we must impose the restriction

$$r_1 > r_2r_3r_4/(\sqrt{2r_2r_3r_4(r_2 + r_3 + r_4)} + r_2r_3 + r_3r_4 + r_4r_2).$$

A solution to this problem was given by Professor Gill in *Lady's and Gentleman's Diary*, 1884, p. 66. The final answer can be given in various forms. For example (see N. A. Court, *Modern Pure Solid Geometry*, Ex. 21, p. 210),

$$1/R^2 - (1/R)\sum(1/r_1) + \sum(1/r_1^2) - \sum(1/r_1r_2) = 0,$$

or (Professor F. Soddy's form, given in *Nature*, vol. 137, p. 1021, June 20, 1936),

$$[\sum(1/r_1) + 1/R]^2 = 3[\sum(1/r_1^2) + 1/R^2].$$

If R_0 is the radius of the sphere circumscribing the four given spheres, then it is interesting that

$$1/R = \sum(1/r_1) + 1/R_0.$$

Also solved by W. B. Carver, D. C. B. Marsh and the Proposer.

The Proposer showed that for part (2)

$$R = 2r_1r_2r_3r_4/[(\sum r_1r_2r_3) + \sqrt{12r_1r_2r_3r_4(\sum r_1r_2) - 3(\sum r_1r_2r_3)^2}].$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4673. *Proposed by Paul Erdös, Haifa, Israel.*

Let $a_1 < a_2 < \cdots$; $b_1 < b_2 < \cdots$ be two sequences of integers. Prove that from the sequence $a_i + b_j$ one can always select an infinite subsequence such that no one divides another.

4674. *Proposed by R. Venkatachalam Iyer, Karamana, Trivandrum, India*

The general solution in rational numbers of the equation

$$x^2 + y^2 + z^2 + 2xyz = 1 \tag{1}$$

is given by I. A. Barnett as

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \qquad y = \frac{c^2 + a^2 - b^2}{2ca}, \qquad z = \frac{a^2 + b^2 - c^2}{2ab}.$$

See *A Diophantine Equation Characterizing the Law of Cosines*, this MONTHLY, April 1955, p. 251. Find the general solution of (1) in integers.

4675. *Proposed by D. H. Lehmer, University of California at Berkeley*

Show that

$$\left(\frac{3\cdot 3}{1\cdot 5}\right)^4 \left(\frac{15\cdot 15}{13\cdot 17}\right)^4 \left(\frac{27\cdot 27}{25\cdot 29}\right)^4 \left(\frac{39\cdot 39}{37\cdot 41}\right)^4 \left(\frac{51\cdot 51}{49\cdot 53}\right)^4 \cdots = 12.$$

4676. *Proposed by J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, N. Y.*

Let $f(x)$ be a real valued measurable function defined on a measurable set A with measure μ such that

$$0 < a \leqq f(x) \leqq b < \infty$$

almost everywhere on A. Let $\bar{f}$ be the average value of f and $\tilde{f}$ be the reciprocal of the average value of the reciprocal of f. Find least upper and greatest lower bounds for $\bar{f}/\tilde{f}$.

4677. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

For what values of θ does the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n\theta}\,?$$

SOLUTIONS

A Finite Sum

4211 [1946, 397]. *Proposed by K. L. Chung, Syracuse University*

Prove that

$$\binom{nd}{n} = d(d-1) \sum_{k=0}^{n} \frac{(dk-2)!}{(k-1)!(dk-k)!} \binom{nd-kd}{n-k}.$$

Editorial Note. The equation of this problem occurs as (13) in an article by H. W. Gould, *Generalizations of Vandermonde's Convolution*, this MONTHLY,

this issue, p. 88. For a complete proof and for other similar and related sums the article should be consulted.

A Function Zero Almost Everywhere

4613 [1954, 718]. *Proposed by Joseph Lehner and G. M. Wing, Los Alamos Scientific Laboratory*

Let $\phi(t)$ be integrable over (0, 1) and suppose, for $z=re^{i\theta}$, that

$$F(z) = \int_0^1 e^{\alpha z t}\phi(t)\,dt, \qquad R\alpha > 0,$$

satisfies

$$\int_0^\infty |F(r)|^2 dr < \infty.$$

Show that $\phi = 0$ almost everywhere.

Solution by R. P. Boas, Jr., Northwestern University. If $\phi(t)$ is not almost everywhere zero, it is intuitive that $F(z)$ must grow exponentially in any direction such that $R(\alpha z)>0$, and hence $\int_0^\infty |F(r)|^2 dr = \infty$. This can be made rigorous as follows. $F(z)$ is an entire function of exponential type, bounded for $\arg z = -\arg \alpha \pm \pi/2$. For such a function, a theorem of Ahlfors and Heins states that $r^{-1}\log|F(r)| \to c$ as $r\to\infty$, except for a set E of values of r such that $\int_E r^{-1}dr$ converges; and from a theorem of Paley and Wiener it follows that $c=\lambda|\alpha|\cos(\arg\alpha)$, where λ is the smallest number such that $\phi(z)=0$ almost everywhere in $(\lambda, 1)$. (*Cf.* Boas, *Entire Functions*, 1954, pp. 108, 116.) If $\lambda=0$, $F(z)\equiv 0$ and (by the inversion formula for Fourier or Laplace transforms) we are through. If $\lambda>0$, we have $|F(r)|>e^{br}$, $0<b<c$, for $r>R$, say, except for the set E. If H is the complement of E (for $r>R$) we have

$$\infty > \int_H |F(r)|^2 dr > \int_H e^{2br}dr > \int_H r^{-1}dr,$$

which is impossible since $\int_{E+H} r^{-1}dr = \infty$. Evidently the condition $\int_0^\infty |F(r)|^2 dr < \infty$ can be replaced by much less restrictive ones.

Also solved by the Proposers.

Definite Integral of an Infinite Product

4614 [1954, 718]. *Proposed by H. F. Sandham, Institute for Advanced Studies, Ireland*

Prove that

$$\int_0^1 (1-x)(1-x^2)(1-x^3)\cdots dx = \pi\sqrt{\frac{48}{23}}\,\frac{\sinh \tfrac{1}{3}\pi\sqrt{23}}{\cosh \tfrac{1}{2}\pi\sqrt{23}}.$$

Solution by T. F. Mulcrone, S.J., St. Charles College, Grand Coteau, La. The termwise integration of Euler's expansion

$$x^{1/24}(1-x)(1-x^2)(1-x^3)\cdots = x^{1^2/24} - x^{5^2/24} - x^{7^2/24} + x^{11^2/24} + \cdots$$

leads, as the Proposer shows in his Note (this MONTHLY [1954, 104]), to the definite integral

$$\int_0^1 x^{\alpha+1/24}(1-x)(1-x^2)(1-x^3)\cdots\frac{dx}{x} = \frac{\pi\sqrt{2}\sinh\frac{1}{3}\pi\sqrt{24\alpha}}{\sqrt{\alpha}\cosh\frac{1}{2}\pi\sqrt{24\alpha}}$$

($\alpha > 1/24$). When $\alpha = 23/24$, we have the desired result.

Also solved by W. J. Blundon, Armand Brumer, Leonard Carlitz, Oscar Goldman, C. D. Olds, M. R. Spiegel, T. W. Summers, Chih-yi Wang, and the Proposer. Late solution by G. B. Findley.

A Trigonometric Limit

4615 [1954, 718]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Define

$$F(x) = \cos x \cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4}\cdots.$$

Prove that $F(x)\to 0$ as $x\to+\infty$. Prove in fact that $F(x)$ falls off exponentially.

Solution by N. J. Fine, University of Pennsylvania. For $x>0$, the number of positive integers k such that $\pi/3 \leqq x/k \leqq 2\pi/3$ is at least $(3x/2\pi)-2$. Taking only these factors, we have

$$|F(x)| \leqq 4(\tfrac{1}{2})^{3x/2\pi} = O(e^{-cx}),$$

where $c = (\log 8)/2\pi$. This gives the desired result.

The constant can be improved considerably. Thus, by Vieta's product,

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cdots$$

so

$$F\left(\frac{x}{2}\right) = \frac{\sin x}{x}\cdot\frac{\sin(x/3)}{x/3}\cdot\frac{\sin(x/5)}{x/5}\cdots.$$

Determine n so that $2n-1 \leqq x \leqq 2n+1$. Then

$$\left|F\left(\frac{x}{2}\right)\right| \leqq \frac{1\cdot 3\cdot 5\cdots(2n-1)}{x^n} = \frac{(2n)!}{(2x)^n n!} \sim \frac{1}{\sqrt{2}}\left(\frac{2n}{ex}\right)^n,$$

by Stirling's formula. Thus $F(x/2) = O(e^{-n}) = O(e^{-x/2})$, and $F(x) = O(e^{-x})$.

Also solved by Joseph Lehner, the Proposer, and one whose communication was unsigned. Late solution by Chih-yi Wang.

Compound Normal Distribution

4616 [1954, 718]. *Proposed by A. C. Cohen, Jr., University of Georgia*

Show that the compound normal distribution function

$$f(x) = \frac{1}{2\sigma\sqrt{2\pi}}\left[\exp\left\{-\tfrac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2\right\} + \exp\left\{-\frac{1}{2}\left(\frac{x-m_2}{\sigma}\right)^2\right\}\right]$$

is bimodal if $|m_1-m_2|>2\sigma$ and unimodal otherwise.

Solution by P. J. Burke, Bell Telephone Laboratories, New York City. Write $f(x)=f=f_1+f_2$ where

$$f_1 = \frac{1}{2\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-m_1)^2}{2\sigma^2}\right\}.$$

Take $m_2 \geqq m_1$. Except for the trivial case $m_1=m_2$, the extrema of f lie in the domain $m_1<x<m_2$ since only there does $\operatorname{sgn} f_1' = -\operatorname{sgn} f_2'$. Let $y=f'/f_1$ and let $k(*)$ be the number of distinct zeros of $*$. Then $k(y)=k(f')$. Setting $f'=0$, one has also $y=0$, and after transposing and taking logarithms in the latter equation, one has

$$z \equiv -\log(x-m_1) + \log(m_2-x) - \frac{[m_2^2 - m_1^2 - 2x(m_2-m_1)]}{2\sigma^2} = 0.$$

Since z has a continuous derivative in the domain defined above, $k(z)\leqq 1+k(z')$; and since z' is quadratic, $k(z)=k(f')\leqq 3$. Hence f has at most 3 extrema.

By direct substitution it is found that $x_c=\frac{1}{2}(m_1+m_2)$ is a root of $f'=0$, and by substitution in f'' it is found that $f(x_c)$ is a maximum for $|m_1-m_2|<2\sigma$ and a minimum for $|m_1-m_2|>2\sigma$. Since f is positive and continuous for $-\infty<x<+\infty$, symmetric in $x=x_c$ and asymptotic to the x-axis, therefore f must have two maxima when $f(x_c)$ is a minimum, and when $f(x_c)$ is a maximum, the unique mode must be at $x=x_c$, since f can have no more than three extrema. For $|m_1-m_2|=2\sigma$, $f(x_c)$ is a maximum since $f'''(x_c)=0$ and $f^{iv}(x_c) = -2/[\sigma^5(2\pi e)^{-1/2}]$.

Also solved by W. J. Blundon, P. G. Kirmser, M. S. Klamkin, S. Parameswaran, Nathan Shklov, Chih-yi Wang, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Modern Trigonometry. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Company, 1955. xii+154+Answers+Tables +Index. $3.25.

Plane Trigonometry. By C. R. Wylie, Jr. New York, McGraw-Hill Book Company, 1955. viii+281+Tables+Answers+Index. $4.00.

As far as it was possible to ascertain, the first of the above books is quite different from any of the many trigonometry textbooks now available. Many teachers and scientists were consulted in its preparation, and a lithoprinted edition was used in seventy-six college classes and an unstated number of high school classes. The book would thus seem to aim at both college and high school needs. Problem lists are ample, and answers and *hints* are given at the end of the book for a great number of problems, both odd and even-numbered. Figures also appear in the answers when these are called for in the problems. All figures are well-drawn, and those of the text are usually placed in the exceptionally wide, outer margins of the pages. Curves showing the beauty of geometric form decorate the ends of the chapters following chapter summaries and self-tests for the student. Chapters have no titles, and, while sections start with chapter and section numbers, these numbers are not shown on each page as is now a common custom.

Chapter 1 introduces functional notation and the distance formula as well as reviewing fractions, solution of equations, and Cartesian coordinates. The six trigonometric functions are defined for general angles in Chapter 2, where is found also a brief mention of radians. Chapter 3 deals with four-place logarithms commencing with the definitions and the laws of exponents. Scientific notation is employed to estimate numerical results and to explain significant figures and characteristics. The computer's rule for favoring the even digits is not mentioned, but logarithmic and exponential equations are treated. A folder of four pages with four-place tables of powers, logarithms, and natural functions is supplied. These tables are duplicated at the back of the book where there is also a four-place table of logarithmic trigonometric functions. The greatest departures from conventional treatment occur in Chapters 4 and 9. The former treats the solution of triangles. The student is told how to use his tables. The cosine and sine laws are derived, and the tangent law is stated without proof. The solution of right triangles is not mentioned, nor is it illustrated by solved examples. Nevertheless, problems on this topic are given. The usual four cases of oblique triangles are missing. No illustrative examples are solved with logarithms, and the data of the problems seem to lend themselves to arithmetic. A teacher might, therefore, omit the chapter on logarithms. The same could be said of Chapter 9 which seems to be recreational, giving very brief discussions of several topics such as matrices, angle trisection, the four color problem, and finite geometry.

Chapter 5 treats the graphs of all six trigonometric functions and emphasizes the fact that their arguments are real variables which need not represent angles. The analytic part of trigonometry is usually the most difficult for the student. Placing this near the end in Chapter 6 should have its advantages. The distance formula is used to prove the composite angle formulas, a method recently characterized by a reviewer in this MONTHLY as the first completely satisfactory one in elementary textbooks. Arcsine, arccosine, and arctangent are defined without reference to their graphs; other inverse trigonometric functions are mentioned but not defined. Chapters 7 and 8 contain topics that might be included in a longer-than- usual course. The former deals with polar coordinates, and the latter treats complex numbers and De Moivre's theorem.

When two books are being discussed in the same review, there is an obvious temptation to make comparisons. When these occur, they are not to be construed as derogatory to either book. Our first book has about the average number of pages. Our second exceeds it by one hundred and seventy pages. Hence the latter has more material, and its explanations are, in general, more detailed. Chapter and section numbers appear on each page. Exercises and illustrative examples are printed in smaller type than the text, and answer to the odd-numbered exercises appear at the end of the book. One notices the use of numerous footnotes, and the mention of many connections between trigonometry and plane geometry.

Chapter 2 has a clear discussion of the logic used in proving identities. Chapter 3 with thirty-four pages is entitled *The Solution of Right Triangles*. It uses only arithmetic and has many applied problems. Chapter 4, *The Graphs of the Trigonometric Functions*, has a good motivating discussion of the applications of $y=A \sin (bx-c)$ and $y=A \cos (bx-c)$. A briefer treatment of the graphs of the other trigonometric functions is followed by a section of four pages on polar coordinates. A derivation of the distance formula commences Chapter 5 in which this formula is used to obtain the formulas for functions of composite angles. Chapter 6 on inverse functions begins with a very helpful exposition of inverse functions in general and their graphs. The principal values of the inverse trigonometric functions are explained from their graphs and are indicated by the usual initial capitals. At the beginning of Chapter 7, *Trigonometric Equations*, the author states, "We shall give all solutions wherever possible and then indicate which one of the set is the principal value." The discussion is more complete than usual, and it includes equations with inverse functions and those involving both algebraic and trigonometric functions. Chapter 8, *Logarithms*, starts with a very brief review of exponents followed by the general definition of a logarithm and the laws of operation with logarithms. Logarithmic solution of right triangles is included in the chapter as are change of base, natural logarithms, logarithmic and exponential functions. The solution of equations involving algebraic and exponential functions is illustrated by examples. All logarithmic tables are five-place. Chapter 9 is *Solution of Oblique Triangles*. The distance formula is used to derive the law of cosines which is then employed to solve

cases III and IV arithmetically. The familiar four cases (although not numbered) are solved by logarithms in the usual manner. Mollweide's formula is given as "probably the best independent check." Chapters 10, 11, and 12 contain material for which there is no time in the usual freshman course, but they are well and interestingly written. Their titles are, respectively, *Complex Numbers and De Moivre's Theorem*, *Trigonometric Series*, and *Hyperbolic Functions*.

R. F. GRAESSER
University of Arizona

The Elements of Probability Theory and Some of Its Applications. By Harald Cramér, New York, J. Wiley and Sons, 1955. 281 pages, $7.00.

This book is a revised and extended translation of the author's earlier Swedish text. It is an elementary treatise for beginners who have a modest mathematical background, namely a familiarity with analytic geometry, calculus and some algebra including determinants. Therefore, the book does not aim at a complete and rigorous mathematical development but refers often for proof to other books, primarily to the author's *Mathematical Methods of Statistics* (Princeton University Press, 1946).

The book is divided into three parts. Part I (Chapters 1–4, 44 pages) deals with the foundations. Chapter 1 gives a brief historical survey. In Chapter 2 the author discusses phenomena which exhibit statistical regularity. The aim of the book is to construct a mathematical model for the description and interpretation of such phenomena. Clearly, the level for which the book is intended precludes a rigorous axiomatic treatment of probability theory. Nevertheless the book is written in the spirit of the foundations of Kolmogorov. A student who starts his study of probability theory with this book should gain the proper appreciation for the modern viewpoint and should be able to read more advanced treatises with ease since a motivation for the modern approach is implicitly provided by the present book. In Chapter 3 the author proceeds to discuss some basic properties and concepts of the mathematical theory of probability (the addition and multiplication rules, conditional probabilities, independent events, Bayes theorem) and in Chapter 4 applies these to the study of repeated observations, drawings with and without replacements and simple problems on games of chance.

Part II (Chapters 5–10, 95 pages) has the title *Random Variables and Probability Distributions*. In Chapter 5 random variables are presented as results of random experiments and the concept of a distribution function is introduced. The two most important types of distribution functions, namely, discrete and continuous distributions are studied in greater detail. Some basic ideas connected with multivariate distributions are considered. The usual measures of location, dispersion and skewness, as well as mean values and moments, are defined and Tchebycheff's inequality is proven. Here, as well as in other instances, the author states the theorem in full generality with a proof which applies either to the continuous or the discrete case. The second proof is omitted and is left as an

exercise to the reader. The addition of independent random variables is discussed and generating functions are defined. The problem of the existence of a generating function is stated and it is mentioned that this difficulty could be avoided by introducing characteristic functions. However, no further use is made of characteristic functions since the author wishes to avoid complex variables. The chapter concludes with a discussion of some of the important properties of generating functions. Chapter 6 deals with the binomial distribution and related topics. Bernoulli's law of large numbers and de Moivre's form of the central limit theorem are discussed. The Poisson distribution is introduced as a limiting distribution for the binomial distribution. The normal distribution and a more general statement of the central limit theorem are treated in Chapter 7. Chapter 8 deals with continuous distributions which are related to the normal. Sections of this chapter are devoted to the chi-square distribution, the t-distribution and the Pearson system. The last two chapters of Part II contain essential material concerning bivariate and multivariate distributions.

Part III (Chapters 11–16, 108 pages) has the title *Applications*. In the introductory Chapter 11 the author clarifies the aims and limitations of statistical tests for investigating the agreement between some suggested theoretical model and actual observations. Chapter 12 deals with the descriptive treatment of statistical material. The empirical distribution of the sample is defined and the distinction between the characteristics of the population and the sample characteristics is emphasized. Chapter 13 introduces the idea of sampling distributions. Moments of the sampling distribution of certain characteristics, for example the mean and the variance, are used to estimate the corresponding population parameters. The asymptotic normality of the distribution of a wide class of sample characteristics is mentioned with an appropriate reference to the detailed treatment in the author's *Mathematical Methods of Statistics*. Certain order statistics are given as examples of exceptions. Thus the reader is warned that it is wrong to assume that the distributions of *all* sampling characteristics are asymptotically normal. This is but one example of the author's careful exposition which avoids the evils of "cook-booking" even though he is forced to omit certain proofs which could not be understood by a reader with the assumed mathematical background. This chapter concludes with a derivation of the exact distribution of the mean and of the variance of a sample drawn from a normal population. The subject of Chapter 14 is statistical inference. A very concise and brief discussion of the problem of estimation and of certain properties (unbiasedness, consistency, efficiency) of estimates is given. The maximum likelihood method as well as the method of moments is presented. The concept of confidence intervals is introduced and discussed in some detail for samples from normal populations. This is supplemented by the consideration of cases where it is known that the estimates involved are only asymptotically normal. The chapter closes with some remarks on the Neyman-Pearson theory of testing hypothesis. Fleeting remarks are made concerning sequential analysis, the concept of risk and A. Wald's theory of statistical decision functions. The purpose

of these remarks is to indicate the existence of further developments and to give the reader stimulation and references for the continuation of his studies. Chapter 15 treats the chi-square test, proofs are mostly omitted but interesting illustrative examples are given. Chapter 16 deals with a variety of topics, such as the theory of errors and regression problems (17 pages), the analysis of variance (10 pages), sampling and quality control (9 pages). This chapter gives an idea of the wide scope of applications of statistical methods. There is a brief appendix with some formulae concerning the gamma function, a table of references, answers to the problems and some statistical tables (normal, chi-square, t- and F-distributions). Problems are inserted at the end of almost every chapter. A large number of interesting illustrative examples are worked out in detail throughout the text. Among these, as well as among the problems, one finds also some interesting applications to mortality tables and to insurance problems.

This reviewer regrets that the third part is restricted to the discussion of parametric methods. He feels that the inclusion of some material on nonparametric tests would have been desirable and feasible. He hopes that this will be added to future editions of this excellent book.

The author intended to write a text book for an introductory course which could be presented to students with a modest background and a variety of motives for studying probability theory and its applications. This very difficult aim has been accomplished with great success and this reviewer feels that this text will help to promote better teaching of elementary probability and statistics.

EUGENE LUKACS
Office of Naval Research, Washington, D. C.

An Introduction to Stochastic Processes. By M. S. Bartlett. New York, Cambridge University Press, 1955. 14+312 pages. $6.50.

The question which recurred often to this reviewer while going through Bartlett's book is, "An introduction for whom?" According to the preface, the volume is addressed to applied mathematicians and statisticians, emphasis being on "methods and applications." The author's plan necessitates omission of much rigor and detail. A treatment of this nature could serve as a useful introduction to the subject for many readers (applied mathematicians and statisticians, as well as others) who want to obtain some idea of the applicability of the theory without spending time on a full mathematical treatment à la Doob; however, it seems doubtful that the present volume will prove to be such a useful introduction for many of these readers.

A main criticism of the book is that omission of some rigor and detail should not necessitate hazy (or complete omission of) definitions of words used technically in the text, and which make whole sections difficult (if not impossible) to decipher. For example, most readers will be confused by the remarkable phrasing in the definition of a "nearest neighbour system" on page 35. In the sections on statistical inference which constitute the last part of the book, the uninitiated

will wonder at many undefined phrases like (page 265) "simple consistent (though not fully efficient) equations of estimation," as well as why one should use certain proposed tests of significance (the notion of the power of a test is never mentioned).

Other aspects of the author's style are also disturbing or fatiguing. One is continually referred to an unpublished work by Moyal for material which can be found in published but unmentioned sources. The book has several disturbing errors; for example, the sufficient condition for the central limit theorem stated on page 9 is incorrect. The mathematical level and choice of what is or is not to be abbreviated sometimes seem uneven, but some of this may be a matter of taste.

On the positive side, there is a great variety of applications in the book. The topics considered range from epidemic models and harmonic analysis to sequential analysis and queueing theory, and include many others. Any usefulness of the book seems to this reviewer to be found in this variety of topics. Their treatment suffers from the defects noted above, and the direction of these topics may not prove useful to all readers (for example, readers concerned with applications to engineering and physics will probably find the book of Blanc-Lapierre and Fortet more useful); but Bartlett's book may prove a useful (if incomplete) source of references and summary of results in some areas. Thus, for example, a reader interested in population growth can find in this book a list of various models and results, a sketch of several methods of obtaining such results, and a list of some papers which contain more detailed presentations (that reader will often find the references more readable than the book). However, for an "introduction" to the subject which will give him an understanding of the basic ideas, a reader interested primarily in applications without too much heavy mathematics would still do better to read (*e.g.*) appropriate portions of Feller's book in place of corresponding sections of Bartlett's book. There is no substitute for a clear, precise presentation of the elementary concepts of a subject like this one for a reader who wants to understand fully the applications.

J. KIEFER
Cornell University

NEW BOOKS RECEIVED

Vector Analysis (International Series in Pure and Applied Mathematics). By H. E. Newell, Jr. New York, McGraw-Hill Book Company, 1955. 11+216 pages. $5.50.

The Geometry of Geodesics. By Herbert Busemann. New York, Academic Press, Inc., Publishers, 1955. x+422 pages. $9.00.

A Manual of Problems in Statistics, Revised Edition. By Scott Dayton. New York, Henry Holt and Company, 1955. v+137 pages. $1.95.

Memorial des Sciences Mathematiques. Vol. 128. La sommation des series divergentes. By Marc Zamansky, 1954. Paris, Gauthier-Villars, 79 pages.

Collection de Logique Mathematique, Series A., No. 7, Deux Esquisses de Logique. By J. B. Rosser. Paris, Gauthier-Villars, 1955. 70 pages. 900 fr.

Colloque sur L'Analyse Statistique. (Held at Brussels Dec. 15, 16, 17, 1954). Paris, Georges Thone Liege, Masson et Cie, 1955. 186 pages. 1.900 fr. francs.

Cours de Geometrie Infinitesimale, 5th vol. Geometrie Infinitesimale 2nd Part, Theorie des Surfaces. By Gaston Julia. Paris, Gauthier-Villars, 1955, 145 pages.

Cours de Geometrie Infinitesimale. 4th Vol. Cinematique et Geometrie Cinematique, 2nd part, Etude Approfondie du Mouvement d'un Corps Solide. By Gaston Julia. Paris, Gauthier-Villars, 1955. 88 pages.

Cours de Geometrie Infinitesimale. 3rd Vol. Geometrie Infinitesimale. 1st part. Methodes Generales. Theorie des Courbes. By Gaston Julia. Paris, Gauthier-Villars, 1955. 220 pages.

Theoreme sur les Surfaces D'onde en Optique Geometrique, avec une note sur le Miroir Integral. By Rene Damien. Paris, Gauthier-Villars, 1955. 34 pages. $2.66.

College Algebra and Trigonometry, A Basic Integrated Course. Second Edition. By F. H. Miller. New York, John Wiley and Sons, Inc., 1955. xiv+342 pages. $4.50.

International Encyclopedia of Unified Science, Vol. 1, part 1, and part 2. Edited by Otto Neurath, Rudolph Carnap and Charles Morris. Chicago, Illinois, The University of Chicago Press, 1955. *Part 1*, ix+339 pages; *Part 2*, 342–760 pages. $6.00 each. (If purchased together, the set costs $11.00.)

Astrophysical Quantities. By C. W. Allen. London, England, University of London, The Athlone Press, 1955. xi+263 pages. $10.00. (Distributed in America by John de Graff, Inc., 64 W. 23rd St., New York 10, New York.)

Elementary Statistics for Students of Social Science and Business. By R. C. Sprowls. New York, McGraw-Hill Book Company, 1955. xiii+392 pages. $5.50.

Basic Mathematics for Science and Engineering. By P. G. Andres, H. J. Miser and H. Reingold. New York, John Wiley and Sons, Inc., 1955. vii+846 pages. $6.75.

Shop Mathematics. By C. E. Stout. New York, John Wiley and Sons, Inc., 1955. xi+282 pages. $3.70.

Reflections of a Physicist. Second Edition. By P. W. Bridgman. New York, Philosophical Library, 1955. xiv+576 pages. $6.00.

A Manual of Problems in Statistics, Revised Edition. By Scott Dayton. New York, Henry Holt and Company, 1955. 137 pages.

Analytic Geometry. Third Edition. By Sisam and Atchison. New York, Henry Holt and Company, 1955. xxiv+292 pages. $3.75.

Principles of Mathematics. By C. B. Allendoerfer and C. O. Oakley. New York, McGraw-Hill Book Company, 1955. xv+448 pages. $5.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Bradley University makes the following announcements: Mr. K. E. Lake of Washburn Municipal University has been appointed to an assistant professorship; Assistant Professor Joseph R. Brown has been promoted to an associate professorship.

Lehigh University reports the following: Mr. Gerhard Rayna, previously a programmer at the Forrestal Research Center, Princeton, New Jersey, Dr. Irving Weiss, formerly a statistician at Sylvania Electric Products Company, and Mr. F. O. Wyse, previously an assistant in instruction at Princeton University, have been appointed to instructorships; Assistant Professor Chuan-Chin Hsiung has been promoted to an associate professorship.

At McGill University: Assistant Professor J. E. L. Peck of the University of New Brunswick has been appointed to an associate professorship; Mr. G. D. Findlay, Mr. J. E. LeBel, Mr. J. Michael, Dr. T. V. Narayana, and Mr. L. A. Smith have been appointed Lecturers.

Texas Christian University announces: Dr. Mabel G. Reavis has been appointed to an assistant professorship; Associate Professor L. A. Colquitt has been promoted to a professorship.

Tulane University reports: Professor B. J. Pettis has been appointed Chairman of the Department of Mathematics of the University for the year 1955–1956; Dr. F. B. Wright, Jr., has been promoted to an assistant professorship; Associate Professor A. H. Clifford of Johns Hopkins University has been appointed Head of the Department of Mathematics of Newcomb College; Mr. F. L. Cleaver, formerly a teaching assistant at the University of Miami, Mr. C. P. Gadsden, previously a teaching assistant at the University, and Miss Eugenia I. Trapp, recently a graduate teaching assistant at the University, have been appointed to instructorships; Dr. Morikuni Goto, formerly a research associate at the University of Illinois, has been appointed to an assistant professorship; Assistant Professor Paul Conrad has been granted a Fulbright award for lecturing at the University of Colombo, Ceylon, for the academic year 1956–1957; Mr. A. B. Simon, a graduate fellow, has been awarded a National Science Foundation Fellowship and is continuing his graduate work at the University.

University of California at Los Angeles announces the following: Dr. P. C. Curtis, Jr., has been appointed to an instructorship; Dr. C. B. Tompkins, director of numerical analysis research, has been appointed to a professorship; Assistant Professors L. J. Paige and E. G. Straus have been promoted to associate

professorships; Associate Professor F. A. Valentine has been promoted to a professorship.

At the University of Illinois: Dr. J. R. Buchi of the University of Michigan, Dr. D. L. Burkholder, previously a graduate research assistant at the University of North Carolina, Dr. N. T. Hamilton of Cornell University, and Mr. J. A. Zilber of Johns Hopkins University have been appointed to assistant professorships; Dr. Klaus Krickeberg, previously a lecturer at Universitat Wurzburg, has been appointed Research Associate; Dr. W. M. Zaring, formerly a graduate assistant at the University of Kentucky, has been appointed to an instructorship; Assistant Professors C. R. Blyth and W. A. Ferguson have been promoted to associate professorships; Associate Professor C. W. Mendel has been promoted to a professorship; Dr. Jewell E. Schubert has been promoted to an assistant professorship; Dr. Michio Suzuki, recently a research associate, has been promoted to an assistant professorship.

University of Miami reports the following: Dr. E. F. Low, Jr., formerly an aeronautical research scientist for the National Advisory Committee for Aeronautics, Langley Field, Virginia, and Dr. R. V. Mendenhall, previously a senior mathematician at Vitro Laboratories, Eglin Air Force Base, Florida, have been appointed to assistant professorships; Mr. Frederick Borges has been appointed to an instructorship.

University of Notre Dame announces: Dr. Irving Glicksburg, previously a research associate for the Rand Corporation, and Dr. William Huebsch, previously a National Science Foundation postdoctoral fellow at the Institute for Advanced Study, have been appointed to assistant professorships; Dr. R. C. Taliaferro has been appointed to an associate professorship; Mr. Frederick Bagemihl, formerly a research fellow at the Institute for Advanced Study, has been appointed Visiting Lecturer.

At the University of Pittsburgh: Mr. Samuel Hanna and Mr. Earl McKinney have been appointed to instructorships.

University of Rochester reports: Mr. J. P. Line of Oberlin College and Mr. Edward Wong, formerly a graduate teaching assistant at the University, have been appointed to instructorships; Assistant Professor Walter Rudin has been promoted to an associate professorship; Mr. R. A. Raimi has been promoted to an assistant professorship and is on leave of absence during 1955–1956.

Assistant Professor W. F. Atchison of the University of Illinois has a position at the Rich Electronic Computer Center, Georgia Institute of Technology.

Dr. V. N. Behrns of Consolidated Vultee Aircraft Corporation, Fort Worth, Texas, has a position as an operations analyst with Technical Operations, Arlington, Massachusetts.

Assistant Professor R. L. Beinert of Hobart and William Smith Colleges has been promoted to an associate professorship.

Dr. Stoughton Bell of the University of California has accepted a position as a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor W. G. Brady of Washington and Jefferson College has been appointed Acting Head of the Department of Mathematics.

Mr. R. E. Cornish of Seton Hall University has accepted a position as a programmer with International Business Machines Corporation, Kingston, New York.

Mr. L. J. Derr, formerly of Tulane University, has a position as a mathematical analyst with the Shell Oil Company, New Orleans, Louisiana.

Dr. William H. Durfee of the Operations Research Office has been appointed to an associate professorship at Mount Holyoke College.

Associate Professor L. A. Dye of The Citadel has been promoted to a professorship.

Dr. R. E. Graves of the Goodyear Aircraft Corporation is now a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Dr. J. S. Griffin, Jr., of Tulane University has been appointed to an instructorship at the University of Michigan.

Dr. J. F. Heyda of the Naval Ordnance Plant has accepted a position as a mathematician with the Allison Division of General Motors, Indianapolis, Indiana.

Dr. M. A. Hyman of the Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania, has accepted a position as a research analyst with the Eckert-Mauchly Division of the Sperry Rand Corporation, Philadelphia, Pennsylvania.

Dr. J. R. Isbell of Tulane University is now in military service.

Mr. A. A. Karwath of the State University of Iowa has been appointed to an assistant professorship at the United States Naval Academy.

Mr. Raymond Kassler of Boland and Boyce, Belleville, New Jersey, has a position as a development engineer with the Atlantic Design Company, Newark, New Jersey.

Associate Professor Dora E. Kearney of Upper Iowa University has been appointed to an assistant professorship at Iowa Wesleyan College.

Assistant Professor George Klein of Mount Holyoke College has been appointed to an assistant professorship at Tufts College.

Dr. A. B. Lehman has been appointed a staff member in the Acoustics Laboratory, Massachusetts Institute of Technology.

Assistant Professor W. J. LeVeque of the University of Michigan has been promoted to an associate professorship.

Associate Professor R. J. Levit of the University of Georgia has accepted a position in the Applied Science Division of International Business Machines Corporation, San Francisco, California.

Mr. E. S. Lowry, formerly a student at the University of Toronto, has a position as a programmer at Computing Devices of Canada, Ottawa, Ontario, Canada.

Mr. G. E. Mahoney has a position as a mathematician with the Computer Control Company, Point Mugu, California.

Dr. Imanuel Marx of the University of Michigan has been promoted to an assistant professorship.

Dr. Elliott Mendelson has been appointed to an instructorship at the College of the University of Chicago.

Assistant Professor M. W. Milligan of Adams State College has been promoted to an associate professorship.

Assistant Professor V. H. Morrill of Hardin-Simmons University has been appointed to an acting assistant professorship at Tarleton State College.

Mr. C. R. Morris of the University of Texas has accepted a position as an associate engineer with the Lockheed Aircraft Corporation, Burbank, California.

Mr. G. M. Muller, formerly a physicist at General Electric Company, Richland, Washington, has accepted an appointment as a mathematician at Stanford Research Institute, Menlo Park, California.

Dr. D. V. Newton of the University of Washington is now an applied science representative with the International Business Machines Corporation.

Dr. Albert Nijenhuis, formerly a member of the Institute for Advanced Study, has been appointed Research Associate and Instructor at the University of Chicago.

Mr. S. J. Scott, formerly a student at the University of North Carolina, has a position as a mathematician with the Vitro Corporation of America, Eglin Air Force Base, Florida.

Mr. W. B. Simmons, Jr., recently a student at Knox College, is employed as an associate engineer in the Mathematical Analysis Department, Lockheed Aircraft Corporation, Burbank, California.

Dr. N. B. Smith of San Diego State College has been promoted to an assistant professorship.

Professor Irving Sussman of the University of San Francisco has been appointed to a professorship at California State Polytechnic College.

Mr. O. E. Taulbee, previously an assistant at Michigan State University, has a position as a mathematician in the Systems Analysis Group, Remington Rand, Engineering Research Associates Division, St. Paul, Minnesota.

Assistant Professor L. C. Teng of the University of Wichita has accepted a position as an associate physicist at the Argonne National Laboratory, Lemont, Illinois.

Professor C. J. Thorne of the University of Utah has accepted a position as a research scientist with the Naval Ordnance Test Station, China Lake, California.

Associate Professor R. M. Thrall of the University of Michigan has been promoted to a professorship.

Associate Professor Emeritus Edith I. Atkin of Illinois State Normal University died on August 27, 1955. She was a member of the Association for twenty-nine years.

Professor C. A. Messick, formerly of Park College, died on October 10, 1955. He was a member of the Association for thirty years.

Professor S. B. Myers of the University of Michigan died on October 8. 1955.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 77 persons have been elected to membership by the Board of Governors on applications duly certified.

LINDA ALLEGRI, A.M. (Hunter) Mathematics Teacher, Hunter College High School.

F. W. ANDERSON, Ph.D. (S.U. of Iowa) Instr., University of Nebraska.

CAPT. R. A. AWTREY, U.S.N. (ret.), M.A. (Florida) Instr., University of Florida.

F. C. BARNETT, M.S. (New Mexico) Instr., Rose Polytechnic Institute.

L. C. BARRETT, M.S. (Utah) Instr., University of Utah.

ALEXANDER BECK, B.S. in Ed. (Temple) Teacher, Olney High School, Philadelphia, Pa.

J. M. BEHR, Student, University of California at Los Angeles; Technical Writer, Librascope, Glendale, Calif.

I. O. BENTSEN, B.A. (Hobart) Grad. Student, University of Rochester.

R. E. BRENNAN, B.S., B.A. (Amherst) Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

LOUIS BRICKMAN, Student, Temple University.

F. W. BRINKLEY, JR., B.A. (Denver) Grad. Student, University of Denver.

ARMAND BRUMER, Student, Brandeis University.

H. L. COOK, Ph.D. (Peabody-Vanderbilt) Head, Department of Mathematics, West Texas State College.

B. G. COOPER, B.A. (Stephen F. Austin S.C.) Grad. Asst., Kansas State College.

J. N. CRAWFORD, M.S. (East Texas S.T.C.) Head, Department of Mathematics, Le Tourneau Technical Institute.

C. W. CURTIS, Ph.D. (Yale) Asst. Professor, University of Wisconsin.

A. G. DAVIS, M.A. (Harvard) Asst. Professor, Clarkson College of Technology.

R. V. DIMEO, A.B. (Rutgers) Atlantic City, New Jersey.

R. E. DOWD, M.A. (N.Y.U.) Garnac Grain Co., New York, N.Y.

CHURCHILL EISENHART, Ph.D. (London) Chief, Statistical Engineering Lab., National Bureau of Standards, Washington, D. C.

J. D. ERDWINN, B.A. (Rice) Grad. Asst., Rice Institute.

W. M. FAUCETT, Ph.D. (Tulane) Senior Aerophysics Engr., Consolidated-Vultee Aircraft Corp., Fort Worth, Texas.

CONSTANCE M. FOLEY, M.A. (New Hampshire) Instr., West Virginia University.

I. C. FRANCIS, B.S. (Puget Sound) Seattle, Washington.

D. A. FREEDMAN, Student, McGill University.

D. G. FULTON, Ph.D. (Michigan) Professor, Tufts University.

DAVID GALE, Ph.D. (Princeton) Asst. Professor, Brown University.

J. J. GOODE, Student, Georgia Institute of Technology.

MRS. DOROTHY H. HELMICK, M.S. (Drake) Instr., Drake University.

J. R. HENDRICKS, B.A. (British Columbia) Meteorologist, Aviation Forecast Office, Winnipeg, Manitoba.

A. H. HERRINGTON, A.M. (Duke) Superintendent, Quitman Public Schools, Ga.

F. W. HOAGLAND, Chicago, Illinois.

MRS. RUTH M. HOFSTRA, M.A. (Syracuse) Fellow, University of Tennessee.

J. W. JOY, B.A. (Washington S.C.) Research Asst., Scripps Institution of Oceanography, La Jolla, Calif.

R. C. JURGENSEN, M.S. (Iowa) Chairman, Department of Mathematics, Culver Military Academy, Ind.

T. F. KIMES, B.S. (Ursinus) Teaching Asst., University of Texas.

GEORGE KOLETTIS, JR., M.S. (Chicago) Instr., Northwestern University.

H. E. KREHBIEL, M.S. (Michigan S.U.) Mathematics Teacher, Niles High School, Mich.

J. B. LACKEY, B.S. (Eastern Kentucky S.C.) Part-time Instr., University of Kentucky.

C. L. LAWSON, B.S. (California) Optometry Officer, U. S. Army Hospital, Fort Jackson, S. C.

F. R. LIDDIL, M.A. (Chicago) Instr., Illinois Institute of Technology.

MRS. SYLVIA C. P. LU, B.S. (Taiwan T.C.) Instr., Taipei College of Technology, Taiwan, China.

T. A. MACKEY, Student, Georgetown University.

A. G. MAZZA, B.S. (London) Senior Mathematics Master, Scottish Schools, Alexandria, Egypt.

J. E. MCFARLAND, M.S. (Oregon S.C.) Grad. Asst., Oregon State College.

ELLIOTT MENDELSON, Ph.D. (Cornell) Instr., University of Chicago.

D. W. MILLER, M.S. (Wisconsin) Instr., University of Nebraska.

LOIS B. MITCHELL, A.B. (Vassar) Mathematician, I.B.M. Corp., New York, N. Y.

MRS. SHU-TEH C. MOY, Ph.D. (Michigan) Instr., Wayne University.

N. W. NAUGLE, B.A. (A.&M.C. of Texas) Teaching Fellow, Agricultural and Mechanical College of Texas.

J. H. NIES, JR., B.S. (Georgetown) Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

J. D. OCKERT, M.S. (Miami) Instr., University of Illinois.

R. R. PEREZ, Student, University of Denver.

MARVIN ROSENBLUM, Ph.D. (California) Acting Asst. Professor, University of Virginia.

V. J. RYBA, B.S. (St. Bonaventure) Grad. Student, University of Buffalo.

R. T. SANDBERG, B.A. (Alfred) Teaching Fellow, University of Buffalo.

E. C. SCHLESINGER, Ph.D. (Harvard) Instr., Yale University.

J. E. SCROGGS, M.A. (Houston) Asst., Rice Institute.

S. P. SHEN, A.B. (Assumption) Scholar in Physics, Clark University.

VIRGINIA SHUFORD, M.A. (Appalachian S.T.C.) Professor, Reinhardt College.

H. T. SLABY, Ph.D. (Wisconsin) Instr., Wayne University.

J. R. SMART, Student, San Jose State College.

BERNARD SMILOWITZ, Pittsburgh, Pennsylvania.

D. A. STORVICK, Ph.D. (Michigan) Instr., Iowa State College.

D. J. STRUIK, Ph.D. (Leiden) Professor, Massachusetts Institute of Technology.

PATRICK SUPPES, Ph.D. (Columbia) Asso. Professor, Stanford University.

ROBERT TATES, B.S. (Rochester) Grad. Student, Syracuse University.

SCOTT TAYLOR, Student, University of California, Berkeley.

W. F. TRENCH, A.M. (Pennsylvania) Asst. Instr., Electrical Engineering, University of Pennsylvania.

T. E. WADDILL, B.S. in Ed. (S.T.C., Kirksville, Mo.) Instr., Chillicothe High School, Mo.

R. C. WARNER, B.A. (Toronto) Grad. Asst., University of Rochester.

J. H. WELLS, M.S. (Texas Tech. C.) Teaching Asst., University of Texas.

D. H. WENTWORTH, M.S. (North Dakota) Instr., University of North Dakota.

NANCY E. WILHELM, B.S. (Oklahoma A.&M.C.) Engg. Asst., General Electric Co., Cincinnati, Ohio.

REV. J. J. WINDOLPH, M.S. (Notre Dame) Instr., Quincy College.

MARY E. B. WIRTZ, M.S. (Louisiana S.U.) Instr., McNeese State College.

R. L. YORK, Student, Siena College.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The Fall meeting of the Minnesota Section of the Mathematical Association of America was held at South Dakota State College in Brookings, South Dakota, on October 15, 1955. Professor H. B. MacDougal presided at the morning session, and Professor E. J. Camp, Chairman of the Section, presided at the afternoon session.

There were 35 persons in attendance, including 22 members of the Association.

By invitation of the Executive Committee, Professor P. C. Rosenbloom of the University of Minnesota delivered an address at the morning session entitled "On the Fundamental Theorem of Algebra." Abstract of this address follows:

If the complex numbers are defined in terms of Cauchy sequences of complex rationals, then the statement that the complex number field is algebraically closed takes the form: If $P_n(z) = \sum_{k=0}^{N} a_{kn}z^k$ is a sequence of polynomials with complex rational coefficients such that $a_{Nn}=1$ for all n and $\{a_{kn}\}$ is a Cauchy sequence for $0 \leqq k \leqq N$, then there is a Cauchy sequence $\{z_n\}$ of complex rationals such that $P_n(z_n) \to 0$. It is easy to prove this by elementary algebraic methods if one has already shown that for any non-constant polynomial $P(z)$ with complex rational coefficients there is a complex rational z such that $|P(z)| < \epsilon$. The author has given an elementary algebraic proof of this using methods suggested by analytic function theory. It would be desirable to find a direct elementary proof of this lemma which is really a proposition in elementary number theory.

The following short papers were presented:

1. *An application of curve fitting to a problem in acoustics*, by Professor H. L. Black, North Dakota Agricultural College, introduced by Professor A. G. Hill.

For some types of transducers the pattern function, given theoretically in terms of Bessel functions, is commonly approximated trigonometrically. In a particular case it has been found convenient to assume a form,

$$b(0, \phi) = \cos^k n\phi.$$

By proper choice of the parameters, a good fit to the pattern is obtained for a considerable angular distance from the acoustic axis. In reverberation problems use of $b(0, \phi)$ leads to definite integrals of the Wallis type. Closely related problems give rise to somewhat more general definite integrals which are evaluated in terms of gamma functions.

2. *Two illustrations of the application of game theory*, by Professor R. P. Winter, College of St. Thomas.

This paper presents a brief report on the content of *The Compleat Strategyst* by J. D. Williams. Two problems with solutions as given in this neat primer of game theory are discussed.

3. *An application of the Laplace transform*, by Dean H. M. Crothers, South Dakota State College, introduced by Professor H. B. MacDougal.

In this paper the Laplace transform is applied to obtain a solution of a problem concerning an impact power wrench.

4. *On Hölder's inequality*, by Professor R. J. Dowling, College of St. Thomas.

The following inequality is used in text books to prove Hölder's inequality:

$$ab \leqq \frac{a^p}{p} + \frac{b^q}{q},$$

where a, b, p and q are positive numbers and $1/p+1/q=1$. Geometric interpretations of this inequality were presented.

5. *Some remarks on divisibility*, by Professor F. L. Wolf, Carleton College.

The fact that a rational number can be expressed as a repeating decimal may be used to show that every positive integer divides some number that is a string of 9's followed by a string of 0's. Application of Fermat's little theorem gives easily that every prime greater than 5 divides some number that is a string of 1's. In listing the primes different from 2 and 5 by finding the divisors of 11, 111, 1111, . . . , does each number yield at least one prime not already listed?

6. *Report on the Madison Conference*, by Professor K. W. Wegner, Carleton College.

This conference was different from others sponsored by the National Science Foundation in that its emphasis was on teacher education. Only about a quarter of those attending were college teachers. Over a third of the scheduled meetings was devoted to studies in curriculum and instruction, mainly at the secondary school level.

This report included a presentation of a new proof of the theorem which states that every real positive number has a real kth root; this proof was given by F. Kiokomeister in one of the lecture courses at the Madison Conference.

F. C. SMITH, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The Fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma City, Oklahoma, on October 28, 1955. Professor R. R. Murphy, Chairman of the Section, presided.

There were 124 in attendance, including 84 members of the Association. This meeting is the annual meeting at which papers of special interest to high school and freshman and sophomore college teachers are presented. Research type papers are presented at a later meeting.

The following officers were elected for one-year terms: Chairman, Professor Truman Wester, Central State College; Vice-Chairman, Professor Paul Sanders, Southeastern State College; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma. The business meeting included also a discussion of proceedings at the meeting of section officers at Ann Arbor, Michigan. The group voted to continue meeting in conjunction with the Oklahoma Educational Association in the Fall and to hold the Spring 1956 meeting in Tulsa, Oklahoma. A motion to investigate the possibility of holding the second meeting during the 1956–57 academic year in conjunction with the Oklahoma Academy of Science, if possible, passed.

The following papers were presented:

1. *Classroom notes in elementary integral calculus*, by Professors Simon Green and M. Schwartz, University of Tulsa.

The attempt was made to fill part of the gap arising in most elementary books introducing students to the definite integral. Firstly, the books select regular intervals and secondly they demonstrate the simplest type of problem as a limit of the sum of a set of rectangles bounded from above and below. They leave the more difficult problems as exercises to be performed by the students who usually have not the necessary experience to cope with them. Examples bringing out clearly the independence of the mode of subdivision and solutions of problems of a more difficult nature were demonstrated.

2. *Publications of books and other materials needed in mathematics classes on the high school level*, by Miss Vivian Spurgeon, Nowata High School, Oklahoma.

Publications and other materials needed by the mathematics classes on the high school level were discussed. The materials needed are (1) supplementary materials for class work and (2) enrichment and general background materials. Teachers of high school mathematics should help prepare publications.

3. *A Carnegie Foundation project in modern mathematics for college sophomores—preliminary report*, by Professor R. V. Andree, University of Oklahoma.

The Carnegie Corporation is sponsoring an experiment in the inclusion of some mathematics other than calculus in the college sophomore course and the inclusion of examples from fields other than physics. It is suggested that about $\frac{3}{4}$ of the academic year be spent on calculus, utilizing the (polynomial) calculus contained in many freshman courses. The remaining $\frac{1}{4}$ academic year will be spent on material mainly from abstract algebra. Interested persons were encouraged to participate in the afternoon discussions or to write to the speaker expressing their opinions.

4. *What is the center of a triangle?* by Professor Arthur Bernhart, University of Oklahoma.

Each familiar notable point is introduced by a property which characterizes its central location. Besides the usual minimax properties, the points of view of physics, vector analysis, projection, perspectivity, game theory, and pursuit contribute to the exposition.

5. *Mathematics in industry*, by Mr. R. B. Rice, Phillips Petroleum Company, Bartlesville, Oklahoma.

This paper was presented at the luncheon held in conjunction with the Oklahoma Education Association. The speaker emphasized the greatly enlarged need for competent mathematicians in the petroleum industry.

The afternoon was spent in a discussion of basic problems which have arisen in the development of a new course in sophomore mathematics based on the premise that the student has had an introduction to polynomial calculus in his freshman work, and of what should and should not be included in such a course.

R. V. ANDREE, *Secretary*

ERRATA

1. A. Oppenheim, *Inequalities connected with definite Hermitian forms*, II, vol. 61, 1954, pp. 463–466. The first proof of Theorem 1 is incorrect as pointed out by the reviewer (*Math. Rev.*, vol. 16, 1955, p. 328). The second proof (by the referee) is correct.

2. L. E. Bush, *The William Lowell Putnam Mathematical Competition*, vol.

62, 1955, pp. 558–564. On page 560, problem 3, for $x_n = \sum_{i=n+1}^{\infty} x_i$ read $x_n \leqq \sum_{i=n+1}^{\infty} x_i$.

3. E. T. Parker and A. M. Mood, *Some balanced Howell rotations for duplicate bridge sessions*, vol. 62, 1955, pp. 714–716. The four-table rotation is not balanced as claimed. In the table on page 714, the cyclic permutations at Tables 2, 3, and 4 should begin with 5, 3, and 2 instead of 4, 6, and 7 respectively. The last paragraph of Section 1 should have 3, 4, 5 changed everywhere to 2, 3, 7.

4. Tomlinson Fort, *Linear difference equations and the Dirichlet series transform*, vol. 62, 1955, pp. 641–645. At the bottom of page 642, in II, instead of:

$$\sum_{t=0}^{\infty} \frac{m^{-st}(t+1)}{m^{-s}-1} \quad \text{read} \quad \sum_{t=0}^{\infty} \frac{m^{-s(t+1)}}{m^{-s}-1};$$

$$\frac{m^{-s}}{(m^s-1)^2} \quad \text{read} \quad \frac{m^s}{(m^s-1)^s}.$$

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, April 28, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY, University of Kentucky, Lexington, April 28, 1956.

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1956.

MINNESOTA, Augsburg College, Minneapolis, April 28, 1956.

MISSOURI, Fontbonne College, St. Louis, April 21, 1956.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Oberlin College, Oberlin, April 14, 1956.

OKLAHOMA, Tulsa, March 31 or April 2, 1956.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

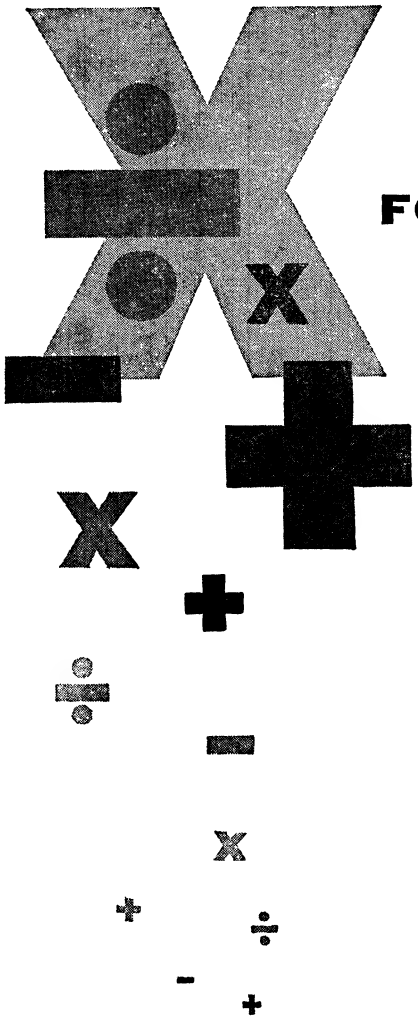
SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April 20–21, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.



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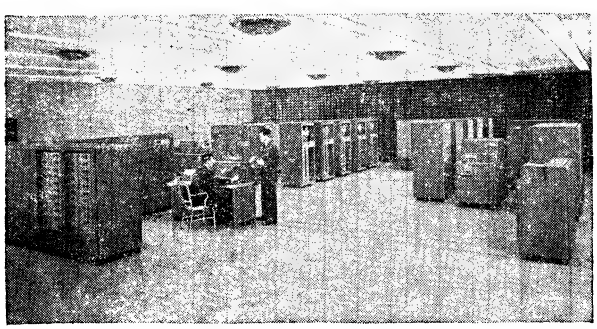
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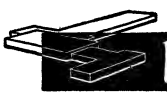
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STATISTICAL DECISIONS*

E. S. KEEPING, University of Alberta

The name of Abraham Wald, who died in an airplane crash in 1950, will always be associated with statistical decision theory, which ranks as one of the three greatest advances in mathematical statistics within comparatively recent times. The first of these advances, started by W. S. Gosset (better known under the familiar pen-name of Student) but due above all to the grand old man among living statisticians, Sir Ronald Fisher, was the development of small sample theory. Exact distributions of the mean, the variance, the correlation coefficient and other statistics, at least for samples from certain types of populations, were obtained, tables were computed, and the experimental statistician was armed with a whole set of techniques, *t*-tests, *F*-tests, analysis of variance, *etc.*, which he has been happily applying ever since.

The second advance, associated particularly with J. Neyman and E. S. Pearson, was the theory of estimation and the testing of hypotheses. The concept of a confidence interval was developed. The statistician was regarded as interested primarily in testing a null hypothesis, such as the hypothesis that two observed samples come from normal populations with the same mean, against some reasonable alternative hypothesis. One of the most important ideas that grew up in this second period was that of the power of a test. A practical development of this theory was the compilation of tables for finding the sizes of samples necessary to discriminate in certain cases, at a given level of confidence, between one hypothesis and another.

Wald's theory of statistical decision functions [1] began a third era in statistical analysis and interpretation. Because the theory was initially developed with great mathematical rigor, so that most applied statisticians found the going pretty rough, the theory did not at first make much headway except in rather exclusive circles. Lately, however, several new books [2] and even two popularizations [3] have appeared, but it is still broadly true that the theory is hardly in a form usable by the practicing experimental statistician. The operations research analysts are, however, vigorously pursuing practical aspects of decision theory. It is with the use of decision theory in statistics that I am particularly concerned to-day, and I want to explain the underlying ideas as simply as I can.

Any statistician, and this includes any experimenter who makes measurements or observations on some quantity subject to random variation, is faced at times with the necessity of making a decision, even if the decision is only what value to adopt for the observed variable. In sequential sampling of a consignment of manufactured articles, the decision may be to recommend either the acceptance or rejection of the lot, or, as a third alternative, to do some more sampling. The practical man would therefore like a useful principle to guide his

* Presented to the Pacific Northwest Section of the Association on June 17, 1955.

choice of a decision, but the application of any such principle must depend largely on the circumstances of the particular problem and often will involve numerical constants which are known only very roughly.

Wald set up a principle which in most cases seems reasonable, even though in some special problems it recommends a course of action which common sense would scarcely approve. A wrong decision will as a rule entail a loss, and that rule which minimizes the expected loss could be regarded as the best. The risk of making a wrong decision can often be reduced to the vanishing point by taking enough observations, but experiments usually cost money, and the risk function which Wald introduced depends on both the cost of experimentation and the cost of making a wrong decision, estimates of these costs being supplied by the experimenter. In order to calculate the expected value of the total loss it is also necessary to know something about the *a priori* probabilities of the various possible states of Nature which might be connected with the different decisions, in particular the various statistical populations which the experimenter might be regarded as sampling, and this sort of information is hard to get.

If we could calculate the total risk under all decision rules, we would naturally choose the rule which minimizes this risk, and this would clearly be an optimum procedure. To avoid the difficulty of unknown *a priori* probabilities there are different methods open to us. One is the way adopted, though apparently with some misgivings, by Thomas Bayes, and later used by Laplace—to assume that, in the absence of any other information, the *a priori* probabilities are all equal. This is the so-called Principle of Insufficient Reason, which was vigorously attacked by Fisher. Wald's method was the *Minimax Principle*, the principle of choosing a rule which minimizes the maximum risk which could occur, whatever the unknown state of Nature might be. This amounts to the rather pessimistic attitude of expecting the worst and acting accordingly, a rule which often turns out to be wise. It is not surprising, however, that various modifications of the minimax principle seem more reasonable in particular circumstances. The general theory of decisions does not stand or fall by the minimax principle.

Statistical decision theory is closely tied up with the theory of games, another topic which has been actively pursued in recent years [4]. The theory of games has obvious applications in warfare and in certain economic situations, but it has lately become apparent that many statistical problems can be regarded as a sort of game, a two-person game which the statistician plays with Nature. His strategy consists in making certain decisions, before or after conducting a series of experiments or trials. Nature, of course, cannot be considered as a conscious opponent who can take advantage of mistakes made by the statistician, but she has a wealth of possibilities in any situation and the statistician wants to know which of these possibilities is the one that actually obtains. In a two-person game against an intelligent opponent the minimax rule is quite reasonable—

play as if your opponent could always be counted on to make the best possible moves in the circumstances, but in games against Nature it is often unduly pessimistic.

In many games the decisions regarding strategies are clear-cut; in such-and-such circumstances one will do so-and-so. These are *pure* decisions. In other games the choice between decisions may depend on a random process like dealing cards or rolling a die. This implies that there is, at a certain stage of the game, a probability distribution over the set of possible decisions, with probabilities governed by the randomizing device, and such decisions are called *mixed.* In the children's game of Snakes and Ladders all moves are mixed, the decision to move a certain number of places, from 1 to 6, depending entirely on the chance throw of a die. In the game of the statistician against Nature, a mixed strategy for Nature is an *a priori* probability distribution of the parameters in a law.

Although in practice two-person games consist of alternate moves by the players, yet in theory each player could formulate his complete strategy for the whole game ahead of time, stating what counter-move he would make to each possible move by his opponent. In chess, White has an initial choice of 20 legal opening moves and makes a decision. Black replies with one of 20 moves. White's second move may depend on which of the 20 Black chooses, and his decision could be specified for each case. It is clear, however, that any actual enumeration of strategies for a whole game is not feasible, except in the very simplest games such as ordinary tic-tac-toe. Yet it is very convenient to think of a game, in the general sense, as a product of two spaces, a space X of all possible strategies by player I, and a space Y of all possible strategies by player II, together with a function $M(x, y)$ which specifies for each x belonging to X and each y belonging to Y what the pay-off will be to player II. Of course there is also a pay-off to player I, but in zero-sum games, with which we are concerned, the total pay-off is zero, so that player I's pay-off is $-M(x, y)$.

If player I uses a mixed strategy, choosing x_i with probability λ_i, then $x = \sum_i \lambda_i x_i$ is an element of X, and similarly $y = \sum_j \mu_j y_j$ is an element of Y. For each pair of strategies x_i, y_j, there will be an outcome r, depending on the rules of the game, or, if there are chance moves in the game, there will be a probability $p_{ij}(r)$ for the outcome r. Associated with each r is also a bounded numerical function $u(r)$ which may be regarded as the monetary gain to player II (or some monotone increasing function of this gain), supposing that the outcome r occurs.

The pay-off to player II is the expected value of $u(r)$, namely

$$M_{ij} = M(x_i, y_j) = \sum_{r \in R} p_{ij}(r)u(r), \tag{1}$$

where R is the space of all possible outcomes.

The game may thus be considered as a matrix (M_{ij}) in which the columns represent strategies of player I, the rows strategies of player II, and the (i, j)th

element the pay-off $M(x_i, y_j)$. Player I chooses a column

		I			
		x_1	x_2	$\cdots$	x_m
II	y_1	M_{11}	M_{21}	$\cdots$	M_{m1}
	y_2	M_{12}	M_{22}	$\cdots$	M_{m2}
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	y_n	M_{1n}	M_{2n}	$\cdots$	M_{mn}

and player II simultaneously chooses a row, neither knowing the other's choice. The pay-off to II is the corresponding entry in the matrix. If either player uses a mixed strategy, he chooses two or more pure strategies with probabilities λ_i or μ_j.

It is not difficult to devise particular pay-off schemes in which different criteria for the best strategy lead to different decisions [5]. An example is the following:

	x_1	x_2	x_3	x_4
y_1	2	2	0	1
y_2	1	1	1	1
y_3	0	4	0	0
y_4	1	3	0	0

The Laplace method assumes that Player I is equally likely to employ any one of his four strategies (in the game of the statistician against Nature, the *a priori* probabilities for all values of an unknown parameter are assumed equal). The expectation of II for strategy y_j is the average gain $(\sum_i M_{ij})/m$, and he should choose the row which maximizes this. In the example, these values are 5/4, 1, 1, 1, so that he should choose y_1.

The minimax principle of Wald states that the player should choose that row for which his smallest possible gain is a maximum. In the four rows of the example the minimum gains are 0, 1, 0, 0, so that player II should prefer strategy y_2. Whatever happens he cannot then gain less than one unit.

Hurwicz [7] has modified Wald's criterion so as to remove some of its pessimistic character. He suggests selecting a number $\alpha(0 \leqq \alpha \leqq 1)$ which may be regarded as measuring player II's optimism. For each row (or probability mixture of rows when mixed strategies are allowed), let m_i be the least value of M_{ij} and M_i the greatest. Player II should choose that row for which $\alpha M_i + (1-\alpha)m_i$ is a maximum. When $\alpha = 0$ this clearly reduces to Wald's criterion. In the example above, the four rows give 2α, $\alpha+(1-\alpha)$, 4α, 3α respectively, so that, as long as $\alpha > \frac{1}{4}$, strategy y_3 is to be preferred to the others.

It was pointed out by Savage [6] that it is often reasonable to minimize what

he called the *maximum regret.* Player II's regret is the difference between the pay-off for his actual strategy y_j and the pay-off he might have got if he had known beforehand what strategy x_i player I would adopt, in which case he would naturally have selected that strategy for which M_{ij} (given i) is a maximum. The regret may be defined as

$$r_{ij} = \max_j M_{ij} - M_{ij}. \tag{2}$$

Player II then chooses the y_j which minimizes the maximum over all i of r_{ij}.

In the example above, when $i=1$, max $M_{ij}=2$, so that $r_{11}=0$, $r_{12}=1$, *etc.* The regret matrix is

	x_1	x_2	x_3	x_4
y_1	0	2	1	0
y_2	1	3	0	0
y_3	2	0	1	1
y_4	1	1	1	1

The maxima of r_{ij} for the different rows are 2, 3, 2, 1, so that on Savage's criterion player II would choose strategy y_4. He is sure then of not missing the boat by more than 1 unit.

Four different criteria lead therefore in this example to four different decisions regarding the choice of strategy. No strategy here dominates all the others in the sense of being at least as good whatever player I does, and definitely better for at least one possibility.

It has been objected to the Savage criterion that if a new strategy is added to the old set (an extra row in the matrix) the principle of minimax regret may dictate a different strategy from the one previously preferred, even though both of these were in the original set. That is, the relative order of preference among the y_j may be altered by an additional y. For example, if to the set

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

we add a third row $(-7\ 4)$, the regret matrix changes from

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 0 & 3 \\ 2 & 2 \\ 9 & 0 \end{pmatrix}$$

and the maximum regret per row changes from

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix}.$$

The principle of minimizing this maximum regret would therefore suggest using strategy 1 *before* the addition of strategy 3, but strategy 2 *afterwards*. However, all the practicable criteria so far devised are subject to somewhat similar criticisms [8].

The above discussion assumes that in the kind of games played, or the kind of statistical decisions made, there is in fact a numerically measurable pay-off, if not in actual dollars and cents, at least in what the economists call "utility." A dollar does not mean the same thing to a rich man and to a poor man, or in other words the utility of a unit of money appears to diminish as one's wealth increases, being perhaps inversely proportional to this wealth (although this is not true of inveterate gamblers who will often gamble more wildly, the less money they have). There has been much argument regarding the validity of the theory of utility, and the major criticisms are dealt with by Savage [2], and by Arrow [11]. In the statistical field, numerical values can very seldom be assigned to the consequences of decisions, except in the sense that a "true" decision (one that is confirmed by subsequent research) is certainly preferable to a false one, so that one could reasonably associate a certain unit loss with a wrong decision and a zero loss with a correct decision. The statistician protects himself to some extent by making assertions which are, as far as he can tell, unlikely to be in error more than a known fraction (say 5%) of the time.

In the game of the statistician against Nature, there will usually be one or more parameters ω, belonging to a parameter space Ω, which the statistician desires to estimate. The possible values of ω are the pure strategies for player I (Nature), and an *a priori* probability distribution for ω is a mixed strategy. In game theory the spaces X and Y are finite, but in statistics the space Ω is usually infinite.

The statistician (player II) performs experiments to spy on Nature. The outcome of an experiment will be a value of z belonging to the sample space Z. An event is a set S of values of z and will have a probability, depending on ω, given by

$$P_\omega(S) = \sum_{z \in S} p_\omega(z). \tag{3}$$

As a result of the experiments the statistician will take an action a belonging to a set A of possible actions. The function $d(z) = a$ which maps the space Z onto A is a *statistical decision function*. The space Z may be supposed divided into disjoint sets S_a, each set corresponding to one action a, so that if z falls in S_a, the action a is taken. A familiar example is the ordinary one-tailed t-test, in which the space is the real axis, divided into two regions, $t > t_\alpha$ and $t < t_\alpha$. If the observed t falls in the former region the null hypothesis is rejected, and if in the latter region it is accepted. The class D of all possible decisions is the class of pure strategies for the statistician. If he flips a coin to decide which of two decisions to adopt, that is a mixed strategy. The class D is often infinite.

The statistician does not usually stand to win anything tangible in his game

with Nature, but he may lose. Experiments cost money or time or both, and a decision based on a false estimate of the parameter ω may be expensive. Although in practice it is often difficult or impossible to estimate with any accuracy the loss which may be caused by wrong decisions, we can suppose for the sake of theory that there is a numerical loss, the magnitude of which, $L(d, \omega)$, depends on the true value ω and the decision d. The pay-off function for the statistician's game is the expected value of this loss, namely

$$M(d, \omega) = \sum_z L(d, \omega) p_\omega(z), \tag{4}$$

in which $d = d(z)$. He should try to act so as to minimize the expected loss.

Occasionally the minimax principle will suggest a different estimate from the customary one. Suppose we wish to estimate the parameter θ (the probability of success), for a binomial distribution, on the basis of our observation that the event has occurred x times in n trials. The usual estimate is, of course, x/n. This estimate is unbiased, since the expectation of x/n, given θ, is precisely θ. It is also the maximum likelihood estimate, since the probability of x/n, given θ, is a maximum when $\theta = x/n$. But it is not the minimax estimate.

If our estimate of θ is $f_n(x)$, our loss may be regarded as proportional to the square of the difference between $f_n(x)$ and the true value. This seems reasonable since, if there were a linear term, the loss would change sign with $f_n(x) - \theta$, so that an error in one direction would represent a gain. The expectation of loss is, for a given θ,

$$kE\{f_n(x) - \theta\}^2 = k\sum_{x=0}^{n} \{f_n(x) - \theta\}^2 p_n(x), \tag{5}$$

where

$$p_n(x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}. \tag{6}$$

If $\lambda(\theta)d\theta$ is the probability, before any trials are made, that θ lies in the interval θ to $\theta + d\theta$, the expectation of loss is

$$L = \int_0^1 kE\{f_n(x) - \theta\}^2 \lambda(\theta) d\theta. \tag{7}$$

An estimate which minimizes this is called a *Bayes estimate.* It may be shown that this estimate is equal to the expectation of θ, given x, which can be calculated by Bayes' theorem if $\lambda(\theta)$ is known. If for example $\lambda(\theta)$ is constant, $f_n(x)$ turns out to be $(x+1)/(n+2)$, an estimate which would minimize our expected loss if we were totally ignorant beforehand of the value of θ, but knew that all possible values were equally likely.

A Bayes estimate is *minimax* if the expectation of loss for a given θ is the same whatever the value of θ. This is so for the estimate $f_n(x) = (x + \frac{1}{2}\sqrt{n})/(n + \sqrt{n})$, which is also a Bayes estimate when $\lambda(\theta) = c\theta^a(1-\theta)^a$, where $a = \frac{1}{2}\sqrt{n} + 1$. The

minimax estimate of θ is therefore $(x+\frac{1}{2}\sqrt{n})/(n+\sqrt{n})$. If for instance we made four trials and observed one success, the ordinary estimate of θ would be $\frac{1}{4}$. The minimax estimate would be $\frac{1}{3}$. The loss for the ordinary estimate is $k\theta(1-\theta)/4$, and that for the minimax estimate is $k/36$. For any value of θ between 0.13 and 0.87 the minimax estimate is the better.

In the field of quality control, there is a practical need for decisions. A factory, for example, buys large numbers of small parts which come in lots and may contain a certain proportion θ of defectives. Complete inspection of the lots is costly, and no inspection at all may permit very poor lots to get by, with resultant trouble and expense to the factory.

Suppose that the parts come in lots of N, that the cost of inspecting an item is c_1 and that of scrapping a defective item is c_2, which is presumably much greater than c_1. There are two possible decisions, d_1 (to accept the lot) and d_2 (to inspect it 100%), the costs of these actions being $N\theta c_2$ and Nc_1 respectively, since with complete inspection no defectives will be purchased. If $\lambda(\theta)$ is the *a priori* probability density for θ, and if the probabilities for d_1 and d_2 are μ and $1-\mu$ respectively, the expected loss is

$$L(\lambda, \mu) = \int_0^1 \lambda(\theta)\{\mu N\theta c_2 + (1-\mu)Nc_1\}d\theta \tag{8}$$

$$= Nc_1 + N\mu\int_0^1 \lambda(\theta)(\theta c_2 - c_1)d\theta.$$

This obviously depends upon λ as well as μ. In practice there will usually be some previous experience to draw upon in estimating λ. If there is no such information the minimax principle suggests the unreasonable strategy $\mu=0$ (that is, never accept a lot without complete inspection), since there is a possibility that $\lambda(\theta)$ might be 0 for $\theta<c_1/c_2$, and $L(\lambda, \mu)$ is then minimized for $\mu=0$. The alternative assumption that $\lambda(\theta)$ is constant gives $L(\lambda, \mu)=Nc_1+\frac{1}{2}N\mu(c_2-2c_1)$, so that the minimum loss is given for $\mu=0$ if $c_2>2c_1$ but for $\mu=1$ if $c_2<2c_1$.

One solution of the difficulty is to inspect n items out of a lot and decide to accept the lot if the number of defectives is less than s but to resort to complete inspection if $x \geqq s$. This amounts to putting

$$\mu = \sum_{x=0}^{s-1} p_n(x) = 1 - P(s, n, \theta), \tag{9}$$

where $P(s, n, \theta)$ is the cumulative binomial probability function, now extensively tabulated [9].

The cost of decision d_1 is $nc_1+(N-n)\theta c_2$, so that the expected loss is

$$L(\lambda, s) = Nc_1 + (N-n)\int_0^1 \lambda(\theta)(c_2\theta - c_1)[1 - P(s, n, \theta)]d\theta. \tag{10}$$

The minimax principle still advises 100% inspection whatever the result of

the sampling. The Laplace principle, $\lambda(\theta) = \text{constant}$, advises a choice of s as near as possible to the value of $(c_1(n+2)/c_2) - \frac{1}{2}$. For $c_2/c_1 = 25$, $N = 1000$, $n = 100$, this gives $s = 4$. The advice is to accept a lot if in a sample of 100 there are not more than three defectives.

Any solution which minimizes $L(\lambda, s)$ for a given $\lambda(\theta)$ is a Bayes solution, and in this problem any Bayes solution boils down to a choice of the integer s. If for example $\lambda(\theta) = 25$ for $0.01 < \theta < 0.05$ and $\lambda(\theta) = 0$ for all other values of θ (that is, θ has a rectangular distribution over an interval of length 0.04), the loss is given by

$$L(s) = Nc_1 + \frac{25(N-n)}{n+1}\left\{\sum_{x=0}^{s-1}\frac{c_2(x+1)}{n+2}[P(x+2, n+2, 0.05) - P(x+2, n+2, 0.01)] - c_1\sum_{x=0}^{s-1}[P(x+1, n+1, 0.05) - P(x+1, n+1, 0.01)]\right\}. \tag{11}$$

For the data mentioned above, the minimum loss is given by $s = 7$. With the fuller information now available on θ it is economical to accept a lot if there are not more than six defectives in a sample of 100.

This problem can also be treated from the point of view of minimax regret. If we suppose that there is in fact a true, but unknown, probability θ for the occurrence of defectives in the lots supplied by the manufacturer, the loss suffered by the purchaser is, per lot,

$$L(s, \theta) = Nc_1 + (N-n)(c_2\theta - c_1)[1 - P(s, n, \theta)] \tag{12}$$

for $s \geqq 1$.

The regret function is the difference between the actual loss and the minimum loss that would be incurred even if the true value of θ were known. Symbolically,

$$r(s, \theta) = L(s, \theta) - \inf L(s, \theta). \tag{13}$$

For each possible s, one can calculate what value of θ would make $r(s, \theta)$ a maximum, and then decide on the particular s which would minimize this maximum regret. For the data given above the calculations suggest that the best strategy in Savage's sense is to take $s = 4$, that is, to accept without inspection a lot with not more than three defectives in a random sample of 100. This example illustrates how the wise decision in such a case may depend not only on the extent of the information available on the value of the unknown parameter but also on the criterion adopted for judging the best decision.

Wald's most well-known contribution to statistical theory is undoubtedly the sampling procedure called *sequential analysis*. The idea was that observations should not all be taken at once, before being analyzed, but that they should be accumulated in successive stages. At each stage the statistician would have to

make a decision, either to take some definite action or to continue with the observations, this decision depending on all the observations available up to that stage. Wald set up a decision rule, the *sequential probability ratio test*, which on certain reasonable assumptions turned out to be remarkably efficient in economizing observations, and which has been quite widely adopted in practice.

We will consider a simplified problem to which sequential analysis is applicable. Suppose we have to decide between two alternative hypotheses H_1 and H_2 of which the *a priori* probabilities are g and $1-g$ respectively. The loss incurred in accepting H_2 if H_1 is true is w_{12}, and the loss in accepting H_1 if H_2 is true is w_{21}. (It is assumed that there is no loss in accepting a *true* hypothesis.) The cost of a single observation is a constant c, which we may take as unity, and all the observations are supposed independent of each other and subject to a common distribution law. The hypotheses concern the value of a parameter p in this law, H_1 being that $p=p_1$ and H_2 that $p=p_2$.

If g is sufficiently small it will be wise to accept H_2 without more ado, and run the risk $R_1=w_{12}g$. Similarly, if g is sufficiently large, it will be wise to accept H_1 with risk $R_2=w_{21}(1-g)$. The whole interval $0\leqq g\leqq 1$ can be divided into three parts, $I_1(0\leqq g\leqq \underline{g})$, $I_2(\underline{g}<g<\bar{g})$, and $I_3(\bar{g}\leqq g\leqq 1)$, such that if g belongs to I_1 we accept H_2, if to I_3 we accept H_1, but if to I_2 we take one or more observations. In this case there will be a sequential test with an average risk less than either R_1 or R_2.

If we take n observations, the *a posteriori* probability of H_1 will be given by Bayes' rule:

$$g_n = gP_{1n}/[gP_{1n} + (1-g)P_{2n}]. \tag{14}$$

where P_{1n} is the probability of the observed set of results on H_1, and P_{2n} the probability of the same set on H_2. The experimentation is stopped as soon as g_n falls either in I_1 or in I_3, the decision being to accept H_2 or H_1 in the respective cases. As long as g_n falls in I_2 the sampling continues.

Writing $W_n=P_{2n}/P_{1n}$, we can easily verify that this procedure is exactly the same as Wald's probability-ratio test, the ratio W_n being computed at each stage, and sampling continued as long as $B<W_n<A$, where A and B are two numbers, depending on g, $\underline{g}$ and $\bar{g}$,

$$A = \frac{1-\underline{g}}{\underline{g}}\cdot\frac{g}{1-g}, \qquad B = \frac{1-\bar{g}}{\bar{g}}\cdot\frac{g}{1-g}. \tag{15}$$

For convenience we can suppose that hypotheses H_1 and H_2 correspond to symmetric binomial dichotomies, p_1 being the proportion of "successes" under H_1, and p_2 the proportion under H_2, with $p_2=1-p_1=q_1$. We can take $p_1<p_2$. If s_n is the number of successes in n trials, and therefore $n-s_n$ the number of failures,

$$\frac{P_{2n}}{P_{1n}} = \frac{p_2^{s_n} q_2^{n-s_n}}{p_1^{s_n} q_1^{n-s_n}} = \left(\frac{q_1}{p_1}\right)^{2s_n-n}. \tag{16}$$

Let x be an observed variable, which takes only the values 1 for a success and -1 for a failure. Then, after n observations,

$$\sum_{i=1}^{n} x_i = s_n - (n - s_n) = 2s_n - n, \tag{17}$$

so that, from (16) and (17),

$$W_n = P_{2n}/P_{1n} = u^z,$$

where $u = q_1/p_1$, and $z = \sum x_i$. The probability ratio test is therefore equivalent to noting z after each observation and continuing as long as $-b < z < a$, where $-b = \log B/\log u$ and $a = \log A/\log u$. When $z \geqq a$ we accept H_2, and when $z \leqq -b$ we accept H_1. Since z will always be an integer, the test will end when $z = a'$ or $z = -b'$, where a' stands for the smallest positive integer $\geqq a$.

When $g = \underline{g}$, $A = 1$ and therefore $a = 0$. The risk of accepting H_2 without observations is $w_{12}\,\underline{g}$. This risk ought to be equal to that of the sequential test with $a' = 1$, $b' = b^*$ where b^* is the smallest positive integer greater than or equal to

$$-\log\left(\frac{1-\bar{g}}{\bar{g}} \cdot \frac{\underline{g}}{1-\underline{g}}\right) \Big/ \log u.$$

Similarly, if $g = \bar{g}$, we have $b = 0$, $(b' = 1)$ and $a' = b^*$, and the risk for this sequential test should be equal to $w_{21}(1 - \bar{g})$.

If L_1 and L_2 are the probabilities of accepting H_1 as a result of a sequential test with limits a' and $-b'$, given that H_1 is true and that H_2 is true respectively, it can readily be shown that

$$\begin{aligned} L_1 &= (u^{a'+b'} - u^{b'})/(u^{a'+b'} - 1), \\ L_2 &= (u^{a'} - 1)/(u^{a'+b'} - 1). \end{aligned} \tag{18}$$

Also, if the expected numbers of observations required to lead to a decision in these two cases are E_1 and E_2 respectively,

$$\begin{aligned} E_1 &= [(a' + b')L_1 - a']/(q_1 - p_1), \\ E_2 &= [a' - (a' + b')L_2]/(q_1 - p_1). \end{aligned} \tag{19}$$

The risk involved in the sequential test is

$$R = g[E_1 + w_{12}(1 - L_1)] + (1 - g)[E_2 + w_{21}L_2]. \tag{20}$$

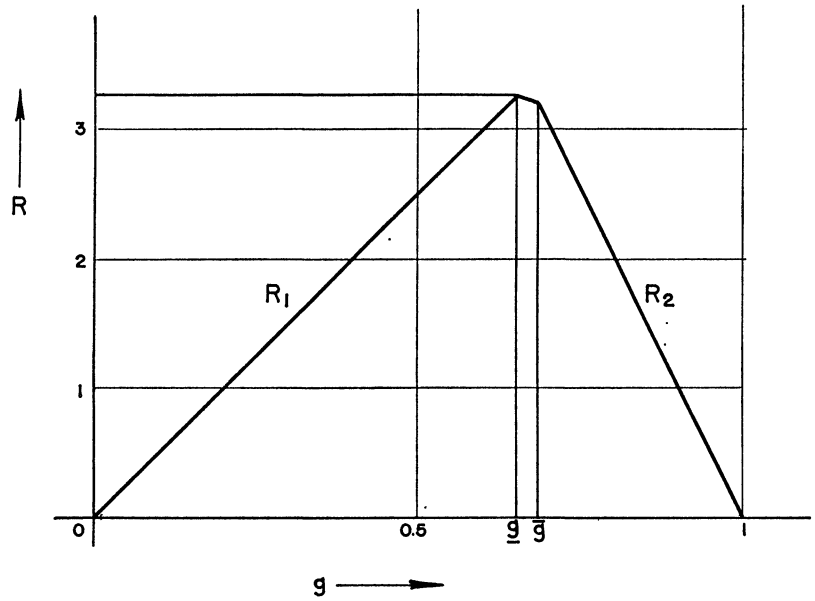
Since a' and b' are integers, the coefficients of g and $1 - g$ will change abruptly at discrete values of g, so that the curve of R will consist of straight line segments. The values of $\underline{g}$ and $\bar{g}$ are given by

$$w_{12}\underline{g} = \underline{g}\left[\frac{(1+b^*)L_1 - 1}{q_1 - p_1} + w_{12}(1 - L_1)\right] + (1 - \underline{g})\left[\frac{1 - (1+b^*)L_2}{q_1 - p_1} + w_{21}L_2\right] \tag{21}$$

and

$$w_{21}(1 - \bar{g}) = \bar{g}\left[\frac{(b^*+1)L_1 - b^*}{q_1 - p_1} + w_{12}(1 - L_1)\right] + (1 - \bar{g})\left[\frac{b^* - (b^*+1)L_2}{q_1 - p_1} + w_{21}L_2\right], \tag{22}$$

where in (21) L_1 and L_2 are found by putting $a'=1$, $b'=b^*$ in (18) and in (22) by putting $a'=b^*$, $b'=1$. The method of solution is to guess a value of b^*, find $\underline{g}$ and $\bar{g}$ and see whether b^* then satisfies its definition. If we take, as a numerical example, $p_1=\frac{1}{3}$, $p_2=\frac{2}{3}$, $w_{12}=5$, $w_{21}=10$, we find the equations are satisfied by $b^*=1$, $\underline{g}=13/20$, $\bar{g}=17/25$. The curve (20) here consists of a single segment. Since x must be either 1 or -1, the sequential test will terminate after one observation.



The maximum risk is $R=13/4$, corresponding to $g=\underline{g}=13/20$. This is the best strategy for Nature.

The statistician employing the minimax principle can either accept H_2 with risk $5g$ or use the sequential test with limits 1 and -1 with risk given by (20).

These risks agree at $g=\underline{g}$ but not for other values. However, he can employ a mixed strategy, accepting H_2 with probability μ and doing the test with probability $1-\mu$, μ being so chosen that the total risk, $\mu(5g)+(1-\mu)R$, is independent of g. For the data used in the example above it turns out that $\mu=\frac{1}{4}$. The minimax strategy consists, then, in using some randomizing device, such as a table of random numbers, to decide either (with probability $\frac{1}{4}$) on accepting H_2 without taking any observations or (with probability $\frac{3}{4}$) on doing a sequential test with limits 1 and -1.

Further examples of dichotomies like the above, and also of trichotomies, in which the various *a priori* distributions are represented by regions in a triangle, are given by Arrow, Blackwell and Girshick [10].

It may be noted that the sequential process is analogous to a game between two players, A and B. At each trial either A pays B one dollar (probability p_1) or B pays A one dollar (probability p_2). A starts with capital a' dollars and B with capital b' dollars. The game ends when either A or B is ruined. The quantity z is the net amount paid by A to B. The probability that B will eventually be ruined is L_1, as given by (18).

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ON A CLASS OF DETERMINANTS ASSOCIATED WITH A MATRIX

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Let A be an $n \times n$ matrix with elements in a field F and let $g(x)$, the minimal polynomial of A, be of degree s. We shall say that A is diagonalizable if there is a non-singular $n \times n$ matrix P with elements in some extension field of F such that PAP^{-1} is in diagonal form.

Let Tr denote the trace function, and for each positive integer m define

$$\Delta_m = \det [\mathrm{Tr}\ (A^{i+j})] \qquad 0 \leqq i, j \leqq m - 1.$$

There are several interesting relationships between determinants of this type and certain properties of the matrix A. For example, in a recent paper by H. Flanders and the author [1] the following theorem was proved:

THEOREM 0. *If F is of characteristic zero then A is diagonalizable if and only if $\Delta_s \neq 0$.*

Actually this theorem is a consequence of the following more general property of these determinants:

THEOREM 1. *Suppose A has exactly r different characteristic roots. Then:*

1. *If F is of characteristic zero then r is equal to the maximum value of m for which $\Delta_m \neq 0$;*
2. *If F is of characteristic p different from zero and if m is a positive integer such that $\Delta_m \neq 0$ then $m \leqq r$ (such values of m exist at least when n is not divisible by p). In particular, if $n < p$ then again r is equal to the maximum such value of m.*

Proof. Let $\lambda_1, \cdots, \lambda_r$ be the distinct characteristic roots of A of respective multiplicities $n_1, \cdots, n_r$ ($n_1 + \cdots + n_r = n$) and let V be the (non-vanishing) Vandermonde determinant of $\lambda_1, \cdots, \lambda_r$. Then, as in the proof of Theorem 0, we see that

$$\Delta_r = n_1 n_2 \cdots n_r V^2 \tag{1}$$

while for $m > r$ we have

$$\Delta_m = n_1 n_2 \cdots n_r \begin{vmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \lambda_1 & \cdots & \lambda_r & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{r-1} & \cdots & \lambda_r^{r-1} & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_r & \cdots & \lambda_r^{r-1} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}$$

which vanishes since there are $m - r$ columns of zeros in the first determinant on the right and $m - r$ rows of zeros in the second.

The conclusions of the theorem follow from these equalities.

This theorem provides a computational method of evaluating the number r of distinct characteristic roots of a given matrix A. For, $r \leqq s \leqq n$, so that by utilizing the sequence $\{\Delta_1, \Delta_2, \cdots, \Delta_s\}$ if s has been previously computed, or the sequence $\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ otherwise, one determines r as the maximum subscript for which a non-zero value is obtained.

There are certain inconveniences to this method however, in that there may exist integers m_1, m_2 with $m_1 < m_2$ such that $\Delta_{m_1} = 0$ but $\Delta_{m_2} \neq 0$. Therefore in using either sequence the computations cannot be ended as soon as a value of zero is obtained, but must be continued until the entire sequence of values has been evaluated.

However in particular cases this circumstance can be avoided. For example, suppose F is contained in the field of complex numbers. Let A^* be the transposed conjugate matrix of A and for each positive integer m define

$$\Delta_m^* = \det [\mathrm{Tr}\ (A^i A^{*j})] \qquad 0 \leqq i, j \leqq m - 1.$$

We then have:

THEOREM 2. *Suppose A is normal (i.e. $AA^* = A^*A$). If A has exactly r different characteristic roots then $\Delta_m^* \neq 0$ for $m \leqq r$ while $\Delta_m^* = 0$ for $m > r$.*

Proof. Let $\lambda_1, \cdots, \lambda_n$ be the characteristic roots of A. Then the characteristic roots of A^* are $\bar{\lambda}_1, \cdots, \bar{\lambda}_n$. Further we can consider A and A^* as defining linear transformations on an n-dimensional complex unitary vector space M. By the normality condition on A we see that M has an orthonormal base $x_1, \cdots, x_n$ such that $Ax_k = \lambda_k x_k$, $A^* x_k = \bar{\lambda}_k x_k$, $1 \leqq k \leqq n$. Thus for any non-negative integers, i, j, $A^i A^{*j} x_k = \lambda_k^i \bar{\lambda}_k^j x_k$ so that $\mathrm{Tr}\ (A^i A^{*j}) = \sum_{k=1}^{n} \lambda_k^i \bar{\lambda}_k^j$. Or, if $\lambda_1, \cdots, \lambda_r$ are the distinct characteristic roots of A, with respective multiplicities $n_1, \cdots, n_r$ $(n_1 + \cdots + n_r = n)$, we have $\mathrm{Tr}\ (A^i A^{*j}) = \sum_{k=1}^{r} n_k \lambda_k^i \bar{\lambda}_k^j$.

Now suppose $m \leqq r$, and define the $m \times r$ matrix

$$C_m = \begin{pmatrix} n_1^{1/2} & \cdots & n_r^{1/2} \\ n_1^{1/2}\lambda_1 & \cdots & n_r^{1/2}\lambda_r \\ \cdots & \cdots & \cdots \\ n_1^{1/2}\lambda_1^{m-1} & \cdots & n_r^{1/2}\lambda_r^{m-1} \end{pmatrix}.$$

We have then, as in the proof of Theorem 1

$$\Delta_m^* = \det (C_m C_m^*)$$

(noting that $C_m C_m^*$ is an $m \times m$ matrix since C_m is $m \times r$ and C_m^* is $r \times m$). But $C_m C_m^*$ is the Gramian matrix of the vectors $(n_1^{1/2}, n_1^{1/2}\lambda_1, \cdots, n_1^{1/2}\lambda_1^{m-1})$, $\cdots$, $(n_r^{1/2}, n_r^{1/2}\lambda_r, \cdots, n_r^{1/2}\lambda_r^{m-1})$ and therefore the rank of $C_m C_m^*$ is equal to the rank of C_m. Considering only the first m columns of C_m we obtain a matrix whose

determinant is equal to

$$(n_1 n_2 \cdots n_m)^{1/2} \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_m \\ \cdots & \cdots & \cdots \\ \lambda_1^{m-1} & \cdots & \lambda_m^{m-1} \end{vmatrix},$$

which is different from zero since $\lambda_1, \cdots, \lambda_m$ are distinct. Thus rank $C_m C_m^*$ $=$rank $C_m = m$, so $\Delta_m^* = \det(C_m C_m^*) \neq 0$.

If $m > r$ an argument similar to that in the proof of Theorem 1 shows that $\Delta_m^* = 0$. This proves the theorem.

Returning now to the general situation where F is an arbitrary field, we have the following theorem (which includes Theorem 0) as a consequence of Theorem 1, and in particular, of equation (1).

THEOREM 3. *The determinants Δ_m have the following properties:*

1. *Δ_n is equal to the discriminant of the characteristic polynomial of A.*
2. *Suppose A has exactly r different characteristic roots $\lambda_1 \cdots, \lambda_r$ of respective multiplicities $n_1, \cdots, n_r (n_1 + \cdots + n_r = n)$ and let D be the discriminant of $g(x)$, a polynomial of degree s. Then $\Delta_s = n_1 n_2 \cdots n_r D$.*

Therefore:

3. *If F is of characteristic zero, a necessary and sufficient condition that A be diagonalizable is that $\Delta_s \neq 0$.*
4. *If F is of characteristic p different from zero, a sufficient condition that A be diagonalizable is that $\Delta_s \neq 0$ (and if $n < p$ this condition is again necessary).*
5. *For any characteristic of F, a necessary and sufficient condition that A have its characteristic roots all distinct is that $\Delta_n \neq 0$, and this condition is also sufficient for the diagonalizability of A.*

Proof. We shall prove assertion 2 first. Clearly $r \leqq s \leqq n$. If $r < s$ then $g(x)$ has a repeated root so $D = 0$. But then by Theorem 1 also $\Delta_s = 0$, so assertion 2 holds in this case. If $r = s$ then assertion 2 follows from equation (1) and the fact that the discriminant of a monic polynomial is equal to the square of the Vandermonde determinant of its roots. Again, if $r < n$ then both Δ_n and the discriminant of the characteristic polynomial of A are equal to zero. If $r = n$ then $n_1 = n_2 = \cdots n_r = 1$. But then we have also $s = n$, so assertion 1 follows from assertion 2.

To prove assertions 3 and 4 we recall that A is diagonalizable if and only if $g(x)$ has distinct roots, *i.e.*, if and only if $D \neq 0$. If $\Delta_s \neq 0$ this condition is satisfied. Conversely, suppose $D \neq 0$. If F is of characteristic zero then $\Delta_s \neq 0$. If F is of characteristic p different from zero then the condition $n < p$ implies that $n_i < p$ for $i = 1, \cdots, r$, so again $\Delta_s \neq 0$.

Assertion 5 is immediate.

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DIOPHANTUS OF ALEXANDRIA

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1. Introduction. The name of Diophantus of Alexandria is immortalized in the designation of indeterminate equations and the theory of approximation. As is perhaps more often the rule than the exception in such cases, the attribution of the name may readily be questioned. Diophantus certainly did not invent indeterminate equations. Pythagoras was credited with the solution $(2n+1, 2n^2+2n, 2n^2+2n+1)$ of the equation $x^2+y^2=z^2$; the famous Cattle Problem of Archimedes is far more difficult than anything in Diophantus, and a large number of other ancient indeterminate problems are known. Further, Diophantus did not even consider the most common type of problem called by his name, the linear equation or system of equations to be solved in integers.

Nevertheless, on at least three grounds the place of Diophantus in the development of mathematics is secure. On all the available data he was the first to introduce systematic algebraic procedures to the solution of non-linear indeterminate equations and the first to introduce extensive and consistent algebraic notation representing a tremendous improvement over the purely verbal styles of his predecessors (and many successors). Finally, the rediscovery of the book through Byzantine sources greatly aided the renaissance of mathematics in western Europe and stimulated many mathematicians, of whom the greatest was Fermat. (Much of Fermat's work is known from notes written in his copy of Diophantus.) [1]

Of Diophantus as an individual we have essentially no information. A famous problem in the Greek Anthology indicates that he died at the age of 84, but in what year or even in which century we have no definite knowledge. He quotes Hypsicles and is quoted by Theon, the father of Hypatia. Now Hypsicles, in the introduction to his book, the so-called Book XIV of Euclid, places himself within a generation or so of Apollonius of Perga whose time is definitely established by the rulers to whom he dedicates his works. Thus we may put Hypsicles in the early or middle part of the second century B.C. with reasonable accuracy [17]. Theon, on the other hand, definitely saw the eclipse of 364 A.D. [10]. Within this gap of five hundred years, historians are at liberty to place Diophantus wherever he best fits their theories of historical development [10, 14]. The majority follow [2] and, on the basis of a dubious reference by the Byzantine Psellus (C.1050), assign him to the third century A.D.

2. The Arithmetic. The surviving work of Diophantus consists of six books (sometimes divided into seven) of the *Arithmetic* and a fragment of a work on polygonal numbers. The introduction to the *Arithmetic* promises thirteen books. The position and content of the missing six or seven books is a matter of conjecture. (The reader is reminded that a "book" is a single scroll and represents the material contained in twenty to fifty pages of ordinary type.)

These books may be summarized as follows: Book I—Determinate systems

of equations involving linear or quadratic methods. Books II–V—Equations and systems of equations, the majority of which are quadratic indeterminate although Books IV and V contain a selection of cubic equations, determinate and indeterminate. Book VI—Equations involving right triangles. All books consist of individual problems and their solutions in positive rationals. In the ordering of the problems some consideration has been given to relative difficulty and interrelation of material, but the over-all impression is of a disconnected assortment.

3. Notation. The numerical notation used by Diophantus is, of course, the standard Hellenistic notation which uses the letters of the Greek alphabet with three archaic letters added to give 27 different symbols; [6, 7] the first nine stand for units, the second for tens and the last for hundreds. Thus, for any further notation, either non-alphabetic symbols or monogrammatic characters were required.

There is a single symbol for the unknown quantity. This may be a monogram for αριθμος. The symbol for "minus" is apparently a monogram for the root of λειψις [5]. Addition is indicated by juxtaposition. The powers of the unknown are designated by easily recognizable monograms, the square by Δ^{υ} for δυναμις, the cube by K^{υ} for κυβος. Higher powers are formed from these by addition, *i.e.*, the fifth power is considered as square-cube. To avoid ambiguity it is necessary to have a special symbol for the zero-order terms also, a monogram for μοναδος, and to write all the negative terms together. Thus, if we adopt an equivalent set of conventions in English, retaining Arabic numerals and the letter x for the indeterminate, the expression: $6x^4+23x^3-2x^2+x-5$ would appear as $SS^q6C^u23X1MS^q2U^n5$.

Fractions were represented either in the inverse position to the present day or by inserting the word for "divided by" between the numerical expressions on the same line. Reciprocals of integers and negative powers of the unknown are designated by a special symbol placed after the number or power.

The most important limitation of this notation is the restriction to one unknown. Since practically all the problems require the determination of several quantities, a considerable part of Diophantus' work lies in the reduction to a single quantity. Further, no general solution in expressed parameters is possible. Even if a general method is indicated, it must be restricted in its presentation to a specific numerical case.

A particular problem will illustrate the situation. In problem 1, Book IV, it is desired, in modern terms, to solve the system: $x^3+y^3=a$, $x+y=b$. Essentially the method is to let $x=z+b/2$, $y=b/2-z$. Substitution in the first equation now yields a binomial quadratic. Let us look at this problem in a translation as bald as possible:

To partition a given number into two cubes of which the sum of the sides is given: Let the number to be partitioned be 370 and the sum of the sides U^n10. Let the side of the first cube be $x1U^n5$, the latter term of which is half the sum

of the sides. Therefore, subtracting, the side of the other cube is $U^n 5Mx$. Then the sum of the cubes will be $S^q 30 U^n 250$. This is equal to $U^n 370$ as is given and x becomes $U^n 2$. As to the original numbers, the first side will be $U^n 7$ and the second, $U^n 3$. The first cube, 343; the second, 27.

4. Diophantine algebra. With this problem in mind let us turn to some aspects of Greek and Babylonian mathematics. A number of tablets [15, 16], both old Babylonian (1800–1600 B.C.) and Seleucid (300 B.C. and later), exist which teach the solution of equations which can be reduced to the forms $x+y=a$; $xy=b$ [10, 12, 13]. Again Euclid's Elements II, 5, 6 can best be viewed as giving solutions to these problems [12]. In modern notation, the procedure in both cases is to write $x=a/2+z$, $y=\pm(a/2-z)$; $xy=b=\pm(a^2/4-z^2)$; $z=\sqrt{a^2/4\pm b}$. Now Diophantus in I (27, 30) considers the same equations, solves them the same way and applies the basic idea repeatedly as in the quoted problem. Other examples can be followed in a similar way, *e.g.*, $x^2+y^2=a$, $xy=b$. (See [13] for a complete discussion of quadratic equations in antiquity.)

Let us now compare the treatment in the three cases: The tablets consist of lists of problems of varying complexity each framed in specific numbers and quantities. The problems are not "practical" nor in any sense rigorously geometric; men are added to days, lengths to areas, areas are multiplied, *etc.* It is clear that the basic thought is purely algebraic. The problems are so set that the solutions are positive integers or terminating sexagesimal fractions such that the roots can be obtained from tables of squares, but from other tablets we learn of approximations to non-terminating rationals like 1/7 or irrationals like $\sqrt{2}$.

The Euclidean problems are cast in the form of propositions about line segments, squares and rectangles. Their generalizations in II, 28, 29, concern parallelograms. The propositions are general and the result is deduced rigorously from the postulational basis. The results are line segments which may well be incommensurable with the original segments; *i.e.*, "irrational" answers are acceptable.

In Diophantus the problems are formulated in terms of abstract numbers but a "number" is always positive rational. The solutions are worked out in terms of particular numerical examples. This procedure may be considered analogous to carrying out a geometrical construction in terms of particular line segments and, indeed, Diophantus probably intended that his problems should be read in this manner. There is, however, no pretense at postulational development. No general propositions are stated even where the solution implies them. Restrictions on the choice of initial values are not always given; in the case of I, 27, we are informed that $a^2/4-b$ must be a square but in the problem in the previous section no restriction is mentioned. The most reasonable conclusion is that he did not know the form of the restriction or did not know how to express numbers that satisfied the restriction. The authors of [5] and [4] disagree but on the naive ground that since he did come up with workable numbers 370 and 10, he must have had some way of generating them. The answer is obvious; he

generated them from the answer.

Like the Babylonians, Diophantus had no qualms about adding areas and lengths (See VI, 19 in 5) although, to be precise, he says that he adds "the number in the area" to "the number in the length." His algebraic technique is tremendously advanced beyond anything we possess of the Babylonians. The complicated cubic and higher degree equations and the indefinite equations are not even suggested in Babylonian algebra. The latter had examples of binary cubics and a few other higher degree equations soluble by tables; they also knew general forms for Pythagorean numbers and obtained solutions of $x^2-2y^2=1$ but this is as far as our present evidence takes them. Even in the quadratic case there may be a difference [13]. When a quadratic is to be solved, Diophantus makes some effort to choose the variable so that a binomial equation results, but if this is not practicable, the general quadratic formula (positive sign before the radical) is used without further comment. The question is still at issue whether the Babylonians ever solve a quadratic without bringing it into some normal form involving a known sum or difference and product.

It is useless even to try to guess what proportion of the advanced problems and methods are Diophantus' own. Most modern historians postulate a continuous underlying tradition of oriental algebraic methods in Greek mathematics rather than a sudden invasion in the Roman period. If this be so, texts and problem lists would certainly have existed. It is probable that the *Arithmetic* was in good part a compilation of such a quality that the predecessors were no longer held in repute. There are traces of the Diophantine notation elsewhere; Heron (60 A.D.) used the same minus sign for example, but no evidence exists that the semi-algebraic notation or the general methods it permitted were used before the publication of the *Arithmetic*.

To sum up, the basic algebraic approach in Diophantus is Babylonian. The generality and abstraction is Greek. The work may be viewed as an episode in the decline of Greek mathematics [12] or as the finest flowering of Babylonian algebra [10].

5. Indeterminate problems. In giving translations of several illustrative problems, I have avoided the usual practice of direct translation. Instead, I adhere carefully to the method of the original while replacing the particular numbers used by parameters. The rationale may thus be conveyed with less verbal explanation than if the presentation were given in its original special form. At the same time, the full power of the method is apparent.

II. 9: If $n=a^2+b^2$, find other representations of n as the sum of two squares: Modify a to $x+a$. The corresponding modification of b may be written $(rx-b)$. Here x is the unknown, a, b and r were assigned specific values.

$$\begin{aligned} n = a^2 + b^2 &= (x + a)^2 + (rx - b)^2 \\ &= (r^2 + 1)x^2 + (2a - 2br)x + a^2 + b^2. \end{aligned}$$

Thus $x=(2br-2a)/(r^2+1)$ where r may be any rational such that the re-

quired quantities are positive.

Note the clever choice of the unknown; fixing x and solving for r would leave a condition on x still to be met; b^2-x^2-2ax would have to be made a square. Again, if a is increased by a fixed amount and the unknown is taken as the corresponding decrease in b, the result does not come out at once. The choice of $rx-b$ instead of $b-rx$ was dictated solely by the numerical values selected which happened to make $b-rx$ negative. (But see V, 24, below.) Euler wrote $a^2+b^2=(a+rx)^2+(b-qy)^2$ which results in a more symmetric solution but this concept is foreign to Diophantus' notation and the solution above is quite general.

Here would be a perfect opportunity to state a proposition instead of a problem. A proof of the theorem: "Any number which is the sum of two squares can be represented as such in an infinite number of ways," is contained in the solution above.

III. 6: Find three numbers whose sum is a square and such that the sum of any two is a square:

$$\begin{aligned} x+y+z &= t^2 \\ x+y &= u^2 \\ y+z &= v^2 \\ x+z &= w^2. \end{aligned}$$

Here Diophantus assigns a definite value to w, or, in modern notation, lets it play the role of the parameter. He then chooses an unknown, r, restricting it as follows: Let $t=r+1$, $u=r$, $v=r-1$. Then $z=2r+1$, $y=r^2-4r$, $x=4r$ and $w^2=6r+1$. Thus $r=(w^2-1)/6$ where w is an arbitrary rational exceeding 5 (so that y is positive). So $x=(2w^2-2)/3$; $y=(w^2-1)(w^2-25)/36$; $z=(w^2+2)/3$.

This problem was chosen to illustrate two points. First Diophantus is not interested in generality except as an incidental by-product. A considerable increase in generality can be obtained merely by replacing $r\pm1$ by $r\pm s$ in the solution and this possibility could easily have been indicated by the addition of a single phrase. Second, the choice of wording of the problems is often peculiar from a modern viewpoint. This problem is clearly equivalent to the single equation: $u^2+v^2+w^2=2t^2$.

Incidentally, using methods available to Diophantus but probably exceeding his control of notation, a much more general solution of this equation is available than the system $(u, v, w, t)=((w^2-1)/6, (w^2-7)/6, w, (w^2+t)/6)$ given above. The equation being homogeneous, it will be more convenient to solve in integers. Let $w=rs$, $u=s^2-p$, $v=s^2-q$, then

$$s^4+(r^2/2-p-q)s^2+(p^2+q^2)/2=t^2.$$

The left hand side is a perfect square if $p^2+q^2=2k^2$, $r^2/2-p-q=2k$. The first of these is the problem of finding three squares in arithmetic progression. It does not occur specifically in the *Arithmetic*, probably because it is too simple in the

rational case, reducing essentially to $a=b=1$ in the problem above. It will be more convenient to take a solution derived from the solution given to the Pythagorean equation by Euclid. If $X^2+Y^2=Z^2$, $(X+Y)^2+(X-Y)^2=2Z^2$. Thus $p=-m^2+2mn+n^2$, $q=m^2+2mn-n^2$, $k=m^2+n^2$; $r^2=4(m+n)^2$ and $r=2(m+n)$. Thus $(u, v, w, t)=(s^2+m^2-2mn-n^2, s^2-m^2-2mn+n^2, 2(m+n)s, s^2+m^2+n^2)$. The previous solution is obtained by setting $m=2$, $n=1$ and dividing by 6.

V. 24: Find a solution of $x^4+y^4+z^4=t^2$.

If $t^2=(x^2-m)^2$, $x^2=(m^2-y^4-z^4)/2m$. Thus an integer m must be found so that $(m^2-y^4-z^4)/2m$ is a square. Let $m=y^2+z^2$ so $x^2=y^2z^2/(y^2+z^2)$. Thus y^2+z^2 must be a square, say $(y+r)^2$. Then $y=(z^2-r^2)/2r$. Thus

$$(x, y, z, t)=\frac{z^3-r^2z}{z^2+r^2}, \quad \frac{z^2-r^2}{2r}, \quad z, \quad \frac{z^8+14z^4r^4+r^8}{4r^2(z^2+r^2)^2}.$$

This example has been chosen for three reasons. First, it is of great historical interest. To this problem Fermat appended a note: "Why does Diophantus not ask for the sum of *two* biquadrates to be a square? This is, indeed, impossible . . . ." Later, Euler conjectured that it was also impossible to find three fourth powers whose sum was a fourth power; *i.e.*, to replace t^2 by t^4. This question remains unsolved.

Second, the problem indicates what happens when the notation is insufficient. First, the chosen unknown is x; m, y, z are assigned specific values to indicate that they play the role of parameters. But the problem cannot be completed, so the author turns to a sub-problem in which y is the unknown.

Finally, the problem contains a curious case of indifference to sign. The quantity x^2-m is, in fact, the negative root of $t^2=(x^2-m)^2$. Since only the square is used, no harm is done but we must remember that, to Diophantus, the quantity x^2-m, which he used, did not exist. The reader may find it interesting to see why x^2+m was not used by trying it. Why $m-x^2$ which is positive and produces the same result was not preferred is a matter for conjecture.

VI. 19: Find a right triangle such that its area added to one of its legs is a square while the perimeter is a cube.

First form the triangle $(2x+1, 2x^2+2x, 2x^2+2x+1)$. The perimeter is $4x^2+6x+2=(4x+2)(x+1)$. Since it is difficult to make a quadratic a cube, consider in turn the triangle $(2x+1)/(x+1)$, $2x$, $2x+1/(x+1)$ obtained by dividing through by $x+1$. The perimeter is $4x+2$ and the area is $(2x^2+x)/(x+1)$. Adding $(2x+1)/(x+1)$ to the latter we have $2x+1$. Thus $4x+2$ is required to be a cube and $2x+1$ a square. The obvious value for $2x+1$ is 4. Thus $x=3/2$ and the triangle is 8/5, 3, 17/5.

It is not clear whether or not Diophantus implies the more general solution $2x+1=4r^6$, $x=(4r^6-1)/2$; probably not.

This problem is illustrative of the rather peculiar problems considered throughout Book VI and of the complete freedom from geometrical considera-

tions. To Euclid such phrases as "the sum of one side and the area" would have been shocking nonsense.

6. An approximation problem. In V. 9 it is required as a sub-problem to find two squares, both exceeding 6, whose sum is 13. Since we have, in the first example of the preceding section, a general method of partitioning a number into two squares when one such partition is given, it is merely necessary to set the two values equal, solve for the parameter and approximate this solution in rationals. If this is done with $a=2$ and $b=3$, we find $r=5+\sqrt{26}$. Approximating by $r=10$, $13=(258/101)^2+(257/101)^2$ and it is readily seen that the conditions are met.

Of course, Diophantus could not do this since the parameters were not expressed. He first finds a number slightly greater than $\sqrt{13/2}$. $13/2=26/4$; if $\sqrt{26}<5+1/x$, $x^2<10x+1$; let $x=10$, then $\sqrt{13/2}\sim 51/20$. Now $51/20=3-9/20=2+11/20$. Thus we wish to find a number near $1/20$ such that $(3-9y)^2+(2+11y)^2=13$. Then $y=5/101$ and the squares are precisely those obtained above.

The problem is typical of the approximate methods used. To approximate the nth root of a rational, first write it in the form p/q^n by multiplying numerator and denominator by the necessary integer to make the denominator a perfect nth power. Then multiply p by the nth powers of successive integers until pa^n is sufficiently close to a perfect nth power, say b^n. The approximation is then b/aq. To improve an approximation a_1 to $\sqrt{a}$, set $(a_1+1/x)^2=a$ and approximate x.

7. Transmission of Diophantus. When the Arabs overran the Southeastern Mediterranean in the 7th century, they came into possession of manuscripts of works which had been published in sufficiently large editions to survive the wars attendant on the breakup of the Roman Empire and the lack of interest in learning of the early Christians. Among these was the *Arithmetic* or at least a portion of it. Translations and commentaries were published in Arabic. These have all been lost; their only trace is in bibliographers' references. When the Arabs formulated their own algebra, they apparently appealed directly to the basic Oriental tradition previously cited. The beginnings of an algebraic notation and the abstract numbers are nowhere to be seen. With the sole exception of the problems mentioned in section 3 as common to the whole ancient world (Diophantus—I, 27–30) not one problem from the *Arithmetic* is found in the algebra of Al-Khwarizmi or, as far as is known, in any other basic Oriental text [13]. Probably the Arabs found Diophantus too impractical for their utilitarian mathematics and the Hindus, if they ever saw the *Arithmetic*, were interested in other problems such as the theory of linear indeterminate problems.

In the other reservoir of learning, Byzantium, the manuscripts of Diophantus lay almost unnoticed for eight centuries. We do not know when the missing books were lost but the part which we now possess escaped the sack of Constantinople by the Crusaders in 1204 and later in the same century M. Planudes and G. Pachymeres wrote commentaries on the first part of the *Arithmetic*. At

some time, probably in the course of the emigration of the Byzantine scholars during the Turkish conquests, copies were brought to Italy and Regiomontanus saw one there between 1461 and 1464.

The first translation to Latin was made by W. Holzmann who wrote under the Greek version of his name, Xylander. This translation was published in 1575. Meanwhile, Bombelli, in 1572, distributed all the problems in the first four books among problems of his own in a text on algebra. Bachet, borrowing liberally from Bombelli and Holzmann made another translation in 1621 and a second edition was published in 1670 including Fermat's marginal notes. In the next two centuries various translations were made into modern languages which were based primarily on the editions just mentioned by Holzmann and Bachet. Finally in 1890, P. Tannery prepared a definitive edition of the Greek text with a translation into simple mathematical Latin using modern numerical and algebraic notation. From this work the three excellent translations listed in the bibliography have been prepared. The references to the last two paragraphs are the commentaries in [2] and [6], particularly the latter which has been followed rather closely.

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ON THE ZEROS OF A CUBIC RECURRENCE

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A cubic recurrence $(T)=(T_1, T_2, T_3, \cdots)$ is defined by real numbers T_0, T_1, T_2, and

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \qquad (n = 0, 1, 2, \cdots),$$

where P, Q, R are real numbers. We shall assume that the roots u, v, w of $f(z)=0$, where

$$f(z) = z^3 - Pz^2 + Qz - R$$

is the *companion* of (T), are nonzero, real, and have distinct absolute values. If $T_k=0$, we call k a *zero* of (T). If $r\geqq 0$ is fixed and (T') is defined by $T_n'=T_{n+r}$, we call (T') a *translate* of (T). Every translate of (T) has the same companion as (T). We shall also assume that $(T)\neq(0, 0, 0, \cdots)$ which holds (1) if and only if no three consecutive T_n are zero, or (2) if and only if it holds for every translate of (T). Then we may show that there are real numbers $(U, V, W)\neq(0, 0, 0)$ and such that

$$T_n = Uu^n + Vv^n + Ww^n \qquad (n = 0, 1, 2, \cdots).$$

THEOREM. *The cubic recurrence (T) has no more than three zeros.*

Morgan Ward [1] recently proved this result under the restriction that u, v, w were integers co-prime in pairs. Our proof is based in part on the ideas of Ward, but is more direct; the essential tool is the familiar Descartes' rule of signs.

We begin with the following lemma.

LEMMA 1. *If $T_0=T_r=T_s=0$ with $s>r>0$, then $x=u$, v, w are roots of*

$$g(x) = (w^r - u^r)x^s - (w^s - u^s)x^r + u^rw^r(w^{s-r} - u^{s-r}) = 0. \tag{1}$$

Proof. The determinant of the matrix

$$C = \begin{bmatrix} 1 & u^r & u^s \\ 1 & v^r & v^s \\ 1 & w^r & w^s \end{bmatrix}$$

is zero, since, by hypothesis, $[U, V, W]C=[0, 0, 0]$. The lemma follows on suitable expansion of this determinant.

LEMMA 2. *If $T_0=T_r=T_s=0$ with $s>r>0$, then $d=(r, s)$ is odd.*

Proof. First note that u, v, w cannot all be positive. For Descartes' rule permits $g(x)=0$ at most two positive real roots. Now suppose that d is even. With

$u'=u^d$, $v'=v^d$, $w'=w^d$, $x'=x^d$, $r'=r/d$, $s'=s/d$, we find that $x'=u'$, v', w' are distinct positive real roots of the equation obtained by priming all of the letters of (1).

Let w be the root of $f(z)=0$ of greatest absolute value. Since replacing T_n by $(-1)^nT_n$ changes the signs of the roots of the companion without changing the zeros, we may assume that $w>0$. (See [1]).

LEMMA 3. *Let* $T_0=T_r=T_s=0$ *with* $s>r>0$. *Then*

(1) *if* r *and* s *are both odd,* $f(z)=0$ *has exactly one positive root,*

(2) *if* r *is even and* s *is odd, two roots of* $f(z)=0$ *are positive,*

(3) r *odd and* s *even is not possible.*

Proof. We have seen that not all the roots of $f(z)=0$ can be positive. We may assume that $u<0$ so that $-w<u<0$. The signs of the coefficients of $g(x)$ in the three cases are: (1) $+--$, (2) $+-+$, (3) $+--$, while the signs of the coefficients of $g(-x)$ are (1) $-+-$, (2) $--+$, (3) $++-$. Descartes' rule and Lemma 1 now yield the stated conclusions.

Proof of the theorem. We see that (T) cannot have three even (or three odd) zeros since a suitable translate of (T) would then violate Lemma 2. Thus (T) has at most four zeros and no odd (even) zero is between two even (odd) zeros or a suitable translate of (T) would violate case (3) of Lemma 3. There remains the possibility that (T) has four zeros k_1, k_2, k_3, k_4 with $0\leqq k_1<k_2<k_3<k_4$ and $k_1\equiv k_2$ and $k_3\equiv k_4\not\equiv k_1$ modulo 2. This occurs if and only if some translate of (T) has four zeros 0, r, s, t with $0<r<s<t$ and with r even, s and t odd. But then Lemma 3 applied to $T_0=T_s=T_t=0$ and to $T_0=T_r=T_s=0$ yields a contradiction: $f(z)=0$ has exactly one positive root and $f(z)=0$ has two positive roots. The proof is complete.

Concluding remark. For non-degenerate real recurrences (T) of order n, the analogue of Lemma 2 (with an analogous proof) limits the number of zeros of (T) to $2n-2$. No doubt there is an analogue of Lemma 3 which will reduce this crude bound.

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MATHEMATICAL NOTES

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AN EIGENVALUE INEQUALITY FOR THE PRODUCT OF NORMAL MATRICES

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Let s_{ij} be n^2 real numbers satisfying

$$s_{ij} \geqq 0; \qquad i,j = 1, \cdots, n \tag{1}$$

$$\sum_{k=1}^{n} s_{kj} = 1; \qquad j = 1, \cdots, n \tag{2}$$

$$\sum_{k=1}^{n} s_{ik} = 1; \qquad i = 1, \cdots, n. \tag{3}$$

Let ϕ_{ij} be any n^2 real numbers and set

$$f(s) = \sum_{i,j=1}^{n} \phi_{ij} s_{ij}.$$

Let p denote any permutation of the numbers $(1, 2, \cdots, n)$. It is known [1] that

$$\min_{p} \sum_{i=1}^{n} \phi_{ip(i)} \leqq f(s) \leqq \max_{p} \sum_{i=1}^{n} \phi_{ip(i)}. \tag{4}$$

The inequality (4) may be directly applied to obtain the following

THEOREM. *If A and B are n-square normal matrices with eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$, $i=1, \cdots, n$, then*

$$\min_{p} \operatorname{Re} \sum_{i=1}^{n} \lambda_i(A)\lambda_{p(i)}(B) \leqq \operatorname{Re} \sum_{i=1}^{n} \lambda_i(AB) \leqq \max_{p} \operatorname{Re} \sum_{i=1}^{n} \lambda_i(A)\lambda_{p(i)}(B). \tag{5}$$

Proof. First note that

$$2 \operatorname{Re} \sum_{i=1}^{n} \lambda_i(AB) = \operatorname{tr} (AB + B^*A^*).$$

Let U and V be unitary matrices such that

$$U^*AU = \operatorname{diag} (\lambda_1(A), \cdots, \lambda_n(A)) = \operatorname{diag} (\lambda(A))$$

and

$$V^*BV = \operatorname{diag} (\lambda_1(B), \cdots, \lambda_n(B)) = \operatorname{diag} (\lambda(B)).$$

Then

$$\begin{aligned}\operatorname{tr}(AB + B^*A^*) &= \operatorname{tr}(U \operatorname{diag}(\lambda(A))U^*B + B^*U \operatorname{diag}(\bar{\lambda}(A))U^*) \\ &= \operatorname{tr}(U\{\operatorname{diag}(\lambda(A))(U^*BU) + (U^*B^*U)\operatorname{diag}(\bar{\lambda}(A))\}U^*) \\ &= \operatorname{tr}(\operatorname{diag}(\lambda(A))U^*V \operatorname{diag}(\lambda(B))V^*U \\ &\quad + U^*V \operatorname{diag}(\bar{\lambda}(B))V^*U \operatorname{diag}(\bar{\lambda}(A))).\end{aligned}$$

Let $T = V^*U$, $T^* = U^*V$ and then

$$\begin{aligned}(T^* \operatorname{diag}(\lambda(B))T)_{ii} &= \sum_{k,\alpha=1}^{n} T^*_{ik}(\operatorname{diag}(\lambda(B)))_{k\alpha}T_{\alpha i} \\ &= \sum_{k=1}^{n} T^*_{ik}\lambda_k(B)T_{ki} \\ &= \sum_{k=1}^{n} |T_{ik}|^2\lambda_k(B).\end{aligned}$$

We set $|T_{ik}|^2 = s_{ik}$ and note that since T is unitary (1), (2) and (3) are satisfied. Proceeding, we have

$$\begin{aligned}\operatorname{tr}(AB + B^*A^*) &= \sum_{i=1}^{n} \lambda_i(A)(T^* \operatorname{diag}(\lambda(B))T)_{ii} + \sum_{i=1}^{n} \bar{\lambda}_i(A)(T^* \operatorname{diag}(\bar{\lambda}(B))T)_{ii} \\ &= \sum_{i=1}^{n} \lambda_i(A) \sum_{k=1}^{n} \lambda_k(B)s_{ik} + \sum_{i=1}^{n} \bar{\lambda}_i(A) \sum_{k=1}^{n} \bar{\lambda}(B)s_{ik} \\ &= \sum_{i,k=1}^{n} (\lambda_i(A)\lambda_k(B) + \bar{\lambda}_i(A)\bar{\lambda}_k(B))s_{ik}.\end{aligned}$$

If we set $\phi_{ik} = \lambda_i(A)\lambda_k(B) + \bar{\lambda}_i(A)\bar{\lambda}_k(B)$ it is clear that (4) applies and we obtain the inequality (5).

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SEQUENCES OF 0's AND 1's AND TOEPLITZ METHODS OF SUMMABILITY

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It has been shown by Schur [2] that no regular matrix summability method sums all bounded sequences. In fact, no regular summability method sums all the sequences consisting of 0's and 1's [1], [3]. On the other hand, there exist many methods which sum all (that is, both) sequences of 0's and 1's in which no pairs of consecutive 0's or consecutive 1's occur. This suggests that we consider the next simplest case of sequences of 0's and 1's containing no triplet of consecutive 0's or of consecutive 1's.

THEOREM. *For each regular Toeplitz matrix* $A=(a_{mn})$, *there exists a sequence* $\{s_n\}$ *of* 0*'s and* 1*'s, containing no triplet of consecutive* 0*'s or of consecutive* 1*'s, and such that* $\{s_n\}$ *is not summable by* A.

Proof. Since every regular matrix is equivalent, for bounded sequences, to a matrix such that, for every m, there are i_m and j_m, $i_m<j_m$, with $a_{mn}=0$ if $n<i_m$ or $n>j_m$ and $\lim_{m\to\infty} i_m=\infty$, we may suppose that A has this property. Moreover, by deleting a set of rows from A we obtain a matrix $B=(b_{mn})$ such that the above $\{i_m\}$ and $\{j_m\}$ may be taken to be multiples of 3 and $i_1<j_1<i_2<j_2<\cdots<i_k<j_k<\cdots$. Since the B transform of every sequence is a subsequence of its A transform, it is evident that every sequence which is A summable is also B summable, so that it suffices to consider the matrix B.

For $k=4p$, $p=1, 2, \cdots$ and for $i_k\leqq n<j_k$, we define s_n by repetitions of the triplet (1, 0, 0). In the same way, for $k=4p+1$, $k=4p+2$, $k=4p+3$ respectively and $i_k\leqq n<j_k$, we define s_n by repetitions of the triplets (0, 1, 0), (0, 0, 1) and (1, 1, 0). The remaining entries in $\{s_n\}$ may be made so that no consecutive triplet of 0's or of 1's appears.

We assume that B sums $\sigma=\{s_n\}$ as well as the four sequences

$$\sigma_1=\{1,0,0,1,0,0,\cdots\},\qquad \sigma_2=\{0,1,0,0,1,0\cdots\},$$
$$\sigma_3=\{0,0,1,0,0,1,\cdots\},\quad\text{and}\quad \sigma_4=\{1,1,0,1,1,0,\cdots\},$$

for otherwise the theorem would be proved. It now follows that all five sequences are summed by B to the same value $c=c_1=c_2=c_3=c_4$. But it is evident that $c_1+c_2=c_4$ and $c_3+c_4=1$, so that $2c=c$ and $2c=1$. This contradiction establishes the theorem.

Hill [1] has shown that the set of sequences of 0's and 1's summable by any regular method, when considered as a set of real numbers in binary expansion, is of the first category. Since the set of 0's and 1's of our theorem is of the first category, Hill's result can not be applied.

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ON COMPLEMENTARY GRAPHS

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1. Introduction. We are concerned in this note with proving certain relations concerning the chromatic numbers of complementary linear graphs. As the terminology suggests, the discussion is motivated by the map-coloring problem. A *graph*, for our purposes, is a finite set of abstract elements, called nodes, and certain unordered pairs of distinct nodes, called edges. The number of nodes is

called the *order* of the graph. A *complete* graph on n nodes is the set of all $n(n-1)/2$ edges and the n nodes determining these edges. Two graphs G and G' having the same n nodes are called *complementary* if each edge determined by the n nodes is in either G or G' but not in both.

A *k-coloring* of a graph G is an assignment to each node of one of the numbers $1, 2, \cdots, k$ in such a way that if the edge (a, b) is in G, the nodes a and b are assigned different numbers. The smallest k for which a graph G admits a k-coloring is called the *chromatic number* of G. Clearly if G has n nodes, $1 \leqq k \leqq n$. We denote the chromatic number of the complementary graph G' by k'.

2. Bounds. Our principal result is the following: If G and G' are complementary graphs on n nodes whose chromatic numbers are k, k' respectively, then

$$2\sqrt{n} \leqq k + k' \leqq n + 1 \tag{1}$$

$$n \leqq kk' \leqq \left(\frac{n+1}{2}\right)^2, \tag{2}$$

and these are the best possible bounds, in the sense that each bound is assumed for infinitely many values of n. The lower bound in (2) has been established by Zykov [1].

Proof. Consider a k-coloring of G, and let n_i be the number of nodes assigned the i-th color, for $i=1, 2, \cdots, k$. Then $n_1+n_2+\cdots+n_k=n$ and $\max n_i \geqq n/k$. Now if two nodes are assigned the same color in coloring G, the edge they determine must be in G', and in coloring G' they must be assigned different colors. Therefore $k' \geqq \max n_i \geqq n/k$, or $kk' \geqq n$. Then $(k-k')^2 \geqq 0$ implies $(k+k')^2 \geqq 4kk'$ and $k+k' \geqq 2\sqrt{n}$. This establishes the lower bounds in (1) and (2).

To show that $k+k' \leqq n+1$, we use induction on n. The result is evident for $n=1$, and we assume that $k+k' \leqq n+1$ holds for complementary graphs on n nodes. To the sum of G and G', which consists of the complete graph on these n nodes, we adjoin a node P. Of the n edges emanating from P, let q be adjoined to G and the remaining $n-q$ to G'. If H and H' are the graphs so determined (each of order $n+1$), and these have chromatic numbers l and l', respectively, then

$$l \leqq k + 1, \qquad l' \leqq k' + 1.$$

These inequalities are evident from the fact that the addition of a node to a graph and some or all edges from this node to nodes of the graph can increase the chromatic number by at most one. Then $l+l' \leqq n+2$ except possibly when $l=k+1$, $l'=k'+1$. In this event, the graph H has the property that if node P and all edges of H emanating from P are deleted, the chromatic number is decreased. Such a point is termed a *critical point* of the graph, and a simple argument shows that $q \geqq k$. Similarly, $n-q \geqq k'$ and thus $k+k' \leqq n$, from which $l+l' \leqq n+2$. This completes the induction. Finally, the inequality $4kk' \leqq (k+k')^2$ implies that $kk' \leqq ((n+1)/2)^2$, establishing the remaining upper bound.

The following three examples show that each of the bounds established is

actually assumed for infinitely many values of n. Example 1. Let G be a complete graph on n nodes. Then $k=n$, $k'=1$, so $k+k'=n+1$, $kk'=n$. Example 2. Let $n=q^2$, and consider q groups of q nodes each, where the nodes in each group bear the same color, q colors being used. The edges of G are then defined as those edges which join nodes of different colors. Then $k=q$, $k'=q$, so $k+k'=2q=2\sqrt{n}$, $kk'=q^2=n$. Example 3. Let $n=2t-1$, and separate the nodes of G into two groups containing t and $t-1$ nodes, respectively. The edges of G are those of the complete graph of order t determined by the first group of t nodes. Then $k=t$, $k'=(t-1)+1=t$, $k+k'=n+1$, and $kk'=t^2=((n+1)/2)^2$.

No graph with $n>1$ has $k+k'=2\sqrt{n}$ and $kk'=((n+1)/2)^2$ holding simultaneously. A simple geometric argument shows that when either of the two equalities just mentioned occurs, then $k=k'$ must hold.

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ON THE PROPAGATION OF ERROR BY MULTIPLICATION

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In applied mathematics the traditional approach to the analysis of the error in a result obtained by computation with approximate numbers is in terms of the maximum error possible [5]. The purpose of this paper is to illustrate a more statistical approach, which is presented briefly in Householder [2], and discussed more fully by Inman [3].* The general approach to be found in these references is the development of a distribution function which associates with an error of given size the probability of its being exceeded.

We illustrate this statistical point of view for the case of the multiplication of two approximate numbers. More precisely we have the following:

Problem: Given the numbers a and b, with respective errors ϵ and ϵ' which are on the range $-\frac{1}{2}$ to $+\frac{1}{2}$; to find the distribution function of the error E in ab, on the assumption that all values of ϵ and ϵ' are equally likely.

Since $(a+\epsilon)(b+\epsilon')=ab+a\epsilon'+b\epsilon+\epsilon\epsilon'$, the probability that E exceeds a fixed value K is equal to the $P[a\epsilon'+b\epsilon+\epsilon\epsilon'>K]$. If we choose an ϵ, then by solving the inequality $E>K$ it follows that $\epsilon'>(K-b\epsilon)/(a+\epsilon)$. Hence, the probability that simultaneously ϵ lies on a small interval of length $\Delta\epsilon$ and ϵ' lies on the interval $[(K-b\epsilon)/(a+\epsilon),\frac{1}{2}]$ is roughly equal to the product

$$(\Delta\epsilon)(\tfrac{1}{2}-(K-b\epsilon)/(a+\epsilon)).$$

With certain necessary modifications the essential plan of proof is to integrate this probability for ϵ ranging from $-\frac{1}{2}$ to $\frac{1}{2}$. Thus apparently, $P[E>K]$ is equal to the expression

* Specific applications to differential equations and matrix theory can be found in Rademacher [4], and Goldstine and von Neumann [1].

$$\int_{-1/2}^{1/2}\left(\tfrac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon. \tag{1}$$

However, since ϵ' lies between $-\frac{1}{2}$ and $\frac{1}{2}$, the quantity $(K-b\epsilon)/(a+\epsilon)$ in the integrand of equation (1) must satisfy the inequality $-\frac{1}{2}\leqq(K-b\epsilon)/(a+\epsilon)\leqq\frac{1}{2}$. Upon solving for ϵ we obtain $(2K-a)/(2b+1)\leqq\epsilon\leqq(2K+a)/(2b-1)$. Thus, if $\epsilon>(2K+a)/(2b-1)$ we must replace $(K-b\epsilon)/(a+\epsilon)$ by $-\frac{1}{2}$, and similarly if $\epsilon<(2K-a)/(2b+1)$ we must replace $(K-b\epsilon)/(a+\epsilon)$ by $\frac{1}{2}$. More precisely $P[E>K]$ seems to take the form:

$$\begin{aligned}&\int_{-1/2}^{(2K-a)/(2b+1)}(\tfrac{1}{2}-\tfrac{1}{2})d\epsilon+\int_{(2K-a)/(2b+1)}^{(2K+a)/(2b-1)}\left(\tfrac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon\\&\qquad+\int_{(2K+a)/(2b-1)}^{1/2}[\tfrac{1}{2}-(-\tfrac{1}{2})]d\epsilon.\end{aligned} \tag{2}$$

However, if $K<(2a-2b-1)/4$ it is impossible for $(K-b\epsilon)/(a+\epsilon)$ to be greater than $\frac{1}{2}$, and the correction shown in (2) for the range $-\frac{1}{2}$ to $(2K-a)/(2b+1)$ becomes unnecessary. Similarly if $K>(2b-2a-1)/4$ then $(K-b\epsilon)/(a+\epsilon)$ cannot be less than $-\frac{1}{2}$, and the correction indicated in equation (2) for the range $(2K+a)/(2b-1)$ to $\frac{1}{2}$ is not needed. Thus we have finally the distribution function of E defined as follows:

$$\frac{-2a-2b+1}{4}\leqq K\leqq\frac{2a-2b-1}{4};$$

$$P[E>K]=\int_{-1/2}^{(2K+a)/(2b-1)}\left(\frac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon+\int_{(2K+a)/(2b-1)}^{1/2}d\epsilon$$

$$\frac{2a-2b-1}{4}\leqq K\leqq\frac{2b-2a-1}{4};$$

$$P[E>K]=\int_{(2K-a)/(2b+1)}^{(2K+a)/(2b-1)}\left(\frac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon+\int_{(2K+a)/(2b-1)}^{1/2}d\epsilon$$

$$\frac{2b-2a-1}{4}\leqq K\leqq\frac{2a+2b+1}{4};$$

$$P[E>K]=\int_{(2K-a)/(2b+1)}^{1/2}\left(\frac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon.$$

Upon performing these integrations we have:

$$P[E>K]=\frac{a+b}{2}+\frac{3}{4}+K-(K+ab)\log_e\frac{4(K+ab)}{(2a-1)(2b-1)};$$

$$\frac{-2a-2b+1}{4}\leqq K\leqq\frac{2a-2b-1}{4}$$

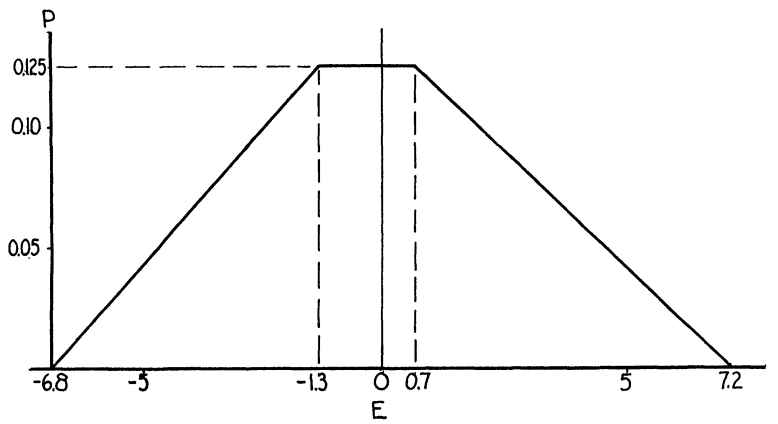
$$P[E > K] = a + \frac{1}{2} - (K + ab)\log_e \frac{2b+1}{2b-1};$$

$$\frac{2a - 2b - 1}{4} \leqq K \leqq \frac{2b - 2a - 1}{4}$$

$$P[E > K] = \frac{a+b}{2} + \frac{1}{4} - K - (K + ab)\log_e \frac{(2a+1)(2b+1)}{4(K+ab)};$$

$$\frac{2b - 2a - 1}{4} \leqq K \leqq \frac{2a + 2b + 1}{4}.$$

A general notion of the geometric nature of this function can be obtained from the graph below:



Graph of the frequency distribution of E for $a=6$, and $b=8$.

From these results it is readily computed that about half the time the error E in the product (6) (8) will lie outside the range $(-2.13, 2.09)$. Five percent of the time it will lie outside the range $(-5.32, 5.61)$, and about one percent of the time it will be less than -6.09 or greater than 6.50.

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CLASSROOM NOTES

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THE HEINE-BOREL THEOREM

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Mathematics is replete with "covering" theorems of one type or another, but the Heine-Borel theorem is perhaps the most well-known of them all. Virtually everyone who has taken a course in advanced calculus has encountered it, and has seen its applications in the proofs of theorems on uniform continuity, the boundedness of continuous functions, and so on.

The theorem is stated explicitly a little farther on; but informally, it says that if a family $\mathfrak{F}$ of open sets covers a certain set A, then there exists a finite subcollection $\mathfrak{G}$ of $\mathfrak{F}$ which also covers A. Here we propose what is thought to be a new proof of this theorem, one which requires the student to know or accept only the barest essential metrical properties of euclidean space and the real number system, and which has the philosophical and psychological advantage of proving the existence of the required covering family constructively, by actually picking out its members one by one. Although the chosen setting is euclidean space, the proof is equally valid in any compact metric space. The method does not apply in arbitrary topological spaces; but in these cases, elegant proofs can be found in suitable books. These naturally require correspondingly advanced maturity of the student, which we specifically exclude. Our procedure consists of selecting one of the "largest" sets (in a sense made precise below) of the given family; then from a certain subset of those remaining, one of the largest of that set is chosen, and so on. All that remains is to ascertain that the covering is completed after a finite number of such steps.

We introduce some definitions and notation. We let E_n denote n dimensional euclidean space; when points are mentioned, these will be understood to be points of E_n. If A and B are sets in E_n, then $A-B$ will denote the set of those points which are in A but not in B; the union of a number of sets will be denoted by $+$ or $\sum$ with appropriate indices; we write $A \subset B$ to denote the fact that A is contained in B. For any two points p and q, we let $\delta(p, q)$ denote the euclidean distance between them. For any given positive number η and any given point q, the set of those points p such that $\delta(q, p) < \eta$ is called the *open sphere of center q and radius η*. A set G in E_n is called *bounded* if it is contained in some open sphere. A set G of points is called *open* if and only if every point q in G is the center of some open sphere $S(q, \eta)$ which is contained in G. It is easy to see that every open sphere is an open set. A point p is called a *point of accumulation* or a *limit point* of a set F in E_n if and only if every open sphere centered at p contains infinitely many points of F. The point p may or may not belong to F. In case

every limit point p of F does belong to F, we say that F is *closed*. Finally, if $\mathfrak{R}$ is a family of sets in E_n, and A is a set in E_n, we shall say that $\mathfrak{R}$ *covers* A if and only if A is contained in the union of $\mathfrak{R}$; that is, every point of A lies in at least one member of $\mathfrak{R}$.

A few properties of euclidean space and the real number system are needed in what follows. We assume known, or postulated, that every bounded infinite set of points in E_n has a limit point. We assume that the properties of least upper bounds are understood; in particular, that if γ is the least upper bound of a non-empty bounded set E of positive numbers, then there exists at least one number η in E such that $\eta > \gamma/2$. Finally, we use the fact that if $\alpha > 0$, and A is a set in E_n such that for each pair of different points p, q in A, $\delta(p, q) > \alpha$, then A has no limit point. This is easily seen, for if t is any point in E_n, then the distance between any two points of the sphere $S(t, \alpha/2)$ is less than α. Thus no such sphere can contain more than one point of A; A has no limit points.

We now prove a preliminary lemma which is, in fact, a special case of the main theorem.

LEMMA. *If A is a bounded closed set in E_n, and if $\mathfrak{R}_1$ is a family of open spheres of uniformly bounded radii, such that* (i) *each point in A is the center of at least one sphere in $\mathfrak{R}_1$, and* (ii) *each sphere in $\mathfrak{R}_1$ is centered at some point of A, then there exists a finite subfamily $\mathfrak{R}$ of $\mathfrak{R}_1$ which covers A.*

Proof. We assume that A has infinitely many points or there is nothing to prove. We denote by $r(S)$ the radius of any sphere S in $\mathfrak{R}_1$. If $\mathfrak{L}$ is any non-empty subfamily of $\mathfrak{R}_1$, we denote by $\rho(\mathfrak{L})$ the least upper bound of all numbers of the form $r(S)$, where S is in $\mathfrak{L}$. Since the radii of all spheres in $\mathfrak{R}_1$ are uniformly bounded, we have $0 < \rho(\mathfrak{L}) < \infty$.

We now construct the required family $\mathfrak{R}$. We select a sphere S_1 in $\mathfrak{R}_1$, with radius $r(S_1) > \rho(\mathfrak{R}_1)/2$ and center p_1 in A. If $A \subset S_1$, the lemma is proved. If not, then the set $A - S_1$ is non-empty, and we let $\mathfrak{R}_2$ denote the non-empty family of those spheres in $\mathfrak{R}_1$ whose centers are points of $A - S_1$. We select a sphere S_2 in $\mathfrak{R}_2$ with radius $r(S_2) > \rho(\mathfrak{R}_2)/2$ and center at p_2 in $A - S_1$. If $A \subset S_1 + S_2$, then the family consisting of S_1 and S_2 satisfies our requirements; otherwise we repeat the process on the non-empty family $\mathfrak{R}_3$ of those spheres in $\mathfrak{R}_1$ whose centers are points of $A - (S_1 + S_2)$, and continue thus inductively. In this way, we obtain a finite or infinite sequence of spheres $S_1, S_2, \cdots, S_m, \cdots$ with centers at $p_1, p_2, \cdots, p_m, \cdots$ respectively, drawn from the subfamilies $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_m, \cdots$ of $\mathfrak{R}_1$, in such a way that for each positive integer m:

(1) p_1 is in A; p_{m+1} is in $A - \sum_{i=1}^{m} S_i$;

(2) $0 < \rho(\mathfrak{R}_m)/2 < r(S_m) \leqq \rho(\mathfrak{R}_m)$.

By condition (1), for each pair of integers μ, ν, $\mu < \nu$ we have

(3) p_μ is the center of S_μ, p_ν is outside S_μ; $\delta(p_\mu, p_\nu) \geqq r(S_\mu)$.

We assert that the process outlined above must terminate for some integer N; that is, $A \subset \sum_{i=1}^{N} S_i$. Suppose the contrary, if possible. Let E denote the set of points $p_1, p_2, \cdots$. From (3), no two coincide, hence E is an infinite subset of

the bounded set A. Then E has at least one limit point p, which is *a fortiori* a limit point of the closed set A, so that p is also in A.

Either p belongs to some sphere S_m or not. If p were in S_μ, say, then since S_μ is open and p is a limit point of E, there must exist infinitely many points of E lying in S_μ; in particular there must be one such point p_ν with index $\nu > \mu$, contradicting (3). If p were not in any sphere S_m, then p must be itself the center of a sphere S belonging to each of the families $\mathfrak{R}_1, \mathfrak{R}_2, \cdots$. Hence, using (2) and (3), for $\mu, \nu = 1, 2, \cdots, \mu \neq \nu$, we should have

$$0 < r(S) \leqq \rho(\mathfrak{R}_\mu) < 2r(S_\mu); \qquad \delta(p_\mu, p_\nu) > r(S)/2 > 0.$$

This would mean E could have no limit points. This contradiction forces the conclusion $A \subset \sum_{i=1}^{N} S_i$ for some integer N; the family $\{S_1, S_2, \cdots, S_N\} = \mathfrak{R}$ covers A as required.

THEOREM. (Heine-Borel). *If $\mathfrak{G}_1$ is a family of open sets covering the bounded closed set A in E_n, then a finite subfamily $\mathfrak{G}$ of $\mathfrak{G}_1$ covers A.*

Proof. We let $\mathfrak{R}_1$ denote the family of open spheres of radii not exceeding 1, whose centers are points of A, and each of which is contained in some member of $\mathfrak{G}_1$. $\mathfrak{R}_1$ clearly satisfies the hypotheses of the preceding lemma. We thus find a finite family of spheres $S_1, S_2, \cdots, S_N$ in $\mathfrak{R}_1$ covering A. For each such sphere S_i we pick some set G_i in $\mathfrak{G}_1$ which contains S_i. The finite family $\{G_1, G_2, \cdots, G_N\} = \mathfrak{G}$ clearly covers A.

WHAT IS A POINT OF INFLECTION?

A. W. WALKER, University of Toronto

Has the curve $y = x^4$ a point of inflection at the origin? At present, the answer depends on whether we consider the matter from the point of view of calculus or algebraic geometry.

As Ewing [1] has remarked, considerable care is required in expressing the necessary and sufficient conditions for an inflection point in the language of the calculus, and inconsistencies are found when various texts are compared. However, the generally accepted view is surely the one expressed by von Mangoldt [2] in the section of the *Encyklopädie* on the application of calculus to curves and surfaces, namely, that a point of inflection is any point where a plane curve crosses both its tangent and its normal. (The curve may consist of two parts with different equations, joining, without discontinuity in the direction of the tangent line, at the point in question. An inflection point is not necessarily a point of zero curvature, and conversely.) According to this definition, the curve $y = x^4$ has, of course, no inflection points.

In algebraic geometry we are concerned to a considerable extent with complex or abstract fields, in which the above definition has no meaning.* The general viewpoint here (expressed with lack of clarity and consistency by many

* The writer is indebted to the referee for this significant remark.

texts in English) seems to be the one clearly stated by R. J. Walker [3]. After remarking that the terms *flex* and *point of inflection* are identical, he defines a flex of a plane curve as a non-singular point at which the tangent line has n coincident intersections with the curve, where $n \geqq 3$. Berzolari [4], in the section of the *Encyklopädie* on higher plane algebraic curves, refers to an *ordinary* inflection point when $n=3$, and to a *higher order* inflection point when $n>3$; in particular, the term *undulation point* is used when $n=4$. This is so for the curve $y=x^4$ at the origin. (It is instructive to consider $y=x^4-k^2x^2$ and let k approach zero.)

The only known reference to the fact that there are (in the real field) these two conflicting definitions is a short footnote in de la Vallée Poussin [5], which does not appear in the first edition but was added later. If the dual interpretation is to continue, it would avoid confusion if writers of future texts on both subjects followed this lead, preferably including a reference to [2] and [4] or their equivalents. An alternative would be the universal adoption in algebraic geometry of the concise term *flex*, defined as above. A flex would be either an inflection point (n odd) or an undulation point (n even), extending the scope of this latter term; the distinction would be significant only in the real field. (Salmon [6] remarks: "Cramer calls those points at which the tangent meets the curve in an odd number of consecutive points, points of visible inflection, to distinguish them from points of undulation, which do not, to the eye, differ from ordinary points on the curve.")

References

1. This MONTHLY, vol. 45, 1938, p. 681.
2. Encyk. der Math. Wiss., III, 3, p. 9.
3. R. J. Walker, Algebraic Curves, Princeton, 1950, p. 71.
4. Encyk. der Math. Wiss., III, 2(1), p. 320.
5. De la Vallée Poussin, Cours d'Analyse Infinitésimale, 7th ed., 1930, Vol. 1, p. 242.
6. Salmon, Higher Plane Curves, 3rd ed., 1879, p. 37.

HODOGRAPH OF A CENTRAL ORBIT

KULDIP SINGH, National Defence Academy, Dehra Dun, India

Let a particle move under a central force $-f(r)(\mathbf{r}/r)$ per unit mass where r is the distance of the particle from the center of force, $\mathbf{r}$ is the vector from the center of force to the particle, and $(\mathbf{r}/r)$ is the unit vector in this direction. Then the equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2} = -f(r)\frac{\mathbf{r}}{r}.$$

Now $r^2(d\theta/dt)=h$, therefore

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{r^2}{h}f(r)\frac{\mathbf{r}}{r}\frac{d\theta}{dt}.$$

Let $r=g(\theta)$ be the polar equation of the orbit with center of the force as the pole. If $\mathbf{R}$ is the position vector of any point on the hodograph (*i.e.*, $\mathbf{R}$ is the velocity vector), then

$$\frac{d\mathbf{R}}{dt} = -\frac{1}{h}F(\theta)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\frac{d\theta}{dt},$$

where $F(\theta)=r^2f(r)$, $r=g(\theta)$. Hence,

$$\mathbf{R} = -\frac{1}{h}\int F(\theta)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})d\theta + \mathbf{C},$$

which is the equation of the hodograph.

Example. A particle describes a conic under a central force to the focus; show that the hodograph is a circle.

Since

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^2}\frac{\mathbf{r}}{r},$$

$$\frac{d\mathbf{R}}{dt} = -\frac{\mu}{h}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\frac{d\theta}{dt}.$$

Therefore

$$\mathbf{R} = -\frac{\mu}{h}(\sin\theta\mathbf{i} - \cos\theta\mathbf{j}) + \mathbf{C}.$$

Hence the hodograph is the circle

$$\mathbf{R}^2 - 2\mathbf{R}\cdot\mathbf{C} + \left(\mathbf{C}^2 - \frac{\mu^2}{h^2}\right) = 0$$

whose center has the position vector $\mathbf{C}$ and whose radius is μ/h.

N. B. The procedure adopted in the above example can be usefully applied to most of the examples on hodograph.

PRINCIPAL VALUES OF CERTAIN INVERSE TRIGONOMETRIC FUNCTIONS

C. B. Read and Ferna Wrestler, University of Wichita

Students often feel the definitions of the principal values of the inverse trigonometric functions are purely arbitrary. Texts in trigonometry may fail to point out that alternative definitions exist; they sometimes justify the definitions given on the basis of later use in the calculus.

The calculus student in turn will be fortunate if his text explains why the particular definitions used were selected. He may for example wonder why $-\pi/2 \leqq \text{Arc sin } x \leqq \pi/2$, rather than $\pi/2 \leqq \text{Arc sin } x \leqq 3\pi/2$. The first choice results in a positive slope throughout the interval, in agreement with the usual

value of the derivative. This explanation may not be a satisfactory reason why, for $x \leqq -1$, we define $-\pi \leqq \text{Arc sec } x < -\pi/2$ rather than $\pi/2 < \text{Arc sec } x \leqq \pi$ since the second choice would give a positive slope throughout while the first does not. Some calculus texts give the second definition; to many students this seems more consistent since it agrees with the definition for Arc cos x.

Two suggestions are offered in justification of the choice, when $x \leqq -1$, $-\pi \leqq \text{Arc sec } x < -\pi/2$. If in the usual derivation of the derivative of Arc sec x we let $x = \sec y$, then $dx/dy = \sec y \tan y = x \tan y$. If $x \geqq 1$ and $0 \leqq y < \pi/2$, then $\tan y = \sqrt{x^2-1}$; if $x \leqq -1$ and $\pi/2 < y \leqq \pi$, then $\tan y$ is $-\sqrt{x^2-1}$, hence $dy/dx = \pm 1/x\sqrt{x^2-1}$, requiring the ambiguous sign. However, if when $x \leqq -1$ we define $-\pi \leqq y < -\pi/2$, $\tan y$ is $\sqrt{x^2-1}$ and for all cases $dy/dx = +1/x\sqrt{x^2-1}$ (dy/dx is not defined at $x = \pm 1$).

As the second suggestion, consider the area bounded by $y = 1/(x\sqrt{x^2-1})$ between $x = -2$ and $x = -\sqrt{2}$. Clearly the area is negative but if the principal value of Arc sec x is defined as $0 \leqq \text{Arc sec } x \leqq \pi$, a positive result is obtained, suggesting reasons for another definition.

These explanations may still leave the student (or the teacher) puzzled as to why the "natural" definition Arc sec $x = \text{Arc cos } (1/x)$ should not be adopted. The ambiguity of signs discussed in the first suggestion just presented can be readily removed by defining $0 \leqq \text{Arc sec } x = \text{Arc cos } (1/x) \leqq \pi$ and using absolute values in the formula

$$\frac{d}{dx} \text{Arc sec } x = \frac{1}{|x|\sqrt{x^2-1}}.$$

In the case of integration, we may write, for negative x,

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{d(-x)}{(-x)\sqrt{x^2-1}} = \text{Arc sec } (-x) + C$$
$$= \int \frac{d|x|}{|x|\sqrt{x^2-1}} = \text{Arc sec } |x| + C \qquad x \leqq -1$$

hence for x either positive or negative $\int dx/x\sqrt{x^2-1} = \text{Arc sec } |x| + C (|x| \geqq 1)$.

Use of this formula (analogous to $\int dx/x = \ln |x| + C$) and the definition $0 \leqq \text{Arc sec } x \leqq \pi$ gives, for the integration example presented, a negative result. It would appear, then, that there is really no necessity for an apparently artificial definition.

It would not be out of order to suggest that authors might well devote a little more space to explaining why the particular choice of definition was selected.

Note: The authors wish to thank the referee for a suggestion for expanding and improving this note.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1206. *Proposed by W. E. Briggs, University of Colorado*

If n is a given positive integer, how many integral solutions does

$$1/n = 1/x + 1/y$$

have with x and y positive and unequal?

E 1207. *Proposed by R. D. Gordon, California State Polytechnic College*

Given two mutually perpendicular lines l_1 and l_2 in a plane, and a point Q in the plane located equally distant from l_1 and l_2. Determine the locus of a point P in the plane if its distance from Q equals the sum of its distances from l_1 and l_2.

E 1208. *Proposed by Gretchen Geller, Student, University of California, Davis, Calif.*

Can a closed interval I be covered by an infinite set J of nondegenerate closed intervals in such a manner that no finite subset of J covers I?

E 1209. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F, and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC.

E 1210. *Proposed by Michael Goldberg, Washington, D. C.*

Given two equilateral triangles of edges a and b. Show how to dissect them by straight cuts into a total of no more than six pieces which can be assembled into another equilateral triangle. When the ratio of the larger to the smaller satisfies $a/b \geqq \sqrt{3}$, then five pieces suffice. When $a/b = 4/3$, four pieces suffice.

SOLUTIONS

Good, Better, Best

E 1176 [1955, 493]. *Proposed by J. L. Brenner, State College of Washington*

Given a, b, c, d real and $ad - bc = 1$. Show that

$$Q = a^2 + b^2 + c^2 + d^2 + ac + bd \neq 0, 1, -1.$$

I. *Solution by Calvin Foreman, Baker University.* We have

$$2Q - 2 = (a + c)^2 + (a - d)^2 + (b + d)^2 + (b + c)^2,$$

the right member of which cannot vanish, since then $c = -a = -d = b = -c$ requiring $a = b = c = d = 0$ which contradicts the stipulation that $ad - bc = 1$. Therefore $2Q - 2 > 0$, that is $Q > 1$, from which the desired conclusion follows.

II. *Solution by A. P. Boblett, U. S. Naval Ordnance Laboratory, Corona, Cal.* Since

$$Q - 3/2 = (a + c/2 - 3d/4)^2 + (b + 3c/4 + d/2)^2 + 3(c^2 + d^2)/16$$

it follows that $Q \geqq 3/2$.

III. *Solution by Leonard Carlitz, Duke University.* Actually $Q \geqq \sqrt{3}$, as is seen from the identity

$$Q - \sqrt{3} = (a + c/2 - d\sqrt{3}/2)^2 + (b + c\sqrt{3}/2 + d/2)^2.$$

That we can have $Q = \sqrt{3}$ is seen by taking $a^2 = \sqrt{3}/2$, $b = -a/\sqrt{3}$, $c = 0$, $d = 2a/\sqrt{3}$.

Also solved by G. E. Bardwell, Hüseyin Demir, I. A. Dodes, R. V. Esperti, Harley Flanders, H. M. Gehman, Michael Goldberg, A. J. Goldman, H. E. Gould, E. S. Grable, G. R. Grainger, D. S. Greenstein, W. G. Hanks, Vern Hoggatt, Raymond Huck, A. R. Hyde, L. E. Isenecker, J. D. E. Konhauser, A. E. Livingston, D. C. B. Marsh, W. H. McKenzie, J. D. Miller, F. D. Parker, Kincaid Patterson, Larry Potter and J. T. Robertson (jointly), C. E. Rhodes, D. W. Smith, J. A. Tierney, Gene Uretz, Chih-yi Wang, and Charlotte Yesselman. Late solutions by Servet Aybat, Walter Guber and Sheldon Weinberg (jointly), M. J. Hellman, and M. A. Rashid.

Three Associated Loci

E 1177 [1955, 493]. *Proposed by W. R. Utz, University of Missouri*

Describe the three types of plane loci of points the product of whose distances from a point and a line is constant.

Solution by D. C. B. Marsh, Colorado School of Mines. Let the fixed line be the y-axis and the fixed point the point $(2a, 0)$ and denote the constant product by k. Then

$$x^2y^2 + x^2(x - 2a)^2 = k^2$$

is the equation of the desired locus.

The locus is symmetric in the x-axis, with y-axis as vertical asymptote and real intercepts on the x-axis for abscissae $a + (a^2 \pm k)^{1/2}$ and/or $a - (a^2 \pm k)^{1/2}$ which are real; excluded intervals likewise depend on the relative sizes of k and a^2. (1) If $k < a^2$, the curve consists of two branches asymptotic to the y-axis

and an isolated oval surrounding the fixed point; (2) if $k=a^2$, the curve consists of two branches asymptotic to the y-axis, one of which has a loop on the x-axis; in the third case, (3), where $k>a^2$, the loop has reduced to a point and the curve consists of two asymptotic branches only. A noteworthy similarity exists between these curves and the conchoids of Nicomedes.

Also solved by Michael Goldberg, A. R. Hyde, Larry Wan, and the proposer.

The Proposer has found this problem an interesting one to pose to students in or beyond analytic geometry as it replaces *quotient* in the definition of the conics by *product*. Hyde presented carefully plotted graphs of the three types of loci and further distinguished a critical case for type (3) above. When $a^2<k<3\sqrt{3}a^2/4$ the curve possesses four real horizontal tangents, when $k=3\sqrt{3}a^2/4$ there are two real horizontal (inflectional) tangents, and when $k>3\sqrt{3}a^2/4$ there are no real horizontal tangents.

An Arithmetical Property of the Triangle

E 1178 [1955, 493]. *Proposed by A. J. Goldman, Princeton University*

Prove that there exists a positive constant c with the following property: If T is any triangle whose area exceeds c, then the product of the lengths of the sides of T is greater than the area of T. What is the best possible value of c?

Solution by Calvin Foreman, Baker University. The product P of the lengths of the sides of T is greater than the area K of T if, and only if, the circumradius R exceeds $\frac{1}{4}$ since $P=4KR$. The circumradius is minimal and equals $\frac{1}{4}$ for a triangle of prescribed area when the triangle is an equilateral triangle whose area equals $(3/64)\sqrt{3}$. The desired conclusion follows, and the best value of c is $(3/64)\sqrt{3}$.

Also solved by Fred Cherry, Michael Goldberg, D. S. Greenstein, R. E. Heaton, Vern Hoggatt, A. R. Hyde, Sam Kravitz, C. W. Langley, D. C. B. Marsh, C. M. Sandwick, Sr., Chi-yi Wang, and the proposer. Late solutions by C. H. Borgmeyer, John Christopher, C. F. Pinzka, and J. W. Ross.

Limit of an Iterated Operation

E 1179 [1955, 493]. *Proposed by C. D. Olds, San Jose State College*

Let f be an operator such that $f(z)=(|z|+z)/2$, and define $f^2(z)=f\{f(z)\}$, $\cdots$, $f^n(z)=f\{f^{n-1}(z)\}$. Evaluate

$$\lim_{n\to\infty} f^n(i), \text{ where } i=\sqrt{-1}.$$

I. *Solution by Michael Goldberg, Washington, D. C.* The procedure can be visualized by the use of the Argand diagram. The value f^1 is midway between 1 and i; f^2 is midway between f^1 and $|f^1|$; *etc.* Therefore

$$|f^n| = |f^{n-1}| \cos(\pi/2^{n+1}).$$

The limiting value is real and is evaluated by the well known infinite product

$$\cos \pi/4 \cos \pi/8 \cos \pi/16 \cdots = 2/\pi.$$

II. *Solution by Michel McKiernan and Berthold Schweizer, Illinois Institute of Technology.* The problem may be solved for any z. Let $z=r \exp(i\theta)$. By constructing the vector $f(z)$ in the complex plane it is readily seen that

$$f(z) = r \cos(\theta/2) \exp(i\theta/2),$$

and that, in general,

$$f^n(z) = r\left[\prod_{k=1}^{n} \cos(\theta/2^k)\right] \exp(i\,\theta/2^n).$$

Since (see, *e.g.*, Knopp, *Infinite Series*, p. 228)

$$\prod_{k=1}^{\infty} \cos(\theta/2^k) = (\sin\theta)/\theta$$

we have

$$\lim_{n\to\infty} f^n(z) = (r \sin\theta)/\theta,$$

and in particular,

$$\lim_{n\to\infty} f^n(i) = 2/\pi.$$

Also solved by W. A. Al-Salam, G. B. Charlesworth, Marius Cohn, Hüseyin Demir, A. L. Epstein, Thomas Erber, R. V. Esperti, Calvin Foreman, A. J. Goldman, D. S. Greenstein, J. R. Hatcher, R. E. Heaton, A. R. Hyde, Rufus Isaacs, P. G. Kirmser, J. D. E. Konhauser, D. C. B. Marsh, W. H. McKenzie, C. S. Ogilvy, F. D. Parker, Paul Payette, C. A. Rogers, B. D. Seckler, Eugene Usdin, Chih-yi Wang, J. V. Whittaker, and the proposer. Late solutions by T. Y. Chow, P. B. Johnson, and M. S. Klamkin.

Editorial Note. The limit of the infinite product is apparent from

$$\frac{\sin\theta}{\theta} = \cos(\theta/2)\frac{\sin(\theta/2)}{\theta/2} = \cos(\theta/2)\cos(\theta/4)\frac{\sin(\theta/4)}{\theta/4}$$

$$= \cdots = \cos(\theta/2)\cos(\theta/4)\cdots\cos(\theta/2^k)\frac{\sin(\theta/2^k)}{\theta/2^k}.$$

The problem is intimately tied up with Vieta's famous infinite product for $2/\pi$ and also with the well known quadratrix of Hippias and Dinostratus.

The 2319th Digit in 1000!

E 1180 [1955, 493]. *Proposed by M. S. Klamkin and Alex Kraus, Polytechnic Institute of Brooklyn*

Determine the 2319th digit in the expansion of 1000!.

Solution by J. B. Muskat, Cambridge, Mass. By Stirling's formula we find that the expansion of 1000! has 2568 digits. Since 5 divides 1000! 249 times, and 2 divides 1000! more than 500 times, the expansion of 1000! terminates in a string of 249 zeros. Further, since $2568-249=2319$, the digit we seek is the last non-zero digit in the expansion. Clearly this digit is even, and depends upon the last digits of the factors 1, 2, 3, $\cdots$, 1000 of 1000! after removal of all factors 5. Taking the factors successively in sets of ten and removing the multiples of 5 we have, for the product of the final digits of the remaining factors in each set,

$$1\cdot 2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8\cdot 9 \equiv 6 \pmod{10}. \tag{1}$$

There are 100 of these products, leaving 200 numbers which are divisible by 5, and if 5 is factored out of each, 200! remains. We then have 20 more products like (1), leaving 40 numbers which are divisible by 5, and if 5 is factored out of each of these, 40! remains. We get 4 more products like (1), leaving 8 numbers which are divisible by 5, and if 5 is factored out of each of these, 8! remains. Dividing out 5 once more, we have left

$$1\cdot 2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8 \equiv 4 \pmod{10}.$$

Now $4\cdot 6^{124}\equiv 4 \pmod{10}$, whence 4 is the last digit in $(1000!)/5^{249}$. But 2^{249} must also be divided out to complement the 5^{249}. Since $2^{249}\equiv 2 \pmod{10}$, it now follows that the last digit in $(1000!)/10^{249}$, which is the digit we are seeking, is 2.

Also solved similarly by the proposers. Different (in some cases incorrect or incomplete) solutions were given by G. E. Bardwell, Hüseyin Demir, G. B. Charlesworth, Fred Cherry, Michael Goldberg, A. J. Goldman, A. R. Hyde, C. W. Langley, D. C. B. Marsh, C. S. Ogilvy, Paul Payette, Walter Penney, B. E. Rhoades, C. M. Sandwick, Sr., and Chih-yi Wang. Late communications from H. F. Bennett, John Christopher, Jerome Manheim, and Harold Saks.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BT E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4670 [1956, 47]. *Correction.* For $(0, \infty)$ read $[0, \infty)$.

4671 [1956, 47]. *Correction.*

The factor a^{2n+1} should appear in the right member of the required relation.

4678. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Evaluate

$$\tan^{-1} 1 - \tan^{-1} \tfrac{1}{3} + \tan^{-1} \tfrac{1}{5} - \cdots.$$

4679. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

4680. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Solve the following generalization of Clairaut's differential equation

$$y - xy' + \frac{x^2y''}{2!} + \cdots (-1)^{n-1} \frac{x^{n-1}y^{(n-1)}}{(n-1)!} + (-1)^n \frac{x^nF(y^{(n)})}{n!} = G(y^{(n)}).$$

4681. *Proposed by Jack Klugerman, Evans Signal Laboratory, Belmar, N. J.*

Given a real symmetric matrix A which is triple diagonal, *i.e.*, it has a diagonal, an upper diagonal, a lower diagonal, and the remaining elements are zero; if c is the eigenvalue with highest multiplicity m, then there must be at least $m-1$ zeros in the upper diagonal.

4682. *Proposed by R. C. Lyness, Preston, England*

(a) Prove that when the series

$$1 + \sum_{r=1}^{\infty} \binom{r\alpha}{r-1} \frac{x^r}{r}$$

is convergent, its sum, y, satisfies $y = 1 + xy^\alpha$.

(b) Prove also that

$$\sum_{r=1}^{\infty} \binom{r\alpha + \beta - 1}{r-1} \frac{x^r}{r} = \frac{y^\beta - 1}{\beta}.$$

SOLUTIONS

An Invalid Inequality

4603 [1954, 571]. *Proposed by H. S. Shapiro, New York University*

Given $x_i \geqq 0$, $i = 1, 2, \cdots, n$. Establish

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geqq \frac{n}{2},$$

equality occurring only if all denominators are equal.

Note by F. H. Northover, Memorial University of Newfoundland. I am indebted to Professor M. J. Lighthill, F.R.S., of Manchester University for the following counter-example.

Let $n=20$ and take $x_1, x_2, \cdots, x_{20}$ (in that order) to be $1+5\epsilon$, 6ϵ, $1+4\epsilon$, 5ϵ, $1+3\epsilon$, 4ϵ, $1+2\epsilon$, 3ϵ, $1+\epsilon$, 2ϵ, $1+2\epsilon$, ϵ, $1+3\epsilon$, 2ϵ, $1+4\epsilon$, 3ϵ, $1+5\epsilon$, 4ϵ, $1+6\epsilon$, 5ϵ; where ϵ is small. We can easily establish

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{19}}{x_{20}+x_1}+\frac{x_{20}}{x_1+x_2}=10-\epsilon^2+O(\epsilon^3),$$

and so, for ϵ small enough, the expression is less than $n/2=10$.*

Simultaneously Vanishing Real Functions

4617 [1954, 719; 1955, 186]. *Proposed by Albert Wilansky, Lehigh University*

Let $\{u_n\}$, $\{v_n\}$ be real sequences, $n=2, 3, 4, \cdots$, such that

$$\sum (u_n^2+v_n^2)<1.$$

Let f, g be real continuous functions of two real variables x, y, $x^2+y^2\leqq 1$, such that

$$f(\cos\theta, \sin\theta)=\cos\theta-\sum (u_n\cos n\theta+v_n\sin n\theta)$$

$$g(\cos\theta, \sin\theta)=\sin\theta-\sum (v_n\cos n\theta-u_n\sin n\theta).$$

Show that f, g vanish simultaneously at some point (x, y) for which $x^2+y^2<1$.

Solution by the Proposer. The hypothesis on u_n, v_n should read $\sum(u_n^2+v_n^2)^{1/2}<1$. With this change, let

$$h(z)=\begin{cases} z-\sum_{n=2}^{\infty}(u_n+iv_n)z^{-n} & \text{for } |z|\geqq 1 \\ f(x, y)+ig(x, y) & \text{for } |z|=|x+iy|\leqq 1. \end{cases}$$

Then h is continuous on the complex plane, $h(z)/z\to 1$ as $z\to\infty$, and $h(z)$ has no zeros for $|z|\geqq 1$ since the hypotheses ensure that, for such z, $|h(z)|>|z|-1\geqq 0$. By a well-known result (*e.g.*, problem 4475 [1953, 271]), h has a zero. This gives the result. The method can obviously be applied to more complicated examples.

Also solved by A. J. Goldman and J. J. Kohn who remark that, with the original hypotheses it can only be shown that f, g vanish simultaneously at some point (x, y) for which $x^2+y^2\leq 1$.

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* The proposer submitted proof that the inequality is true for $n=3$, 4, but confessed that he had not found a proof for greater values. C. R. Phelps proved the case $n=5$. Nine alleged "proofs" for general n were received by the editor.

Hermitian Matrices

4618 [1955, 45]. *Proposed by A. Oppenheim, University of Malaya, Singapore*

$A=(a_{ik})$, $B=(b_{ik})$ are two non-negative definite Hermitian matrices of order n. C_n is a matrix whose elements are $a_{ik}^n+b_{ik}^n$. If a, b, c_n are the determinants of A, B, C_n, prove that $c_n \geqq (a+b)^n$.

Solution by the Proposer. Let $E=(e_{ik})$, $F=(f_{ik})$ be non-negative definite Hermitian matrices. Then $G=(e_{ik}\cdot f_{ik})$ and $H=(e_{ik}+f_{ik})$ are both non-negative definite Hermitian matrices and their determinants satisfy the inequalities

$$g \geqq ef, \tag{1}$$

$$h^{1/n} \geqq e^{1/n} + f^{1/n}. \tag{2}$$

(See *Journal London Math. Soc.* 5(1930), 114–119 for references and proofs.) Let $A=(a_{ik})$, $B=(b_{ik})$ and write

$$A_m = (a_{ik}^m), \qquad B_m = (b_{ik}^m), \qquad C_m = A_m + B_m, \qquad (m = 1, 2, \cdots).$$

If the respective determinants are a_m, b_m, c_m, then, by repeated application of (1),

$$a_m \geqq a^m, \qquad b_m \geqq b^m,$$

and, by (2),

$$c_m^{1/n} \geqq a_m^{1/n} + b_m^{1/n} \geqq a^{m/n} + b^{m/n}.$$

In particular, with $m=n$, $c_n \geqq (a+b)^n$, the inequality of the problem.

A Generalization of Ptolomey's Theorem

4619 [1955, 45]. *Proposed by V. F. Ivanoff, San Francisco, California*

Let L_i, $i=1, 2, \cdots, 5$, be five lines in space. Denote the angle between L_i and L_j by (ij). Prove that the determinant

$$\left| \sin^2 \frac{(ij)}{2} \right| = 0 \qquad (i, j = 1, 2, \cdots, 5).$$

Solution by Louise A. Wolf, University of Wisconsin. We have

$$\sin^2 \frac{(ij)}{2} = \frac{1-\cos(ij)}{2} = \tfrac{1}{2}(1 - x_ix_j - y_iy_j - z_iz_j),$$

where x_i, y_i, z_i are the direction cosines of L_i. Thus

$$|A| = \left| \sin^2 \frac{(ij)}{2} \right| = \frac{1}{32} \left| 1 - x_ix_j - y_iy_j - z_iz_j \right|,$$

which can be expressed as the product of the two singular determinants

$$|A| = \frac{1}{32}\begin{vmatrix} 1 & x_1 & y_1 & z_1 & 0 \\ 1 & x_2 & y_2 & z_2 & 0 \\ 1 & x_3 & y_3 & z_3 & 0 \\ 1 & x_4 & y_4 & z_4 & 0 \\ 1 & x_5 & y_5 & z_5 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -x_1 & -x_2 & -x_3 & -x_4 & -x_5 \\ -y_1 & -y_2 & -y_3 & -y_4 & -y_5 \\ -z_1 & -z_2 & -z_3 & -z_4 & -z_5 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

and hence $|A| = 0$.

Also solved by Robert Breusch, William Kruskal, Bela Sz.-Nagy, H. A. Osborn, M. F. Smiley, J. A. Tierney, Chih-yi Wang, and the Proposer.

Editorial Note. The Proposer notes that the corresponding determinant of the fourth order vanishes if $L_1, \cdots, L_4$ lie in a plane. In fact the problem can be traced to the following problem in Salmon, *Conic Sections* (1879), p. 134: To find the relation connecting the mutual distances of four points on a circle. The answer is $|(ij)^2| = 0$, where (ij) is the distance between points P_i and P_j. What about more than three dimensions?

An Infinite Radical

4620 [1955, 45]. *Proposed by Herman Hanisch, New York University*

Evaluate

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}}}.$$

Solution by N. J. Fine, University of Pennsylvania. For $x > 0$, let

$$f_n(x) = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + \cdots + (x+n-1)\sqrt{1 + (x+n)}}}}.$$

Clearly $f_n(x)$ increases with n. But

$$\begin{aligned} x + 1 &= \sqrt{1 + x(x+2)} \\ &= \sqrt{1 + x\sqrt{1 + (x+1)(x+3)}} \\ &= \cdots \\ &= \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + \cdots + (x+n-1)(x+n+1)}}} > f_n(x). \end{aligned}$$

Hence $f(x) = \lim_{n\to\infty} f_n(x)$ exists, and $f(x) \leqq x+1$. Also

$$f(x) > \sqrt{1 + x\sqrt{1 + x\sqrt{1 + x\sqrt{\cdots}}}} = \frac{x + \sqrt{x^2+4}}{2} > x.$$

Thus $x < f(x) \leqq x+1$. Let $\delta(x) = x+1-f(x)$. Then $0 \leqq \delta(x) < 1$. From the easily derived functional equation

$$f^2(x) = 1 + xf(x+1)$$

we get

$$(x + 1 + f(x))\delta(x) = x\delta(x+1).$$

Hence

$$0 \leqq \frac{\delta(x)}{x} \leqq \frac{\delta(x+1)}{x+1} \leqq \cdots \leqq \frac{\delta(x+n)}{x+n} < \frac{1}{x+n} \to 0.$$

So $f(x) = x+1$ and the given expression $= f(2) = 3$.

Also solved by Robert Breusch, W. B. Carver, M. S. Cohen and Cornelius Groenewoud, Gerald Freilich, Emil Grosswald, H. M. Gurk, Edgar Karst, M. A. Kirchberg, Eugene Levin, D. J. Newman, F. H. Northover, F. R. Olson, Walter Penney, C. R. Phelps, W. G. Preble, H. D. Ruderman, Daniel Shanks, M. R. Spiegel, O. E. Stanaitis, T. Vij, Vladeta Vučković, J. V. Whittaker, and Koichi Yamamoto.

Editorial Note. Yamamoto mentions that the problem was proposed by S. Ramanujan. See *Collected Papers*, Cambridge 1927, p. 323, Question 289 (*Journ. Ind. Math. Soc.*, III, 90). For this and similar problems, refer to a paper, *On infinite radicals*, by Aaron Herschfeld, this MONTHLY, August–September 1935, pp. 419–429.

A Diophantine Problem

4621 [1955, 45]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For each of the following equations

$$x(3x + 1)(6x + 1) = ky^2, \qquad k = 1, 2, \cdots, 6,$$

find the solutions in positive integers or prove it insoluble.

Solution by W. J. Blundon, Memorial University of Newfoundland. Since x, $3x+1$, $6x+1$ are relatively prime in pairs, their square-free factors are given by the following table:

Case	k	x	$3x+1$	$6x+1$	*Case*	k	x	$3x+1$	$6x+1$
I	1	1	1	1	VI	5	5	1	1
II	2	2	1	1	VII	5	1	5	1
III	2	1	2	1	VIII	5	1	1	5
IV	3	3	1	1	IX	6	3	2	1
V	4	1	1	1	X	6	6	1	1

If $6x+1 = 5m^2$, or $3x+1 = 5m^2$, or $3x+1 = 2m^2$, then $2m^2 \equiv 1 \pmod 3$ which is clearly impossible. Therefore cases III, VII, VIII, IX all fail, and it follows that

$3x+1$ and $6x+1$ are perfect squares. Let $6x+1=X^2$. Then the original equation becomes

$$X^6 - X^2 = 12ky^2 \qquad k = 1, 2, 3, 4, 5, 6.$$

Let a be the product of the square-free factors of $12k$. Then the equation reduces to the form $x^4-1=aZ^2$, where a is given by

Case	I	II	IV	V	VI	X
a	3	6	1	3	15	2

Now it is known that $X^4-Y^4=aZ^2$ has no integral solutions (other than $X=Y$, $Z=0$) when $a=1$, 2, 3. Therefore cases I, IV, X fail. The only integral solution for $X^4-Y^4=15Z^2$ with $Y=1$ is (2, 1, 1) which is clearly inadmissible here. In the remaining case II, $X^4-Y^4=6Z^2$ has only one solution when $Y=1$, namely (7, 1, 20). Thus the given equations have no solutions unless $k=2$, and for this value, the unique solution $x=8$.

Also solved (partially) by D. C. B. Marsh and R. Venkatachalam Iyer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Universal Mathematics. Part I. Functions and Limits.* Preliminary Edition. By the 1954 Summer Writing Group of the University of Kansas. Lawrence, Kansas, Student Union Book Store. 1954. 10+310 pages. Planographed. $2.75.

This book is based upon an outline prepared by a committee of the Mathematical Association of America which was appointed to study the undergraduate program in mathematics and to make recommendations relating thereto. *Part I, Functions and Limits*, encompasses only the first semester's work of the freshman year and is intended for a course meeting three times per week. It is to be followed by *Part II, Structures in Sets*, which is to constitute the second semester's work. As the name implies, *Universal Mathematics* is intended to be used in a basic course in mathematics for all students, including engineering students. It is designed for students with normal high school preparation consisting of at least two years of high school mathematics, including intermediate algebra.

The book deviates sharply from other books on first year college mathematics both in its organization and in its content. In organizing their material the

* *Editorial Note:* In connection with this review the reader should refer to the following account by Professor Duren of his experiences in teaching this book at Tulane.

authors have given parallel presentations of almost every topic covered, one treatment being essentially intuitive, and the other a formal and more difficult treatment. It is not intended that the formal treatment be followed continuously but that appropriate selections of theorems and proofs be made by the instructor.

More emphasis is placed upon concepts and ideas than is customary in books at this level and the authors recognize that the average student with only normal preparation will not acquire from this book alone the manipulative techniques which are necessary for the further study of calculus and analytic geometry. The authors suggest that this situation be met by adding a "Technical Laboratory" course to the curriculum of the freshman year. This course would meet two hours per week throughout the year and would be intended for engineering students and other students who plan to take calculus and analytic geometry during their sophomore year.

The many departures from tradition in this book are not suggested by the chapter headings, which are as follows: I Coordinate Systems. II Scientific Measurements. III Functions. IV Limits. V Derivatives. VI Integrals. VII The Logarithmic and Exponential Functions. VIII Summary. In addition there is a fifteen page introduction and an eight page appendix. The Introduction contains an interesting discussion of the relationship of mathematical theories to the natural universe. The difference between empirical and mathematical proofs, the role of intuitive reasoning in mathematics, and the need for further mathematical theories are also discussed in the Introduction. Some of the ideas in the Introduction are followed up in the second chapter, where it is explained how real numbers and vectors are used for the measurement of physical quantities.

Throughout the book a serious effort has been made to present real mathematics at an elementary level. This is exemplified in the first chapter, where a careful discussion of the real number line by means of postulates is given. The real numbers are represented as infinite decimals and the postulate for a least upper bound is invoked to insure their existence. Further properties of real numbers, based upon the postulates for a complete ordered field, are given in the appendix. The treatment of real numbers paves the way for a discussion of approximate measurements and intervals, and for a discussion of the plane and its Cartesian coordinates. Vectors are introduced by means of parallel displacements and play a prominent role in the mathematical description of the plane. The usual analytic Euclidean geometry of the plane is by-passed in favor of a geometry of graphs (in which distances not measured along the axes, angles, circles, and arc length are undefined concepts). A relation is defined as a subset of the pairs which form the Cartesian product of two sets, and graphs of various kinds of relations are constructed.

Functions are introduced as special (single-valued) relations. A careful distinction is made between a function f and its functional value $f[a]$ at a. Brackets are used to indicate functional values whereas parentheses designate composite

functions. The letter x, when placed in parentheses after another function, is used to represent the identity function on a set X, so that if a is an element of X we have $f(x)[a]=f[x[a]]=f[a]$, and there is no logical distinction between f and $f(x)$. Polynomial functions and the algebra of these functions are discussed with reasonable thoroughness. A minor criticism might be made that it is neither proved nor stated that a polynomial function is expressible uniquely, and that without such uniqueness the concept of the degree of a polynomial is not well-defined. Convex sets on a line and in the plane, as well as convex functions, are also discussed in the chapter on functions.

The theory of limits which is used is based upon the Moore-Smith definition of a limit, which depends upon the properties of partially ordered sets. One might believe that these concepts would prove to be bitter pills for most beginning students. However this may be, and the reviewer is not too sanguine, the pills are sugar-coated with good illustrative examples and meticulous explanations. The more usual epsilon definition of a limit is given in the guise of a theorem. Limits of functions and continuous functions are defined and discussed for functions whose domains are convex sets. The theory of limits which is adopted leads to an elegant theory of the derivative and the definite integral.

The divided difference of a function over a neighborhood in its domain, which in a sense is a generalization of the familiar difference quotient, is fundamental to the authors' definition of a derivative. This divided difference is in fact an interval in which each point designates a difference quotient. The derivative is then defined as the final residue which is contained in all of the divided differences which are generated by a certain convergent limit process. If the limit process does not converge the final residue is a proper interval rather than a point, in which case the derivative does not exist. Even in this case the divided difference is a useful concept for the study of functions. These ideas are of course expressed in a precise mathematical form in the book. Standard topics such as the mean value theorem, maxima and minima, rates, motion, and inverse functions are in no way neglected.

The chapter on integration begins with a discussion of partitions, step functions, areas bounded by step functions, and Riemann sums. The Riemann sum which is defined, and which is fundamental for the definition of the definite integral, is not the classical Riemann sum, but is defined as a certain interval in which each point designates a classical Riemann sum. This makes possible the definition of a definite integral as a Moore-Smith limit of Riemann sums. Proofs of the fundamental theorem of the integral calculus, the substitution theorem, and the theorem for the existence of a definite integral of a continuous function are then given with an economy of effort. Existence theorems for definite integrals of monotone functions and sectionally integrable functions are seldom or never discussed in elementary courses, but they are proved in this book. The logarithmic function is defined by means of a definite integral and e is defined by the equation $\ln[e]=1$. It is shown that e^x is the inverse function of $\ln(x)$ and the properties of $\ln(x)$ and e^x are derived.

The authors have indicated on the cover of the book that it is "a book of experimental text materials." Trial and experiment will determine how much of the material can be effectively incorporated into courses designed for a large number of beginning students. The formal exposition, which is in a fairly concise mathematical style, will probably require, for the most part, more intellectual effort and innate mathematical ability than can reasonably be expected of most beginning students. But under the guidance of a experienced teacher the more able and ambitious students can study much of this material profitably. A larger number will be able to profit from the more discursive intuitive treatment upon which most of the class work is supposed to be based. This treatment makes use of diagrams, figures, and good illustrative examples wherever these aids are helpful. Both the formal and intuitive treatments include adequate lists of well chosen exercises. In the opinion of the reviewer the book would be easier to read if the formal parts were placed in appendices at the ends of the various chapters instead of being interspersed with the intuitive parts.

In spite of the general excellence of the exposition the reviewer believes that even the intuitive treatment will be difficult for most students to follow and that the instructor will not always find it easy to present the material effectively. But, as Professor Alfred Whitehead once wrote: "Whenever a textbook is written of real educational worth, you may be quite certain that some reviewer will say that it will be too difficult to teach from it. Of course it will be difficult to teach from it. If it were easy the book ought to be burned; for it cannot be educational."

Whether or not he expects to experiment with a course organized along the lines of this book the teacher of mathematics will do well to examine the book carefully. Its treatment of classical subject matter from a fresh and modern viewpoint will give him a deeper insight into his subject which cannot fail to have an impact upon his teaching.

H. P. EVANS
University of Wisconsin

TULANE EXPERIENCE WITH UNIVERSAL MATHEMATICS, PART I

The University of Kansas and the Social Science Research Council supported in 1954 a Kansas Summer Writing Group which wrote a book of experimental text materials called: *Universal Mathematics, Part I*, which was an effort to put into concrete form some of the results of the studies of the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America. This hastily written preliminary edition was given a mass trial at Tulane University as a universal mathematics course for all first year students who do not have to take intermediate algebra. Some 750 students in engineering, liberal arts, and business administration were involved. About 28 instructors, ranging from graduate teaching assistants to full professors, took part in the trial. Recently, a staff meeting was held to discuss the results.

The book is not yet suitable as a textbook and caused considerable difficulty to students and instructors. The main trouble is that students cannot read it. However, the chairman's proposal to replace it with one of the existing mathematical analysis texts met with unanimous opposition. Most of the staff wanted radical revisions of the book. Moreover, the failure rates using this text were higher than normal, although the conclusion that this was due entirely to the text is unwarranted since the quality of freshman classes was poorer than usual.

One thing is clear; and that is the fact that *Universal Mathematics* is not adaptable to students whose high school background in mathematics is scant. It was not intended to be "universal" in the sense that everybody should take this course at the beginning of the first college year, but universal in the sense that is for everybody who continues mathematics after acquiring competence in intermediate algebra and geometry. Thus some students are ready for *Universal Mathematics* while still in school and others are not ready for it when they enter college. It was written to what the Committee judged to be realistic prerequisites for a beginning first year college student whose high school training properly qualifies him for the traditional first college courses in mathematics.

In judging the value of *Universal Mathematics, Part I*, it should be emphasized that it was not intended to be a textbook in the proper sense. For one thing, it includes entirely too much material, too many different ideas, for a textbook. A proper textbook should be much simpler. This book was intended as a kind of mother of textbooks. Another aspect of it, which cannot be overlooked, is the unusual method by which it was uncompromisingly written. The book was written with two relatively complete and independent presentations. First a formal account was written to establish the mathematical structure of the work. After the formal account, with its postulates, definitions, theorems, and proofs was complete, an intuitive presentation was written to follow it closely in the same order. The two accounts appear side by side in the book. The intuitive account is used in the classroom, with the formal one as a reference. This method has several advantages and disadvantages. With the complete mathematical theory established and made dominant by principle, the textbook writer's usual freedom to choose his material and to choose the order of presentation of his topics is severely limited. It requires much more imagination to find appropriate motivations and interpretations for young students if mathematical theory determines the formulations of concepts and their order of introduction, while serious teaching purpose dictates that the concepts shall be well motivated and supported by meaningful exercises so as to become part of the student's working mental equipment. However, it turns out that if the really general and abstractly simple form of the theoretical ideas is chosen then the interpretation becomes surprisingly easy and rich.

Thus the advantages of the dual presentation, mathematical and intuitive, lie in the fact that every intuitive or incomplete exposition will definitely permit

expansion and elaboration in a correct formal theory which has been set down in axioms. Moreover the "hard" theory is out of the way so that the "easier" intuitive account can be read by itself, with the student and teacher, instead of the textbook writer, determining when they shall turn to the more austere formal theory; yet the formal theory fits with the intuitive account both in notation and structure so that this desirable transition is facilitated. The Tulane experience indicates that this was realized in practice. A disadvantage of the method is that the first stage of intuitive simplification here presented is still too formal. It needs at least two more stages of simplification and many more exercises. Another aspect of the same difficulty is concerned with the compromise between a correct formal mathematical language and the common classroom jargon of elementary mathematics. This method of writing does not facilitate the making of such a compromise and results in difficult reading even on the intuitive side.

The excess of ideas had interesting effects. The early introduction of relations and vector concepts which was a kind of side issue seemed to appeal to many instructors and students, though others did not like it. In general, one class would take hold of one set of ideas, and another class a different one. Some instructors found the Introduction very stimulating and used it as a part of the text material. The radical method of presenting limits was well liked by some and disliked by other instructors. One set of ideas seems to have been almost universally disliked, and that was the treatment of the divided difference as an interval of numbers. The chapter on scientific measurement was omitted to shorten the course. The fact that the book presents more ideas than is considered desirable caused unfavorable first reactions on teachers. Mathematics textbooks are required by convention to be as empty as possible of ideas, certainly empty in comparison with a textbook in chemistry, such as that of Pauling, or a first college textbook in physics. The Tulane trial seems to indicate that, although an idea-packed book requires that the teacher help the students to read the book, it turns out to be less boring to him in the end and also more interesting to the better students, without causing excessive trouble for the slow student. Why don't we use textbooks in mathematics which have a high density of ideas? What could it hurt?

Some of the defects of the book as a textbook could be obviously remedied by some worked-out examples, some answers to the problems, and more exercises. The method of paging two parallel accounts could be much improved, though it is more inconvenient to a casual reader than to a student who studies the book.

One of the Committee's general efforts seems to have been fairly well realized. This was an effort to rogue out unessential prerequisites to the study of calculus. An important one of these was the reexamination of the relation of the Euclidean analytic geometry to the calculus. The *Universal Mathematics, Part I* was written with a consistent "graphs" geometry, which is the geometry of the Cartesian product of the real line and the real line without the imposition

of the invariant Euclidean metric in the plane. Perhaps, teachers will feel that this is a doctrinaire position, but it seems in trial to make things actually easier. The modified Menger notation scheme seems to cause no particular difficulty if one does not attempt to explain it, and it may make teachers feel more honest.

There is still a serious difficulty in getting through the foundation of the elementary calculus. The method of writing so far has resulted in an excessive dwelling upon the real numbers system in the beginning although it was the intention of the authors to find a correct but brief foundation for calculus. A simpler start must be found. But despite the over-formality of this writing, the calculus theory presented does represent a real simplification.

At Tulane, as everywhere else, when the students actually get to elementary calculus they find it interesting and comparatively easy. In short, the Tulane experience indicates that elementary calculus is more appropriate for beginning college students than college algebra, or trigonometry, or analytic geometry. The morale of the students is better, and the technical difficulties with it are less than in the conventional subjects. Moreover, there is still reason to hope that students will learn more algebra by studying a well simplified calculus than they will in a course in algebra.

The treatment of the exponential and logarithmic functions before the trigonometric function worked out well. The fact that the calculus was imbedded in the "graphs" geometry dictates a postponement of the study of the trigonometric functions until one returns to Euclidean geometry. This analytic trigonometry can be postponed to the beginning of the second year calculus without causing any difficulty in most programs.

Thus *Universal Mathematics, Part I*, belongs to a growing trend in first year college mathematics to start with calculus as soon as possible and on as simple a foundation as possible. Whether one continues with more calculus and geometry in the second semester or whether one turns to sets and integers, as *Universal Mathematics, Part II*, proposes to do, is a matter to be determined by what is the most valuable in actual experiences, rather than by opinion. The Committee is preparing *Part II* which is formally sets and integers and intuitively "choice and chance" to make a comparison by experience possible.

W. L. DUREN, JR.
Tulane University

Differential and Integral Calculus. By H. M. Bacon. Second Edition. McGraw-Hill Book Company, New York, 1955. vii+547 pages. $6.00.

Calculus. By William L. Hart. D. C. Heath & Company, Boston, 1955. xiii+626 pages. $5.50.

The first edition of Bacon's book was published in 1942 but apparently was not reviewed in this MONTHLY at that time. The author states that several sections have been rewritten for the second edition but he does not claim to have added new material or to have changed the order of topics. This book is typical

of a large number of texts designed for use in traditional introductory courses devoted primarily to the development of formal skills. It deserves to be ranked among the best of its class. While individual sections are well written and show the result of careful planning in the light of much classroom experience, the general organization of the topics could be improved. For example, the presentation of some sixty pages of material on the computation of anti-derivatives before giving any motivation for such activity has obvious disadvantages as does a long chapter on limits placed at the very beginning of the text. (The author mentions in the preface that this chapter on limits can be omitted.) Solid analytic geometry is presupposed and the discussion of partial derivatives and multiple integrals is rather brief. The book closes with a chapter on the elements of differential equations.

Several factors combine to render Hart's book an outstanding addition to the list of calculus texts. In the first place it is more extensive than most books. At least three semesters would be needed to present all of the material which it contains. This is due in part to the fact that in addition to rather full but more or less standard chapters on differential equations and hyperbolic functions there are three chapters devoted to advanced topics in partial differentiation and their applications in which are treated matters such as envelopes, implicit differentiation in general systems, elementary differential geometry of space curves and surfaces, Taylor's series for functions of two variables, *etc.*, which are not often found together in one elementary text. The definite integral is introduced as soon as the usual material on the differential calculus has been discussed, and the student is required to compute some of these by elementary summation and limiting processes before any connection with anti-derivatives is mentioned. The sequence of three chapters on methods of integration is interrupted by a chapter in which the major applications of the definite integral are presented. (The fact that the discussion of centroids and moments of inertia is here confined to the case of one dimensional mass distributions, while the discussion of these topics in the case of two and three dimensional distributions is postponed until multiple integrals are available deserves to be noted and approved.) While plane analytic geometry is a prerequisite, such topics as parametric equations and polar coordinates are treated *ab initio*. In fact, unusual emphasis is placed on the parametric form. This yields some useful results, not the least of which is a most convincing intuitive derivation of L'Hospital's rule. There is an excellent chapter on solid analytic geometry, and the emphasis which has been placed on parametric equations makes the transition to space curves a simple matter. A most conspicuous feature is the steady increase in the level of sophistication as the book progresses. At no stage however, is there hesitation in mentioning results which have proofs beyond the level of the book. When this is done the meaning and use of the theorems mentioned are well illustrated by examples.

H. T. MUHLY
State University of Iowa

OBITUARY

WILLIAM DEWEESE CAIRNS

In Memoriam

William DeWeese Cairns was born in Troy, Ohio, on November 2, 1871, and died on July 15, 1955, in Pasadena, California. In 1898 he married Iva M. Crofoot, and there were two children, Mary Catharine Cairns who died in 1938, and Robert W. Cairns who is Director of Research for the Hercules Powder Company of Wilmington, Delaware. After the death of his first wife he married, in 1930, Mrs. Bertha N. Pope of Oberlin, who survives him. He was a member of the Congregational Church.

After preparation in the Troy High School, Cairns entered Ohio Wesleyan University where he received the A.B. degree in 1892. He taught physics in the Troy High School during the years 1894–96, and then went to Harvard University where he received the A.B. degree in 1897 and a year later, as Schattuck Scholar, the degree A.M. After teaching mathematics for a year at the Calumet (Michigan) High School, he became an Instructor in 1899 at Oberlin College, and in 1904 an Assistant Professor. In the fall of 1905, with a two-year leave of absence from Oberlin, he went to Germany and entered Göttingen University from which he received the Ph.D. degree in 1907. His Göttingen dissertation was entitled "*Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen.*" He returned to Oberlin College where he became Professor of Mathematics in 1920 and served as head of the department from 1920 until his retirement in 1939. In 1943 he taught a pre-meteorological course at the University of New Mexico.

To generations of Oberlin College students he was a patient, friendly, and beloved teacher, but members of the Mathematical Association of America will remember Will Cairns as the enthusiastic guiding spirit of that organization through its early years. He served as its Secretary-Treasurer from the beginning in 1916 until he resigned from the office at the end of 1942. On that occasion the late Earle Raymond Hedrick wrote for this MONTHLY:*

"During this time its membership has increased to considerably more than twice the eleven hundred charter members; meetings are held in twenty-two different sections throughout the country; and an ambitious program of periodical and book publication, and of sponsorship of other periodicals and of prize competitions, has come to fruition. Throughout this period of increasing usefulness the now retiring Secretary-Treasurer, William DeWeese Cairns, has been an outstanding leader in the determination of policy and in the conduct of all the Association's affairs. In the extent and value of his services to the Association, he is rivalled only by Herbert Ellsworth Slaught (1861–1937), also an

* A remarkably good likeness of Professor Cairns appears as a frontispiece to volume 50 (1943) of this MONTHLY.

enthusiastic and constant worker for the interests of the Association from the time of its founding. To these two men far more than to any others the Association owes its success and its present strong position."

Unwilling to have Cairns sever his official connection with the organization, the Association immediately in 1943 elected him to a two-year term as President. Then at the annual business meeting in Chicago on November 26, 1944, he was made Honorary President of the Association for life.

Though Professor Cairns' most important contribution to American mathematics was undoubtedly the devoted service which he gave to the Association, he still found some time for other mathematical activity. He was the author of about a dozen papers published in *Annals of Mathematics*, *Bulletin of the American Mathematical Society*, *American Mathematical Monthly*, *Science*, and *Mathematics Teacher*. He was a member of the American Mathematical Society, the Ohio Association of Teachers of Mathematics and Science, and the American Association for the Advancement of Science, and he served for many years as the representative of the Mathematical Association on the Council of the latter organization.

The Association made William DeWeese Cairns an Honorary President "for life," but his death can not terminate our appreciation of his most valuable service, and those of us who had the privilege of knowing him personally may continue to honor his memory.

W. B. CARVER

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SHELL MERIT FELLOWSHIPS FOR HIGH SCHOOL SCIENCE AND MATHEMATICS TEACHERS

The Shell Companies Foundation, Inc., has announced a broad program of recognition fellowships for high school teachers of science and mathematics.

Through the program, worked out with the cooperation of leading educational associations, Shell will underwrite summer seminars at Stanford and Cornell Universities for sixty teachers yearly. The Fellowship recipients, chosen on the basis of merit and demonstrated leadership qualities, will receive travel allowances, all tuition and fees, living expenses on the university campus and $500 in cash to make up for the loss of potential summer earnings. The fellow-

ships were particularly designed for able experienced teachers who ordinarily might seek remunerative summer employment outside the school system. The intensive seminar programs will include graduate-level classes, lectures by outstanding scientists, and visits to modern industrial installations and research laboratories.

Mathematics, physics, or chemistry teachers with five years' experience and known leadership ability will be eligible for the Fellowships. Thirty teachers from west of the Mississippi River will attend the eight-week Stanford program which will be administered by the Stanford School of Education. Thirty from east of the Mississippi will be invited to a similar six-weeks' series of courses at Cornell.

In addition to teachers, present heads of departments or supervisors with good backgrounds in mathematics, chemistry or physics who were previously teachers are also eligible. Full responsibility for the final selection of the sixty will rest with Stanford University and Cornell University.

Requests for Fellowship applications should be sent to Cornell University and Stanford University. The completed form of each applicant must be accompanied by personal recommendations from his high school principal, a faculty member from his most recent college, and someone who can attest to the applicant's leadership talents with young people outside school or with his professional associates.

The Fellowships are in addition to the Shell Companies Foundation's present $350,000 aid-to-education program which includes fifty graduate fellowships and twenty grants in fundamental research in science and engineering at forty-one colleges and universities.

SUMMER INSTITUTES FOR HIGH SCHOOL AND COLLEGE TEACHERS

The National Science Foundation will sponsor three institutes for teachers of mathematics in the summer of 1956. Preliminary announcements of these appear below. In each case housing will be provided in university dormitories and a limited number of stipends will be available for participants.

Iowa State Teachers College. This institute for high school teachers will be held during the period June 18 to July 27 under the direction of Professor Henry Van Engen. Inquiries should be addressed to Professor Van Engen, Iowa State Teachers College, Cedar Falls, Iowa.

University of Michigan. This institute for college teachers will be held during the period June 25 to August 14 under the direction of Professor T. H. Hildebrandt with the assistance of Professor P. S. Jones. The principal lecturers will be: Professor C. C. MacDuffee of the University of Wisconsin, whose title is "Elementary Algebra from an Advanced Viewpoint" and Professor G. B. Price of the University of Kansas whose title is "Limits, Derivatives and Integrals." Inquiries should be addressed to Professor Hildebrandt, University of Michigan, Ann Arbor, Michigan.

Williams College. This institute for high school and college teachers will be held during the period July 2 to August 11 under the direction of Professor Donald E. Richmond. The principal speakers will be Professor B. W. Jones of the University of Colorado who will lecture on "Algebra and Number Theory," and Professor S. C. Kleene of the University of Wisconsin whose title is "Sets, Logic, and Mathematical Foundations." Shorter lecture series will be given by R. H. Bing, "Topology"; W. L. Duren, Jr., "Universal Mathematics"; W. Prenowitz, "Geometric Vector Analysis and the Concept of Vector Space," and others, some from industry. Inquiries should be addressed to Professor Richmond, Williams College, Williamstown, Massachusetts.

SUMMER INSTITUTE FOR HIGH SCHOOL TEACHERS OF MATHEMATICS

The University of Minnesota will conduct a program in the basic and applied aspects of mathematics as these are related to the pedagogical needs of high school teachers. The program will be offered by the Department of Mathematics during the first term of the Summer Session. Scholarships covering tuition and living expenses will be available. Six credits in the Graduate School will be given to those who have been admitted to the Graduate School and who satisfactorily complete the work in the program. These credits will be applicable toward advanced degrees for those whose major subject is Education and whose minor subject is Mathematics and for those whose major subject is Mathematics. Those interested in the institute should write to B. R. Gelbaum, Associate Professor of Mathematics, 119 Folwell Hall, University of Minnesota, Minneapolis 14, Minnesota.

PERSONAL ITEMS

Catholic University announces the following: Mr. Eugene Lukacs, previously head of the Statistical Branch, Office of Naval Research, Washington, D. C., has been appointed to a professorship; Assistant Professor Choy-tak Taam has been promoted to an associate professorship; Interim Associate Professor N. A. Wiegmann has been promoted to an associate professorship.

Fenn College reports: Assistant Professor A. W. Brunson of Michigan College of Mining and Technology has been appointed to an assistant professorship; Mr. David Helacher, formerly a graduate assistant at the State University of Iowa, has been appointed to an instructorship.

Pennsylvania State University announces: Dr. Robert Ellis of the University of Chicago, Dr. P. C. Gilmore, formerly a research associate at the University of Toronto, Assistant Professor Harry Gonshor of the University of Miami, and Dr. G. N. Raney of Brooklyn College have been appointed to assistant professorships; Mr. R. L. Duncan, formerly a graduate assistant at the University, has been appointed to an instructorship; Mrs. Catherine Euwema and Miss Audrey Weber, previously a graduate assistant at the University, have been appointed to part-time instructorships.

At Purdue University: Professor Ralph Hull, head of the Department of Mathematics, is on leave of absence as a consultant at the Ramo-Wooldridge Corporation, Los Angeles, California; Professor Arthur Rosenthal is Acting Head of the Department during the academic year 1955–1956; Assistant Professor O. P. Aggarwal of the University of Washington, Dr. Helen Bozivich, formerly a research associate at Iowa State College, and Dr. W. H. Fleming, previously a mathematician at the Rand Corporation, Santa Monica, California, have been appointed to assistant professorships; Assistant Professors P. E. Irick and J. H. B. Kemperman have been promoted to associate professorships; Dr. Robert Baer, Dr. N. D. Kazarinoff, and Dr. Daniel Waterman have been promoted to assistant professorships; Dr. Chester Feldman of the Hammond Extension Center of the University has been promoted to an assistant professorship; Mr. J. E. Forbes, previously a teaching assistant, has been promoted to an instructorship; Professor C. T. Hazard and Associate Professor C. N. Wunder have retired with the title of Professor Emeritus; Assistant Professor Henry Teicher is on leave of absence for the year 1955–1956 and is in the Department of Mathematical Statistics at Stanford University; Dr. Robert Baer and Dr. Daniel Waterman received research grants for the summer of 1955.

University of New Hampshire reports the following: Mr. F. J. Lorenzen, Jr., previously an instructor at Tilton Academy, New Hampshire, has been appointed to a graduate assistantship; Mr. C. W. Schenck, formerly a lecturer at the University, has been appointed to an instructorship in the Mechanical Engineering Department.

Dr. M. Q. Barton, formerly a teaching assistant at the University of Illinois, has been appointed a research associate at Brookhaven National Laboratory, Upton, Long Island, New York.

Dr. J. S. Bendat, previously a research engineer at Northrop Aircraft Company, Hawthorne, California, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor W. H. Carnahan of Purdue University now has a position with D. C. Heath and Company, Publishers.

Professor Emeritus E. W. Chittenden of the State University of Iowa has accepted a position as mathematician with the Diamond Ordnance Fuse Laboratories, Washington, D. C., for the year 1955–1956.

Dr. A. E. Danese of the University of Tennessee has accepted a position as a mathematician with Eastman Kodak Company, Rochester, New York.

Mr. Reid Haywood, formerly a mathematician at the United States Naval Proving Ground, Dahlgren, Virginia, has a position as a senior computation mathematician at the Glenn L. Martin Company, Baltimore, Maryland.

Assistant Professor Jacob Korevaar of the University of Wisconsin has been promoted to an associate professorship.

Mr. G. N. Landes, previously a mathematical analyst at the Sandia Corpora-

tion, Albuquerque, New Mexico, has been appointed an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. Mark Lotkin of the Radio Corporation of America Service Company, Inc., Patrick Air Force Base, Cocoa, Florida, has accepted a position as a mathematician with the Republic Aviation Corporation, Farmingdale, New York.

Mr. W. C. Lowry, previously an assistant instructor at Ohio State University, has been appointed to an assistant professorship in education at the University of Virginia.

Miss Mardell S. Miskowski, formerly a mathematician at Aberdeen Proving Ground, Maryland, has accepted a position as a research engineer at North American Aviation, Downey, California.

Mr. C. R. Perisho of Mankato State Teachers College has been promoted to an assistant professorship.

Dr. C. M. Petty of Purdue University has a position with the Lockheed Missile Division, Van Nuys, California.

Mr. S. L. Prosser, previously a graduate assistant at the University of Alabama, has been appointed an applied science representative with the I.B.M. Corporation, Atlanta, Georgia.

Dr. Anatol Rapoport of the Center for Advanced Study in the Behavioral Sciences, Stanford, California, has been appointed to an associate professorship in mathematical biology at the Mental Health Research Institute, University of Michigan.

Mr. W. G. Stokes of Vanderbilt University has been appointed Associate Professor and Head of the Department of Mathematics at Austin Peay State College.

Dr. R. G. Stoneham, recently a mathematician at Logistics Research, Inc., Redondo Beach, California, has been appointed to an assistant professorship at the University of California, Santa Barbara College.

Mr. Irwin Stoner, formerly a dynamics engineer at the Bell Aircraft Corporation, Niagara Falls, New York, has accepted a position as an associate mathematician with the Cornell Aeronautical Laboratory, Buffalo, New York.

Miss Virginia R. Swann, previously a graduate student at the University of North Carolina, is now employed as a mathematician at the Naval Ordnance Laboratory, Silver Spring, White Oak, Maryland.

Mr. L. O. Thompson has been appointed to an instructorship at Marshall College.

Assistant Professor S. I. Vrooman of Rensselaer Polytechnic Institute has been appointed to a visiting professorship at the University of Pittsburgh.

Dr. D. T. Walker has been appointed to an assistant professorship at Memphis State College.

Mr. L. F. Babcock of the Woodmere Academy, Long Island, New York, died on August 27, 1955.

Mr. A. D. Pierson, retired chairman of the Department of Mathematics of the Junior College of Kansas City, died on October 26, 1955. He was a member of the Association for thirty-five years.

Professor Emeritus Mary E. Sinclair of Oberlin College died on June 3, 1955. She was a charter member of the Association.

Professor Emeritus M. O. Tripp of Wittenberg College died on November 30, 1955. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-NINTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-ninth annual meeting of the Mathematical Association of America was held at the Rice Institute, Houston, Texas, on Friday, December 30, 1955, in conjunction with the annual meeting of the American Mathematical Society. Three hundred and eighty-two persons were registered, including two hundred and fifty-four members of the Association.

Sessions of the Association were held on Friday morning and afternoon in the Physics Amphitheatre of Rice Institute. Vice-President G. B. Price presided at the morning session, President W. L. Duren at the annual business meeting, and Professor W. T. Reid presided at the remainder of the afternoon session. The Program Committee for the meeting consisted of Wallace Givens, Chairman; W. T. Reid, and C. R. Sherer.

FIRST SESSION OF THE ASSOCIATION

"Some aspects of the theory of knots," by Dr. Guillermo Torres, Institute of Mathematics, University of Mexico.

"Simple unsolvable problems about semi-groups and groups," by Professor W. W. Boone, Catholic University of America and the Institute for Advanced Study.

"Mathematics instruction and the new fields of electronic computation," by Dr. H. R. J. Grosch, Aircraft Gas Turbine Division, General Electric Company.

SECOND SESSION OF THE ASSOCIATION

Annual Business Meeting of the Association.

"Report on current programs to improve science teaching," by Dr. L. W. Cohen, National Science Foundation.

"Some recent discoveries and near misses in topology," by Professor R. H. Bing, University of Wisconsin.

"Stimulating interest of youthful talent in mathematics and other sciences," by Professor H. J. Ettlinger, University of Texas.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday evening in the lounge of the Cohen House, with eighteen members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Duren of the following Nominating Committee for 1956: Ivan Niven, Chairman; H. E. Bray, and M. R. Hestenes.

Professor W. B. Carver was re-elected as a member of the Finance Committee for 1956–1959.

It was voted to hold the Thirty-eighth Summer Meeting at Pennsylvania State University in August 1957 and to hold the 1957–1958 annual meeting (normally scheduled for December 1957) during the latter part of January 1958.

The Board voted to approve the printing of two ninety-six page issues of the MONTHLY during 1956. The publication of two Slaught papers during 1956 was approved if suitable manuscripts are available. The Editor was instructed to omit from the MONTHLY hereafter the names of members attending meetings of the Association and its sections, giving only the number of persons and of members in attendance.

Professor J. C. Oxtoby of Bryn Mawr College was invited to deliver the Hedrick Lectures on "Category and Measure" at the 1956 Summer Meeting.

On the recommendation of the Committee on Sections, the Board voted to hold meetings of section officers annually, preferably at the time and place of the summer meeting; to authorize the establishment of a Northeastern Section of the Association to include the territory previously included in the New England Region; to transfer Huntington (Cabell County), West Virginia, from the Allegheny Mountain Section to the Ohio Section.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, December 30, 1955, in the Physics Amphitheatre of the Rice Institute, Houston, Texas. President W. L. Duren presided.

The Secretary announced the results of the balloting for officers in which 1635 votes were cast: Professor R. V. Churchill of the University of Michigan was elected First Vice-President for the two-year term 1956–1957, and Dr. A. S. Householder of Oak Ridge National Laboratory and Professor M. F. Smiley of the State University of Iowa were elected Governors for the three-year term 1956–1958.

The Association was organized on this same date forty years ago with a charter membership of 1045 persons. Of these, 208 are still living and have maintained continuous membership in the Association. The membership as of December 23, 1955, is 6045.

A report for the Committee on Visiting Lecturers was presented by Professor D. E. Richmond. Professor J. S. Frame reported for the Committee on Employ-

ment Opportunities that the Employment Register had been available throughout the meeting in Anderson Hall. Dean W. L. Duren described the plans of the Committee on the Undergraduate Program, and Professor G. B. Price asked that suggestions be sent to him as chairman of the Committee to Review the Activities of the Association.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday, December 27 through Thursday morning. The Gibbs lecturer was Professor Joseph E. Mayer. Professor G. T. Whyburn delivered his retiring presidential address and an invited address was delivered by Professor S. S. Chern.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: F. E. Ulrich, Chairman; H. E. Bray, A. H. Brown, L. K. Durst, H. M. Gehman, Guy Johnson, Jr., G. R. MacLane, Szolem Mandelbrojt, and J. W. T. Youngs.

Registration headquarters was in the Fondren Library of Rice Institute. Dormitory accommodations were provided in Wiess Hall and meals were available in the Rice Commons. The lounges of the Fondren Library and of the Cohen House (Faculty Club) were open daily. Tea was served every afternoon in the Cohen House. A banquet was held on Wednesday evening in the Commons. On Thursday afternoon a trip was taken to the Baytown Refinery of the Humble Oil and Refining Company, and on Thursday evening there was a concert by the Lyric Art String Quartet in Fondren Library.

President R. L. Wilder of the Society acted as toastmaster at the banquet. President W. V. Houston of Rice Institute spoke on the critical shortage of mathematicians and other scientists, and President W. L. Duren responded for the Association by telling of some of the steps being taken by the Association to attempt to alleviate this shortage. Dr. L. W. Cohen presented a resolution of thanks which was unanimously adopted expressing the deep appreciation of the members of both mathematical organizations to the Faculty and their ladies, to the President and Trustees of the Rice Institute for the excellent arrangements and accommodations provided for the meeting. Both organizations will remember the characteristic Texas hospitality they enjoyed during the Annual Meeting in 1955.

H. M. GEHMAN, *Secretary-Treasurer*

ITINERARIES OF VISITING LECTURERS, 1956

G. B. Price

Univ. of Arkansas and environs	Fayetteville, Ark.	Feb. 13–22
Memphis State College	Memphis, Tenn.	Feb. 23–25
Denison Univ.	Granville, Ohio	March 12–14
Case Institute of Technology	Cleveland, Ohio	March 15–17

Univ. of Detroit and Wayne Univ.	Detroit, Mich.	March 19–23
Michigan Section, M.A.A.	Ann Arbor, Mich.	March 24
Iowa State Teachers College	Cedar Falls, Iowa	April 2–4
St. Olaf Coll. and Carleton Coll.	Northfield, Minn.	April 5–8
Macalester College	St. Paul, Minn.	April 9–10
Beloit Coll. and Univ. of Wisconsin	Beloit and Madison, Wis.	April 11–12
Nat'l. Council of Teachers of Math.	Milwaukee, Wis.	April 13
Univ. of Omaha	Omaha, Neb.	April 16–18
Univ. of Nebraska	Lincoln, Neb.	April 19–21
State Teachers College	Shippensburg, Pa.	May 7–9
Drew Univ.	Madison, N. J.	May 10–12
Hunter Coll. and environs	New York, N. Y.	May 14–16
Skidmore College	Saratoga Springs, N. Y.	May 17
Alfred Univ.	Alfred, N. Y.	May 18

D. H. Lehmer

San Jose State College	San Jose, Calif.	Feb. 20 or 21
Univ. of Redlands	Redlands, Calif.	Feb. 29–March 2
Whittier College	Whittier, Calif.	March 2
Univ. of Arizona	Tucson, Ariz.	March 5–9
New Mexico A and M	Las Cruces, N. M.	March 12–13
New Mexico Inst. of Mining and Tech.	Socorro, N. M.	March 14–16
Univ. of Colorado and environs	Boulder, Colo.	March 19–22
Texas Christian Univ. and environs	Fort Worth, Tex.	March 26–28
Fresno State College	Fresno, Calif.	April 3
Univ. of Oregon	Eugene, Ore.	April 23–25
Oregon State College	Corvallis, Ore.	April 26–27
Univ. of Washington and environs	Seattle, Wash.	April 30–May 4
Montana State Univ.	Missoula, Mont.	May 7–8
State Coll. of Washington and Univ. of Idaho	Pullman, Wash. and Moscow, Idaho	May 9–12
Northwest Nazarene College	Nampa, Idaho	May 14–15

George Pólya

Tulane Univ.	New Orleans, La.	Feb. 17–21
Philander Smith College	Little Rock, Ark.	Feb. 23–25
Univ. of Mississippi	University, Miss.	Feb. 27–28
Alabama Polytechnic Inst.	Auburn, Ala.	March 1–3
Univ. of Miami	Coral Gables, Fla.	March 5–7
Univ. of Florida	Gainesville, Fla.	March 8–10
Univ. of South Carolina	Columbia, S. C.	March 12–15
Southeastern Section of M.A.A.	Athens, Ga.	March 16–17

The visiting lecturers for 1956–57 will be Professors Edwin Hewitt of the University of Washington, Kurt Mahler of the University of Manchester, England, A. W. Tucker of Princeton University, and J. L. Walsh of Harvard University. It is expected that detailed announcements will be sent out April 15 to all mathematics departments having members of the Association. Anyone not receiving an announcement by early in May may write to the Secretary of the Association for details and application blanks.

OFFICERS AND COMMITTEES AS OF JANUARY 1, 1956

OFFICERS

President, W. L. DUREN, JR., University of Virginia (1955–1956)
First Vice-President, R. V. CHURCHILL, University of Michigan (1956–1957)
Second Vice-President, G. B. PRICE, University of Kansas (1955–1956)
Editor, C. B. ALLENDOERFER, University of Washington (1952–1956)
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo (1953–1957)
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo (1953–1957)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

R. E. LANGER, University of Wisconsin (1951–1956)
SAUNDERS MACLANE, University of Chicago (1953–1958)
E. J. MCSHANE, University of Virginia (1955–1960)

Governors at Large

PHILIP FRANKLIN, Massachusetts Institute of Technology (1954–1956)
C. E. SPRINGER, University of Oklahoma (1954–1956)
H. W. BRINKMANN, Swarthmore College (1955–1957)
M. A. ZORN, Indiana University (1955–1957)
A. S. HOUSEHOLDER, Oak Ridge National Laboratory (1956–1958)
M. F. SMILEY, State University of Iowa (1956–1958)

Sectional Governors (July 1, 1953–June 30, 1956)

Illinois, E. B. MILLER, Illinois College
Iowa, E. N. OBERG, State University of Iowa
Louisiana-Mississippi, F. A. RICKEY, Louisiana State University
Maryland-Dist. of Col.-Virginia, S. B. JACKSON, University of Maryland
Michigan, P. S. JONES, University of Michigan
Minnesota, K. O. MAY, Carleton College
Philadelphia, C. O. OAKLEY, Haverford College
Southern California, C. W. TRIGG, Los Angeles City College
Texas, E. R. HEINEMAN, Texas Technological College

Sectional Governors (July 1, 1954–June 30, 1957)

Allegheny Mountain, MORRIS OSTROFSKY, Westinghouse Electric Corporation
Indiana, RALPH HULL, Purdue University
Kentucky, H. H. DOWNING, University of Kentucky
Metropolitan New York, R. M. FOSTER, Polytechnic Institute of Brooklyn
Nebraska, M. A. BASOCO, University of Nebraska
Northern California, W. H. MYERS, San Jose State College
Oklahoma, L. W. JOHNSON, Oklahoma Agricultural and Mechanical College
Rocky Mountain, C. A. HUTCHINSON, University of Colorado
Wisconsin, R. C. HUFFER, Beloit College

Sectional Governors (July 1, 1955–June 30, 1958)

Kansas, C. B. READ, University of Wichita
Missouri, F. F. HELTON, Central College
Northeastern, G. B. THOMAS, JR., Massachusetts Institute of Technology
Ohio, L. L. LOWENSTEIN, Kent State University
Pacific Northwest, IVAN NIVEN, University of Oregon

Southeastern, F. W. KOKOMOOR, University of Florida
Southwestern, M. S. HENDRICKSON, University of New Mexico
Upper New York State, J. F. RANDOLPH, University of Rochester

STANDING COMMITTEES OF THE ASSOCIATION

FINANCE COMMITTEE

W. B. CARVER (1956–1959), J. F. RANDOLPH (1954–1957), H. M. GEHMAN, *ex officio*.

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

TIBOR RADO, *Chairman* (1954–1959), I. S. SOKOLNIKOFF (1951–1956), P. R. HALMOS (1952–1957), SAMUEL EILENBERG (1953–1958), C. G. LATIMER (1955–1960), E. E. FLOYD (1956–1961).

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

EDWIN HEWITT, *Chairman* (1954–1956), MARK KAC (1955–1957), S. C. KLEENE (1956–1958).

JOINT COMMITTEE ON PLACES OF MEETINGS

W. M. WHYBURN, *Chairman* (1954–1956), C. B. MORREY, JR. (1955–1957), R. M. THRALL (1956–1958).

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

L. M. KELLY, *Chairman* (July 1954–June 1957), R. E. GREENWOOD (July 1955–June 1956), LEO MOSER (July 1955–June 1958); L. E. BUSH, *Director* (July 1953–June 1958).

COMMITTEE ON SECTIONS

J. C. POLLEY (1953–1956), C. H. FRICK (1955–1958), EDITH R. SCHNECKENBURGER. *ex officio*.

COMMITTEE ON SLAUGHT MEMORIAL PAPERS

J. L. KELLEY, *Chairman* (1954–1956), H. S. M. COXETER (1955–1957), R. D. SCHAFER (1956–1958), C. B. ALLENDOERFER, *ex officio*.

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES

J. S. FRAME, *Chairman*, H. M. BACON, RAY BERKOWITZ, W. M. HIRSCH, MORRIS OSTROFSKY, G. W. PATTERSON, J. A. WARD.

COMMITTEE ON FILMS FOR CLASSROOM INSTRUCTION

P. S. JONES, *Chairman*, H. H. CAMPAIGNE, MARGUERITE LEHR.

COMMITTEE ON HIGH SCHOOL CONTESTS

T. F. COPE, *Chairman*, R. P. BAILEY, MAX KRAMER, EDITH R. SCHNECKENBURGER, ROTHWELL STEPHENS.

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COMMITTEE TO STUDY THE ACTIVITIES OF THE ASSOCIATION

G. B. PRICE, *Chairman*, C. W. CURTIS, DAVID GALE, ROTHWELL STEPHENS, R. R. STOLL, PATRICK SUPPES, G. B. THOMAS, JR.

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

E. J. MCSHANE, *Chairman*, J. G. KEMENY, G. B. PRICE, A. L. PUTNAM, A. W. TUCKER, W. L. DUREN, JR., *ex officio*.

COMMITTEE ON VISITING LECTURERS

B. W. JONES, *Chairman*, G. B. HUFF, D. E. RICHMOND, R. A. ROSENBAUM.

COMMITTEE ON THE 1956 CHAUVENET PRIZE

P. A. WHITE, *Chairman*, J. D. MANCILL, J. G. WENDEL.

NOMINATING COMMITTEE FOR 1956

IVAN NIVEN, *Chairman*, H. E. BRAY, M. R. HESTENES.

REPRESENTATIVES OF THE ASSOCIATION

On the Policy Committee for Mathematics:
D. E. RICHMOND (1955–1957), W. L. DUREN, JR., *ex officio*, H. M. GEHMAN, *ex officio*.
On the National Research Council:
EINAR HILLE (July 1, 1953–June 30, 1956).
On the Council of the American Association for the Advancement of Science:
L. M. GRAVES (1955–1956), H. J. ETTLINGER (1956–1957).
On the American Council on Education:
W. L. DUREN, JR., *ex officio*, H. M. GEHMAN, *ex officio*.
On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
DAVID BLACKWELL (1954–1956).
On the Committee on Definitions of Electrical Terms:
S. A. SCHELKUNOFF.
On a Committee of the N.C.T.M. on a Vocational Pamphlet:
A. L. PUTNAM.
Associate Editor of Mathematics Student Journal:
IZAAK WIRSZUP (1954–1956).

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.
No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.
No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.
No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.
No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.
No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.
No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.
No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.
No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.
No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, April 28, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY, University of Kentucky, Lexington, April 28, 1956.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1956.

MINNESOTA, Augsburg College, Minneapolis, May 5, 1956.

MISSOURI, Fontbonne College, St. Louis, April 21, 1956.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Oberlin College, Oberlin, April 14, 1956.

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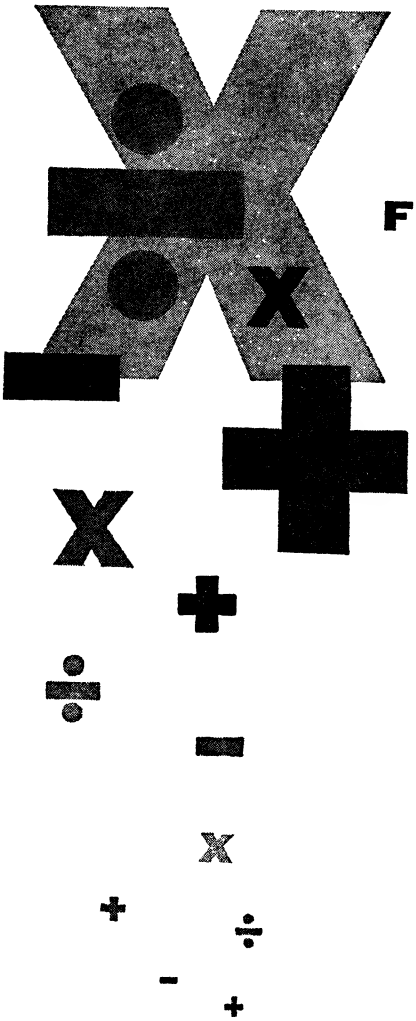
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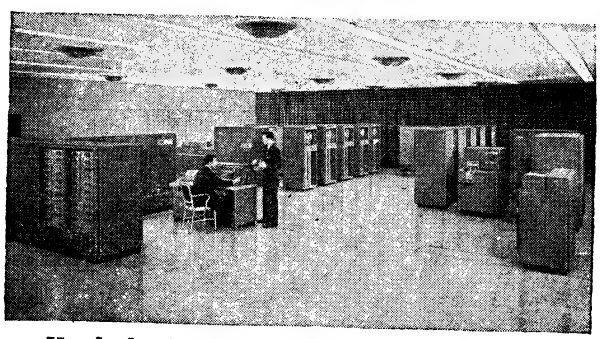
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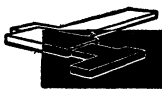
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SOLVABILITY AND CONSISTENCY FOR LINEAR EQUATIONS AND INEQUALITIES*

H. W. KUHN, Byrn Mawr College

1. Introduction. The term *consistent* is in common use in two contexts that appear to be quite different at first sight. It is often applied to systems of linear equations as a synonym for *solvable*. Thus, Dickson says in [3]: "We shall call two or more equations *consistent* if there exist values of the unknowns which satisfy all of the equations." Again, Bôcher writes in [1]: "The equations may have no solution, in which case they are said to be *inconsistent*." Elsewhere, it is applied by logicians to deductive systems as a synonym for *non-contradictory*. Thus, Tarski defines the term in [8]: "A deductive theory is called *consistent* or non-contradictory if no two asserted statements of this theory contradict each other."

Our preliminary purpose is to reconcile these two usages, agreeing informally that "solvable" means "satisfiable" and that "consistent" means "non-contradictory." The reconciliation is brought about by setting forth explicitly the definition of consistency that has been employed implicitly in ordinary treatments of linear equations. This definition is based on a non-effective characterization of the logical consequences of the system, and is almost trivially equivalent to solvability under much more general conditions than are considered here. Therefore, this equivalence adds little or nothing to our knowledge of the special subject of linear equations.

These considerations are in sharp contrast with another use of consistency, which considers only those consequences that can be derived by applying a finite number of algebraic operations to the system. We shall show that the ordinary criteria for the solvability of systems of linear equations follow directly from the latter notion when the only algebraic operation that is allowed is that of forming linear combinations. Whatever novelty there is in this approach consists in viewing the system of equations as a set of postulates added to an underlying logic that includes the laws of real numbers, and then investigating the "methods of proof" appropriate to it.

The main object of this paper is to extend this formulation to systems of linear inequalities. It is somewhat remarkable that the same theorems persist with only minor modifications, namely, with reasonable care in the type of linear combinations that are allowed, and with the use of a process of elimination designed for linear inequalities. The criteria for solvability that are proved in this manner form a basis for the modern disciplines of linear programming and game theory. They also admit important geometric interpretations. These and other applications will be treated in a sequel.

* A preliminary version of this paper was prepared while the author was a consultant to the National Bureau of Standards. The preparation of the present version was supported, in part, by the Social Science Research Council and, in part, by the Office of Naval Research Logistics Project, Princeton University.

2. Systems of linear equations. The system of linear equations

$$\begin{aligned} 3x - 6y + 4z &= 1 \\ -x + 2y - 2z &= 3 \\ x - 2y + z &= 0 \end{aligned}$$

does not have a solution. Despite its triviality, let us examine an argument that might be advanced as a proof of this statement.

Suppose the equations are multiplied by new unknowns, u, v, and w, respectively, and added. Represent this by listing the multipliers to the left of the equations and the sum below, thus:

$$\begin{array}{lccc} u: & 3x & -6y & +4z = 1 \\ v: & -x & +2y & -2z = 3 \\ w: & x & -2y & +z = 0 \\ \hline & (3u - v + w)x + (-6u + 2v - 2w)y + (4u - 2v + w)z = u + 3v \end{array}$$

If the original system were satisfied by numbers $\bar{x}$, $\bar{y}$, and $\bar{z}$, then the same operations would yield

$$(3u - v + w)\bar{x} + (-6u + 2v - 2w)\bar{y} + (4u - 2v + w)\bar{z} = u + 3v$$

and this would be a true statement for all values of u, v, and w. However, if the multipliers $u = 1$, $v = 1$, $w = -2$ are chosen, this says

$$0\bar{x} + 0\bar{y} + 0\bar{z} = 4,$$

a false statement about numbers.

We shall see later how such multipliers can be found by successively eliminating the unknowns. For the moment it is important to remark that they were chosen to satisfy

$$\begin{aligned} 3u - v + w &= 0 \\ -6u + 2v - 2w &= 0 \\ 4u - 2v + w &= 0 \\ u + 3v &\neq 0 \end{aligned}$$

and thus establish the contradiction. This is, of course, the original system transposed in a special way.

So much for motivation; the object of this section is to state precise conditions for the solvability of general *systems* S of linear equations. The systems will be assumed to contain r inhomogeneous equations in n unknowns, $x_1, \cdots, x_n$, and hence will have the form

$$
\text{(S)}\qquad
\begin{aligned}
c_{11}x_1 + \cdots + c_{1n}x_n &= c_1 \\
&\cdots\cdots\cdots\cdots \\
c_{r1}x_1 + \cdots + c_{rn}x_n &= c_r.
\end{aligned}
$$

The c_{kl} and c_k are given real numbers (for $k=1, \cdots, r$ and $l=1, \cdots, n$) and define the system S. The adjective "inhomogeneous" means that no assumption is made about the right hand members c_k.

An indexed set $\overline{X}$ of real numbers $(\bar{x}_1, \cdots, \bar{x}_n)$ is called a *solution* for S if all of the equations

$$c_{k1}\bar{x}_1 + \cdots + c_{kn}\bar{x}_n = c_k \qquad (k = 1, \cdots, r)$$

are true statements. The set of solutions for S is denoted by $\mathfrak{S}(S)$ and S is called *solvable* if $\mathfrak{S}(S)$ is not empty. An equation

$$d_1x_1 + \cdots + d_nx_n = d$$

in the unknowns $x_1, \cdots, x_n$ is called a (logical) *consequence* of S if $d_1\bar{x}_1 + \cdots + d_n\bar{x}_n = d$ is a true equation whenever $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ is a member of $\mathfrak{S}(S)$, in symbols, if $\overline{X} \in \mathfrak{S}(S)$. A curious situation occurs if S is not solvable, Then the definition of a consequence places no restriction on an equation $d_1x_1 + \cdots + d_nx_n = d$ because there are no solutions $\bar{x}_1, \cdots, \bar{x}_n$ to check for equality. Hence we are forced to conclude that, if S is not solvable, then *every* equation in the unknowns $x_1, \cdots, x_n$ is a consequence of S.

Our first definition of consistency will be based on the idea of consequence and the form of the patently false equation, $0\bar{x} + 0\bar{y} + 0\bar{z} = 4$, derived in the example above. Namely, a system S is said to be *inconsistent* if some equation

$$0x_1 + \cdots + 0x_n = d, \qquad \text{with } d \neq 0,$$

is a consequence of S; otherwise it is called *consistent.* (Thus, the contradictory assertions used in this definition are $0 = d$ and $d \neq 0$.)

The first theorem will establish the logical connection between solvability and this notion of consistency. Its statement explains immediately why these terms have often been confused (or used interchangeably) in textbook discussions of linear equations. The triviality of its proof reveals that it is a theorem without special content for linear equations.

THEOREM 1. *A system S is solvable if and only if it is consistent.*

Proof. If S is solvable, choose an $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n) \in \mathfrak{S}(S)$. Then $0\bar{x}_1 + \cdots + 0\bar{x}_n$ is equal to zero by the rules of operating with real numbers and hence $0\bar{x}_1 + \cdots + 0\bar{x}_n = d$ is not a true equation for any $d \neq 0$. Therefore, no $0x_1 + \cdots + 0x_n = d$, with $d \neq 0$, is a consequence of S, and S is consistent.

On the other hand, if S is not solvable, then every equation in $x_1, \cdots, x_n$ is a consequence of S. In particular, $0x_1 + \cdots + 0x_n = 1$ is a consequence of S, and S is inconsistent.

The significant content of the theory of linear equations comes not from the fact that an absurd consequence can always be exhibited when S is not solvable, but from the *form* of this equation and from *how* it can be obtained. To this end, form a scheme analogous to that used in the example, with multipliers appearing at the left of the system and the sum appearing below:

$$\begin{aligned} w_1&: \ c_{11}x_1 + \cdots + c_{1n}x_n = c_1 \\ w_2&: \ c_{21}x_1 + \cdots + c_{2n}x_n = c_2 \\ &\cdots\cdots\cdots\cdots \\ w_r&: \ c_{r1}x_1 + \cdots + c_{rn}x_n = c_r \\ &\overline{\qquad\qquad\qquad\qquad\qquad} \\ &\quad d_1x_1 + \cdots + d_nx_n = d. \end{aligned}$$

The coefficients of the sum are easily read off; they are:

$$\begin{aligned} d_1 &= w_1c_{11} + \cdots + w_rc_{r1} \\ &\cdots\cdots\cdots\cdots \\ d_n &= w_1c_{1n} + \cdots + w_rc_{rn} \\ d &= w_1c_1 + \cdots + w_rc_r. \end{aligned}$$

An equation,

$$d_1x_1 + \cdots + d_nx_n = d,$$

that is formed in this manner, is called a *linear combination* of the equations of S. The multipliers $w_1, \cdots, w_r$ are called the *coefficients* of the linear combination.

With this set of definitions, the central theorem on the solvability of systems of inhomogeneous linear equations is:

THEOREM 2. (a) *Every linear combination of the equations of S is a consequence of S.* (b) *If S is solvable, then every consequence of S is a linear combination of the equations of S. If S is not solvable, then an equation*

$$0x_1 + \cdots + 0x_n = d, \qquad \text{with } d \neq 0,$$

is a linear combination of the equations of S.

Several informal remarks may help to explain the content and intent of this theorem. First, it characterizes the consequences of *solvable* systems S as being exactly the linear combinations. Since "consequence" is a logical notion and "linear combination" is algebraic, this is quite remarkable. Often we may obtain the information that some statement holds whenever other statements are true; however, it is seldom that this forces such a narrow and explicit connection. Secondly, this theorem says that it is always possible to demonstrate the inconsistency of an unsolvable system by exhibiting as a linear combination an equation that is false for all values of the unknowns. Thus both parts of the theorem

yield information through the medium of linear combinations.

The statements of Theorem 2 (b) can be strengthened to constructive assertions, that is, there is a finite computational procedure that will find the multipliers $w_1, \cdots, w_r$ in the linear combinations (and hence the solution to the transposed system) for every case. This procedure is based on the familiar technique for eliminating an unknown from the system.

THEOREM 3. *The process of elimination either yields a solution for S or exhibits an equation*

$$0x_1 + \cdots + 0x_n = d, \qquad \text{with } d \neq 0,$$

as a linear combination of the equations of S.

Although the technique of elimination is known to every algebra student in practice, it will be described here for the sake of comparison with the generalizations to follow. (Actually, "elimination" is something of a misnomer; for formal reasons, we will not eliminate an unknown, but will make all of its coefficients equal to zero.) It is applied only to systems S in which some coefficient c_{kl} is different from zero. For notational convenience, assume that $c_{11} \neq 0$, renumbering unknowns and equations if necessary. Then define a new system S' of r equations in the unknowns $x_1, \cdots, x_n$ by:

$$(S') \qquad \begin{aligned}
&0x_1 + 0x_2 + \cdots + 0x_n = 0 \\
&0x_1 + \left(c_{22} - \frac{c_{21}c_{12}}{c_{11}}\right)x_2 + \cdots + \left(c_{2n} - \frac{c_{21}c_{1n}}{c_{11}}\right)x_n = c_2 - \frac{c_{21}}{c_{11}}c_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&0x_1 + \left(c_{r2} - \frac{c_{r1}c_{12}}{c_{11}}\right)x_2 + \cdots + \left(c_{rn} - \frac{c_{r1}c_{1n}}{c_{11}}\right)x_n = c_r - \frac{c_{r1}}{c_{11}}c_1.
\end{aligned}$$

The formation of this system follows an obvious rule; it is obtained by subtracting c_{k1}/c_{11} times the first equation from the kth equation for $k = 1, \cdots, r$. Thus the coefficient of x_1 in the kth sum equation is $c_{k1} - c_{k1}c_{11}/c_{11} = 0$ for $k = 1, \cdots, r$. The seemingly redundant first equation, $0x_1 + \cdots + 0x_n = 0$, is retained to avoid the logical complications involved in considering void systems. The proof of Theorem 3 is then based on the following three assertions:

1° *Every equation of S' is a linear combination of the equations of S.*

2° *If S' is solvable, then S is solvable.*

3° *More unknowns have all zero coefficients in S' than in S.*

The details of the proof are straight-forward and will not be given. Theorem 2 (b) follows immediately from Theorem 3; however, the details of this derivation are also omitted because we shall prove a more general statement for inequalities in Section 3. At this stage, presenting the proofs might obscure the simplicity of the logical structure of the results. To emphasize this structure, the theorems

will be reproduced in a condensed form to arm the reader for the section that follows.

The core of this section is contained in the diagram:

solution → / contradiction ↓	x_1	x_2	$\cdots$	x_n	
w_1	c_{11}	c_{12}	$\cdots$	c_{1n}	$= c_1$
w_2	c_{21}	c_{22}	$\cdots$	c_{2n}	$= c_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
w_r	c_{r1}	c_{2r}	$\cdots$	c_{rn}	$= c_r$
	$= 0$	$= 0$	$\cdots$	$= 0$	$\neq 0$

It presents a schematic expression of the statement that exactly one of the two systems

$$(S) \qquad \begin{aligned} c_{11}x_1 + \cdots + c_{1n}x_n &= c_1 \\ &\cdots\cdots\cdots\cdots \\ c_{r1}x_1 + \cdots + c_{rn}x_n &= c_r \end{aligned}$$

and

$$(T) \qquad \begin{aligned} w_1c_{11} + \cdots + w_rc_{r1} &= 0 \\ &\cdots\cdots\cdots\cdots \\ w_1c_{1n} + \cdots + w_rc_{rn} &= 0 \\ w_1c_1 + \cdots + w_rc_r &\neq 0 \end{aligned}$$

is solvable. A solution to T means that S is inconsistent. Successive elimination of $x_1, \cdots, x_n$ leads either to a solution for S or to a solution for T.

The relation of solvability to consistency for linear equations should now be clear. Any confusion is caused by two different uses of the word consistency. The first (which we have called "consistency") might be called "consistency with respect to consequences" (following Church [2]) and is the notion that is used interchangeably with "solvability" because Theorem 1 is valid. The second might be called "consistency with respect to linear combinations" and leads to the following formulation of the results: Forming a linear combination is a rule of inference (*i.e.*, never leads from true statements to a false statement) for systems of linear equations. If a system S is solvable then every consequence can be derived in this manner. If S is inconsistent with respect to consequences, then it is inconsistent with respect to linear combinations. Elimination is an effective procedure for deciding whether a system is solvable or inconsistent (in either sense).

3. Systems of linear inequalities. The object of this section is to establish conditions for the solvability of general systems S of linear inequalities. We shall pattern our definitions upon those given for equations and, happily, the same theorems will be found to hold with only slight and obvious modifications.

The systems S under consideration are assumed to contain $p+q>0$ inequalities in the unknowns $x_1, \cdots, x_n$; of these $p \geqq 0$ are assumed to be strict. Thus, they can be written

$$(S) \qquad \begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &> a_i \qquad (i = 1, \cdots, p) \\ b_{j1}x_1 + \cdots + b_{jn}x_n &\geqq b_j \qquad (j = 1, \cdots, q) \end{aligned}$$

where the a_{ik}, a_i, b_{jk}, and b_j $(i=1, \cdots, p; j=1, \cdots, q; k=1, \cdots, n)$ are given real numbers that define the system S. Again, no assumption is made concerning the right hand members a_i and b_j, and the inequalities are called inhomogeneous.

An indexed set $\overline{X}$ of real numbers $(\bar{x}_1, \cdots, \bar{x}_n)$ is called a *solution* for S if all of the inequalities

$$\begin{aligned} a_{i1}\bar{x}_1 + \cdots + a_{in}\bar{x}_n &> a_i \qquad (i = 1, \cdots, p) \\ b_{j1}\bar{x}_1 + \cdots + b_{jn}\bar{x}_n &\geqq b_j \qquad (j = 1, \cdots, q) \end{aligned}$$

are true statements. The set of solutions for S is denoted by $\mathfrak{S}(S)$, and is possibly empty. An inequality,

$$d_1x_1 + \cdots + d_nx_n \mathfrak{R} d,$$

where $\mathfrak{R}$ is one of the relations $>$ or $\geqq$, in the unknowns $x_1, \cdots, x_n$ is called a *consequence* of S if $d_1\bar{x}_1 + \cdots + d_n\bar{x}_n \mathfrak{R} d$ is a true inequality wherever $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ is a member of $\mathfrak{S}(S)$. If S is not solvable then *every* inequality in the unknowns $x_1, \cdots, x_n$ is a consequence of S. A system S is said to be *inconsistent* if the inequality

$$0x_1 + \cdots + 0x_n > 0$$

is a consequence of S; otherwise, it is called *consistent*. The choice of this "standard" contradiction will be explained below.

The logical connection between solvability and consistency holds without change; to emphasize the strict parallel, this theorem and those following it will be given roman numerals corresponding to their counterparts for equations.

THEOREM I. *A system S is solvable if and only if it is consistent.*

Proof. If S is solvable, choose any $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n) \in \mathfrak{S}(S)$. Then $0\bar{x}_1 + \cdots + 0\bar{x}_n$ is equal to zero by the rules for operating with real numbers and hence $0\bar{x}_1 + \cdots + 0\bar{x}_n > 0$ is not a true inequality. Hence $0x_1 + \cdots + 0x_n > 0$ is not a consequence of S and S is consistent.

On the other hand, if S is not solvable, then every inequality in $x_1, \cdots, x_n$

is a consequence of S. In particular, $0x_1 + \cdots + 0x_n > 0$ is a consequence of S and S is inconsistent.

The significant content of the theory of linear inequalities is found again in the form of the consequences that can be derived from S and the manner of their derivation, namely, as linear combinations. We must exercise some care in forming these, but no more than is indicated by the following rules for manipulating inequalities.

RULE 1. Any inequality (strict ($>$) or ordinary ($\geqq$)) still holds if it is multiplied by a *positive* number throughout.

RULE 2. Strict inequality ($>$) holds for the sum of two similarly directed inequalities if and only if strict inequality holds in at least one of the summands. Ordinary inequality ($\geqq$) can always be asserted for the sum.

RULE 3. Strict inequality ($>$) always implies ordinary inequality ($\geqq$).

These three rules make possible a precise description of the nature of the linear combinations that will be used. Rule 1 suggests that the multipliers should be restricted to be *non-negative* (a zero multiplier means that that inequality is not being used). Rule 2 says that the relation holding for the linear combination can be $>$ only if some strict inequality has a positive coefficient. Rules 2 and 3 say that the relation $\geqq$ can always be asserted for the linear combination. Following these precepts, form a multiplier scheme patterned on the scheme used for equations, with *non-negative* multipliers at the left and the sum below:

$$\begin{aligned}
u_0 \geqq 0:&\quad 0x_1 + \cdots + 0x_n > -1\\
u_1 \geqq 0:&\quad a_{11}x_1 + \cdots + a_{1n}x_n > a_1\\
&\quad \cdots\cdots\cdots\cdots\\
u_p \geqq 0:&\quad a_{p1}x_1 + \cdots + a_{pn}x_n > a_p\\
v_1 \geqq 0:&\quad b_{11}x_1 + \cdots + b_{1n}x_n \geqq b_1\\
&\quad \cdots\cdots\cdots\cdots\\
v_q \geqq 0:&\quad b_{q1}x_1 + \cdots + b_{qn}x_n \geqq b_q\\
\hline
&\quad d_1x_1 + \cdots + d_nx_n \mathcal{R} d
\end{aligned}.$$

The insertion of the first line is related to the choice of a standard inconsistent inequality. The coefficients of the sum are easily calculated to be:

$$\begin{aligned}
d_1 &= u_1a_{11} + \cdots + u_pa_{p1} + v_1b_{11} + \cdots + v_qb_{q1}\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
d_n &= u_1a_{1n} + \cdots + u_pa_{pn} + v_1b_{1n} + \cdots + v_qb_{qn}\\
d &= -u_0 + u_1a_1 + \cdots + u_pa_p + v_1b_1 + \cdots + v_qb_q.
\end{aligned}$$

An inequality,

$$d_1x_1 + \cdots + d_nx_n \mathcal{R} d,$$

that is formed in this manner from a system S, is called a *legal linear combination* of the inequalities of S provided that $\mathfrak{R}$ is $\geqq$, *or* $\mathfrak{R}$ is $>$ and some u_i is positive ($i=0, 1, \cdots, p$). The reader should notice that this definition has been framed so as to conform to Rules 1, 2, and 3 and thus to insure the fact that every legal linear combination is a consequence. Another important fact to be noted is that every inequality in the multiplier scheme above can be proved to be a legal linear combination by choosing its multiplier equal to one and all other multipliers equal to zero. In particular, the definitions make $0x_1+\cdots+0x_n>-1$ a legal linear combination of every system.

With these definitions, the central theorem on the solvability of systems of inhomogeneous linear inequalities is:

THEOREM II. (a) *Every legal linear combination of the inequalities of S is a consequence of S.* (b) *If S is solvable then every consequence of S is a legal linear combination of the relations of S. If S is not solvable, then the inequality*

$$0x_1 + \cdots + 0x_n > 0$$

is a legal linear combination of the relations of S.

Proof. Theorem II (a) follows immediately from the definitions and Rules 1, 2, and 3. As before, the statements of Theorem II (b) will be strengthened to constructive assertions; that is, elimination is an effective procedure for finding the multipliers $u_0, u_1, \cdots, u_p, v_1, \cdots, v_q$ in the legal linear combinations for every case. Precisely, the proof of Theorem II (b) will be based on

THEOREM III. *The process of elimination either yields a solution for S or exhibits the inequality $0x_1+\cdots+0x_n>0$ as a legal linear combination of the inequalities of S.*

Proof. The process of elimination is applied only to systems S in which some coefficient a_{il} or b_{jl} is different from zero. Assume that $l=1$, renumbering unknowns if necessary, and set about eliminating x_1 from the system. Separate the inequalities of the system into three classes (Classes I, II, III) according to whether the coefficient of x_1 is positive, negative, or zero. These classes will be distinguished notationally by the number of primes on the indices. Define a new system S' of inequalities in the unknowns $x_1, \cdots, x_n$ by:

$$0x_1 + \qquad 0x_2 + \cdots + \qquad 0x_n > -1$$

$$(S') \quad 0x_1 + \left(\frac{a_{i'2}}{a_{i'1}} - \frac{a_{i''2}}{a_{i''1}}\right) x_2 + \cdots + \left(\frac{a_{i'n}}{a_{i'1}} - \frac{a_{i''n}}{a_{i''1}}\right) x_n > \frac{a_{i'}}{a_{i'1}} - \frac{a_{i''}}{a_{i''1}}$$

for *all* pairs i' and i'' with $a_{i'1}>0$ and $a_{i''1}<0$,

$$0x_1 + \left(\frac{a_{i'2}}{a_i'_1} - \frac{b_{j''2}}{b_{j''1}}\right) x_2 + \cdots + \left(\frac{a_{i'n}}{a_{i'1}} - \frac{b_{j''n}}{b_{j''1}}\right) x_n > \frac{a_{i'}}{a_{i'1}} - \frac{b_{j''}}{b_{j''1}}$$

for *all* pairs i' and j'' with $a_{i'1}>0$ and $b_{j''1}<0$, (*S' continues on next page.*)

(S' *cont.*) $$0x_1 + \left(\frac{b_{j'2}}{b_{j'1}} - \frac{a_{i''2}}{a_{i''1}}\right) x_2 + \cdots + \left(\frac{b_{j'n}}{b_{j'1}} - \frac{a_{i''n}}{a_{i''1}}\right) x_n > \frac{b_{j'}}{b_{j'1}} - \frac{a_{i''}}{a_{i''1}}$$

for *all* pairs j' and i'' with $b_{j'1}>0$ and $a_{i''1}<0$,

$$0x_1 + \left(\frac{b_{j'2}}{b_{j'1}} - \frac{b_{j''2}}{b_{j''1}}\right) x_2 + \cdots + \left(\frac{b_{j'n}}{b_{j'1}} - \frac{b_{j''n}}{b_{j''1}}\right) x_n \geqq \frac{b_{j'}}{b_{j'1}} - \frac{b_{j''}}{b_{j''1}}$$

for *all* pairs j' and j'' with $b_{j'1}>0$ and $b_{j''1}<0$,

$$0x_1 + \quad a_{i'''2}x_2 + \cdots + \quad a_{i'''n}x_n > a_{i'''}$$

for *all* i''' with $a_{i'''1}=0$,

$$0x_1 + \quad b_{j'''2}x_2 + \cdots + \quad b_{j'''n}x_n \geqq b_{j'''}$$

for *all* j''' with $b_{j'''1}=0$.

Although it may seem formidable at first glance, the formation of the system S' follows a simple rule. Namely, the coefficient of x_1 in each inequality in Class I is made $+1$ through the use of the *positive* multipliers $1/a_{i'1}$ and $1/b_{j'1}$, while the coefficient of x_1 in each inequality of Class II is made -1 through the use of the *positive* multipliers $-1/a_{i''1}$ and $-1/b_{j''1}$. By Rule 1, the inequalities are all preserved. Then, *all* pairs of inequalities, one from Class I and one from Class II are added and the inequality type of the sum is assigned by Rule 2. The inequalities of Class III are copied, unchanged; the first inequality, $0x_1+\cdots+0x_n>-1$, is inserted in S' to avoid the logical complications involved in considering the empty system that would arise if all of the inequalities of S were in Class I or II. This characterization of the formation of S' proves:

1° *Every inequality of S' is a legal linear combination of the inequalities of S, and hence is a consequence of S.*

Another manner of viewing the formation of S' is revealed by isolating x_1 on one side of all of the inequalities of Classes I and II, then pairing the results as follows:

$$\frac{a_{i''}}{a_{i''1}} - \frac{a_{i''2}}{a_{i''1}} x_2 - \cdots - \frac{a_{i''n}}{a_{i''1}} x_n > x_1 > \frac{a_{i'}}{a_{i'1}} - \frac{a_{i'2}}{a_{i'1}} x_2 - \cdots - \frac{a_{i'n}}{a_{i'1}} x_n$$

for *all* pairs i' and i'' with $a_{i'1}>0$ and $a_{i''1}<0$,

$$\frac{b_{j''}}{b_{j''1}} - \frac{b_{j''2}}{b_{j''1}} x_2 - \cdots - \frac{b_{j''n}}{b_{j''1}} x_n \geqq x_1 > \frac{a_{i'}}{a_{i'1}} - \frac{a_{i'2}}{a_{i'1}} x_2 - \cdots - \frac{a_{i'n}}{a_{i'1}} x_n$$

for *all* pairs i' and j'' with $a_{i'1}>0$ and $b_{j''1}<0$,

$$\frac{a_{i''}}{a_{i''1}} - \frac{a_{i''2}}{a_{i''1}} x_2 - \cdots - \frac{a_{i''n}}{a_{i''1}} x_n > x_1 \geqq \frac{b_{j'}}{b_{j'1}} - \frac{b_{j'2}}{b_{j'1}} x_2 - \cdots - \frac{b_{j'n}}{b_{j'1}} x_n$$

for *all* pairs j' and i'' with $b_{j'1}>0$ and $a_{i''1}<0$,

$$\frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}x_2-\cdots-\frac{b_{j''n}}{b_{j''1}}x_n \geqq x_1 \geqq \frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}x_2-\cdots-\frac{b_{j'n}}{b_{j'1}}x_n$$

for *all* pairs j' and j'' with $b_{j'1}>0$ and $b_{j''1}<0$.

(Again, the reader should not ignore the possibility that either Class I or Class II might be empty, in which case there would be no such pairs; he should check that each assertion holds in this case, too.) If x_1 is dropped from each of the pairs and the result arranged with unknowns on the left and constants on the right, and if the inequalities of Class III and $0x_1+\cdots+0x_n>-1$ are adjoined to the system, the result is exactly S'.

Suppose $\bar{x}_2, \cdots, \bar{x}_n$ solve S' (note that x_1 can be given any value in this system). Then, arranging S' as in the previous paragraph, the inequalities that are not in S are:

$$(i', i''): \quad \frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}\bar{x}_2-\cdots-\frac{a_{i''n}}{a_{i''1}}\bar{x}_n > \frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}\bar{x}_2-\cdots-\frac{a_{i'n}}{a_{i'1}}\bar{x}_n$$

$$(i', j''): \quad \frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}\bar{x}_2-\cdots-\frac{b_{j''n}}{b_{j''1}}\bar{x}_n > \frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}\bar{x}_2-\cdots-\frac{a_{i'n}}{a_{i'1}}\bar{x}_n$$

$$(j', i''): \quad \frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}\bar{x}_2-\cdots-\frac{a_{i''n}}{a_{i''1}}\bar{x}_n > \frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}\bar{x}_2-\cdots-\frac{b_{j'n}}{b_{j'1}}\bar{x}_n$$

$$(j', j''): \quad \frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}\bar{x}_2-\cdots-\frac{b_{j''n}}{b_{j''1}}\bar{x}_n \geqq \frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}\bar{x}_2-\cdots-\frac{b_{j'n}}{b_{j'1}}\bar{x}_n.$$

The problem of completing $\bar{x}_2, \cdots, \bar{x}_n$ to a solution for S by a proper choice of $\bar{x}_1$ is clearly that of fitting $\bar{x}_1$ into these inequalities as required by the system immediately above them. With an eye to picking out those inequalities of S that restrict the choice of $\bar{x}_1$ most severely, define

$$\alpha' = \max_{i'}\left\{\frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}\bar{x}_2-\cdots-\frac{a_{i'n}}{a_{i'n}}\bar{x}_n\right\},$$

$$\beta' = \max_{j'}\left\{\frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}\bar{x}_2-\cdots-\frac{b_{j'n}}{b_{j'1}}\bar{x}_n\right\},$$

$$\alpha'' = \min_{i''}\left\{\frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}\bar{x}_2-\cdots-\frac{a_{i''n}}{a_{i''1}}\bar{x}_n\right\},$$

$$\beta'' = \min_{j''}\left\{\frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}x_2-\cdots-\frac{b_{j''n}}{b_{j''1}}x_n\right\}.$$

(We shall adopt the convention that a maximum over an empty set is $-\infty$ while a minimum over an empty set is $+\infty$. Thus, if there are no inequalities

in Class I, $\alpha' = -\infty$, *etc.*) It is important to realize these maxima and minima are actually achieved for a proper choice of the indices i', j', i'', and j'' if the sets concerned are non-empty, and so among the inequalities immediately above one can find:

$$\alpha'' > \alpha', \qquad \beta'' > \alpha', \qquad \alpha'' > \beta', \quad \text{and} \quad \beta'' \geqq \beta',$$

(where the inequalities are trivially true if the sets are empty and the convention is applied).

If $\bar{x}_1 > \alpha'$, $\bar{x}_1 \geqq \beta'$, $\bar{x}_1 < \alpha''$, *and* $\bar{x}_1 \leqq \beta''$, *then* $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ *is a solution for* S.

Proof. Since the inequalities in S of Class III are also in S', only those of Classes I and II need be checked. For these,

$$a_{i'1}\bar{x}_1 + a_{i'2}\bar{x}_2 + \cdots + a_{i'n}\bar{x}_n > a_{i'1}\alpha' + a_{i'2}\bar{x}_2 + \cdots + a_{i'n}\bar{x}_n$$
$$\geqq (a_{i'} - a_{i'2}\bar{x}_2 - \cdots - a_{i'n}\bar{x}_n) + (a_{i'2}\bar{x}_2 + \cdots + a_{i'n}\bar{x}_n) = a_{i'}$$

for all i' with $a_{i'1} > 0$,

$$a_{i''1}\bar{x}_1 + a_{i''2}\bar{x}_2 + \cdots + a_{i''n}\bar{x}_n > a_{i''1}\alpha'' + a_{i''2}\bar{x}_2 + \cdots + a_{i''n}\bar{x}_n$$
$$\geqq (a_{i''} - a_{i''2}\bar{x}_2 - \cdots - a_{i''n}\bar{x}_n) + (a_{i''2}\bar{x}_2 + \cdots + a_{i''n}\bar{x}_n) = a_{i''}$$

for all i'' with $a_{i''1} < 0$,

$$b_{j'1}\bar{x}_1 + b_{j'2}\bar{x}_2 + \cdots + b_{j'n}\bar{x}_n > b_{j'1}\beta' + b_{j'2}\bar{x}_2 + \cdots + b_{j'n}\bar{x}_n$$
$$\geqq (b_{j'} - b_{j'2}\bar{x}_2 - \cdots - b_{j'n}\bar{x}_n) + (b_{j'2}\bar{x}_2 + \cdots + b_{j'n}\bar{x}_n) = b_{j'}$$

for all j' with $b_{j'1} > 0$,

$$b_{j''1}\bar{x}_1 + b_{j''2}\bar{x}_2 + \cdots + b_{j''n}\bar{x}_n \geqq b_{j''1}\beta'' + b_{j''2}\bar{x}_2 + \cdots + b_{j''n}\bar{x}_n$$
$$\geqq (b_{j''} - b_{j''2}\bar{x}_2 - \cdots - b_{j''n}\bar{x}_n) + (b_{j''2}\bar{x}_2 + \cdots + b_{j''n}\bar{x}_n) = b_{j''}$$

for all j'' with $b_{j''1} < 0$. This proves that S is solved by $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$.

To choose an $\bar{x}_1$ satisfying these conditions, consider the four exhaustive cases, corresponding to the four possible orderings of α' with β' and α'' with β'', separately:

CASE 1. $\beta' \leqq \alpha'$ and $\alpha'' \leqq \beta''$.
Choose $\bar{x}_1$ so that $\alpha' < \bar{x}_1 < \alpha''$.

CASE 2. $\beta' \leqq \alpha'$ and $\beta'' < \alpha''$.
Choose $\bar{x}_1$ so that $\alpha' < \bar{x}_1 < \beta''$.

CASE 3. $\alpha' < \beta'$ and $\alpha'' \leqq \beta''$.
Choose $\bar{x}_1$ so that $\beta' < \bar{x}_1 < \alpha''$.

CASE 4. $\alpha' < \beta'$ and $\beta'' < \alpha''$.
Choose $\bar{x}_1$ so that $\beta' \leqq \bar{x}_1 \leqq \beta''$.

This proves the crucial fact:

2° *If* S' *is solvable, then* S *is solvable.*

Suppose that the process of elimination is applied several, say h, times to yield successively the systems S, S', S'', $\cdots$, $S^{(h)}$. It is clear that 1° and 2° still hold, with $S^{(h)}$ replacing S'. The only detail that needs verifying, namely, that legal linear combinations of legal linear combinations are legal linear combinations, is obvious and tedious to write out. The termination of the process is insured by a third property of S' relative to S.

3° *More unknowns have all zero coefficients in S' than in S.*

This is clear, since any unknown with all zero coefficients in S still has all zero coefficients in S' while x_1 had a non-zero coefficient in S (some a_{i1} or b_{j1} was assumed non-zero) and has all zero coefficients in S'.

Property 3° insures that the process of elimination must end after a finite number of steps (no more than n) with a system $S^{(h)}$ in which all of the unknowns have all zero coefficients. If all the right hand members of the strict (or ordinary) inequalities are negative (or non-positive) then any set of numbers $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ solves $S^{(h)}$ and hence S is solvable by h applications of 2°. Otherwise, $0x_1 + \cdots + 0x_n > d$ with some $d \geqq 0$, or $0x_1 + \cdots + 0x_n \geqq d$ with some $d > 0$ appears in $S^{(h)}$. In this case, the proof is completed by writing down the two multiplier schemes:

$$1\colon\ 0x_1 + \cdots + 0x_n > d, \qquad \text{with any } d \geqq 0$$

$$\frac{d\colon\ 0x_1 + \cdots + 0x_n > -1}{0x_1 + \cdots + 0x_n > 0}$$

$$1\colon\ 0x_1 + \cdots + 0x_n \geqq d, \qquad \text{with any } d > 0$$

$$\frac{d\colon\ 0x_1 + \cdots + 0x_n > -1}{0x_1 + \cdots + 0x_n > 0}$$

(The reader should verify that the result is a legal linear combination in each case.) Hence, $0x_1 + \cdots + 0x_n > 0$ is a legal linear combination of $S^{(h)}$ and can be exhibited as a legal linear combination of the inequalities of S (and possibly $0x_1 + \cdots + 0x_n > -1$) by at most $h+1$ applications of 1°. This completes the proof of Theorem III.

Before returning to the proof of Theorem II (b), these results will be summarized in a multiplier scheme. This diagram expresses the fact that *exactly* one of the two systems listed there admits a solution. A solution to T means that S is inconsistent. Successive elimination of $x_1, \cdots, x_n$ leads either to a solution for S or to a solution for T.

solution → / contradiction		x_1	x_2	$\cdots$	x_n	
	$u_0 \geqq 0$	0	0	$\cdots$	0	> -1
not	$u_1 \geqq 0$	a_{11}	a_{12}	$\cdots$	a_{1n}	$> a_1$
all	$u_2 \geqq 0$	a_{21}	a_{22}	$\cdots$	a_{2n}	$> a_2$
zero ↓	⋮	⋮	⋮		⋮	⋮
	$u_p \geqq 0$	a_{p1}	a_{p2}	$\cdots$	a_{pn}	$> a_p$
	$v_1 \geqq 0$	b_{11}	b_{12}	$\cdots$	b_{1n}	$\geqq b_1$
	$v_2 \geqq 0$	b_{21}	b_{22}	$\cdots$	b_{2n}	$\geqq b_2$
	⋮	⋮	⋮		⋮	⋮
	$v_q \geqq 0$	b_{q1}	b_{q2}	$\cdots$	b_{qn}	$\geqq b_q$
		$= 0$	$= 0$	$\cdots$	$= 0$	$= 0$

The system S is defined by

$$a_{i1}x_1 + \cdots + a_{in}x_n > a_i \qquad (i = 1, \cdots, p)$$
$$b_{j1}x_1 + \cdots + b_{jn}x_n \geqq b_j \qquad (j = 1, \cdots, q).$$

The system T is defined by

$$u_1a_{1l} + \cdots + u_pa_{pl} + v_1b_{1l} + \cdots + v_qb_{ql} = 0 \qquad (l = 1, \cdots, n)$$
$$-u_0 + u_1a_2 + \cdots + u_pa_p + v_1b_1 + \cdots + v_qb_q = 0$$

with all of the unknowns $u_0, u_1, \cdots, u_p, v_1, \cdots, v_q$ non-negative and not all of the $u_0, u_1, \cdots, u_p$ equal to zero.

Proof of Theorem II(b). The case for systems without solutions is stated directly in Theorem III and so it is only necessary to show that, for solvable systems, every consequence is a legal linear combination. Assume that the strict inequality $d_1x_1 + \cdots + d_nx_n > d$ is a consequence of S. Thus, if $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ is any solution for S, then $d_1\bar{x}_1 + \cdots + d_n\bar{x}_n$ is a number that is *larger* than d. This means the *insolvability* of the system U (to save rewriting the system, multipliers have been included):

$$(U') \qquad \begin{aligned} u_0 \geqq 0&: \quad 0x_1 + \cdots + 0x_n > -1 \\ u_i \geqq 0&: \quad a_{i1}x_1 + \cdots + a_{in}x_n > a_i \qquad (i = 1, \cdots, p) \\ v_j \geqq 0&: \quad b_{j1}x_1 + \cdots + b_{jn}x_n \geqq b_j \qquad (j = 1, \cdots, q) \\ t \geqq 0&: \quad -d_1x_1 - \cdots - d_nx_n \geqq -d. \end{aligned}$$

By Theorem III, there exist non-negative multipliers $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p, \bar{v}_1, \cdots, \bar{v}_q, \bar{t}$ with not all of $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p$ equal to zero, such that

$$\bar{x}_l: \quad \bar{u}_1a_{1l} + \cdots + \bar{u}_pa_{pl} + \bar{v}_1b_{1l} + \cdots + \bar{v}_qb_{ql} - \bar{l}d = 0 \quad (l = 1, \cdots, n)$$

$$-1: \quad -\bar{u}_0 + \bar{u}_1a_1 + \cdots + \bar{u}_pa_p + \bar{v}_1b_1 + \cdots + \bar{v}_qb_q - \bar{l}d = 0.$$

If $\bar{l}$ were known to be positive, then all of these equations could be divided by it, establishing $d_1x_1 + \cdots + d_nx_n > d$ as a legal linear combination of the inequalities of S. Recall that S is assumed to be solvable and let $\bar{x}_1, \cdots, \bar{x}_n$ be a solution. These have been written as multipliers at the left of the last system, using -1 to multiply the last equation. Multiplying and adding,

$$\bar{u}_0 + \sum_i \bar{u}_i(a_{i1}\bar{x}_1 + \cdots + a_{in}\bar{x}_n - a_i) + \sum_j \bar{v}_j(b_{j1}\bar{x}_1 + \cdots + b_{jn}\bar{x}_n - b_j)$$

$$= \bar{l}(d_1\bar{x}_1 + \cdots + d_n\bar{x}_n - d).$$

Since the left side of this equation is positive, the right side is also, and hence $\bar{l} > 0$. Hence $d_1x_1 + \cdots + d_nx_n > d$ is a legal linear combination of the inequalities of S with multipliers $\bar{u}_0/\bar{l}, \bar{u}_1/\bar{l}, \cdots, \bar{u}_p/\bar{l}, \bar{v}_1/\bar{l}, \cdots, \bar{v}_q/\bar{l}$. If the consequence is an ordinary inequality $d_1x_1 + \cdots + d_nx_n \geqq d$, a few changes must be made in the proof. The inequality $-d_1x_1 - \cdots - d_nx_n \geqq -d$ in U should be replaced by $-d_1x_1 - \cdots - d_nx_n > -d$. Using the same notation for multipliers, the same transposed system of equations is obtained, with not all of $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p$, *and* $\bar{l}$ equal to zero. If $\bar{l}$ is assumed zero, we are in the previous case (with not all of $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p$ equal to zero) and can conclude $\bar{l} > 0$ as before, a contradiction. Hence $\bar{l}$ is positive and $d_1x_1 + \cdots + d_nx_n \geqq d$ is a legal linear combination of the inequalities of S with multipliers $\bar{u}_0/\bar{l}, \bar{u}_1/\bar{l}, \cdots, \bar{u}_p/\bar{l}, \bar{v}_1/\bar{l}, \cdots, \bar{v}_q/\bar{l}$. This completes the proof of Theorem II (b).

4. An infinite example. Of course, the results of Section 3 can be paraphrased in logical terms exactly in the form of the conclusion to Section 2. At this point the reader may be led to the erroneous belief that consistency with respect to consequences is always equivalent to consistency with respect to linear combinations for linear systems. The following (infinite) system shows that this is not so.

$$(S) \qquad x > -1/n \qquad (n = 1, 2, 3, \cdots).$$

Clearly $x \geqq 0$ is a consequence of this system, yet is not a linear combination of any number of inequalities of S. We infer that

$$(S_1) \qquad -x > 0, \quad x > -1/n \qquad (n = 1, 2, 3, \cdots)$$

is inconsistent with respect to consequences but is consistent with respect to linear combinations. The natural way out of this dilemma is to add new rules of inference based on limiting operations.

5. Acknowledgments and historical remarks. The author is indebted to T. S. Motzkin, who first suggested that his "transposition theorem" [6] might be viewed as asserting the disjoint alternatives of solvability or contradiction *via* linear combination. He also called attention to a remark of Fourier [5] that elimination was a natural method for solving linear inequalities. However, the

author takes full responsibility for the logical consequences of these ideas.

Elimination, as a method for solving linear inequalities, has been used for theoretical purposes by a number of authors, notably by Dines [4]. However, his treatment seems to conceal rather than emphasize the parallel with equations.

The distinction between logical consequences and provable consequences for logistic systems was first made by Tarski [7]. Our definition of the logical consequences of a system of linear equations or inequalities is in an obvious way parallel to Tarski's definition of logical consequence, but the two are not the same. They might be compared in the following manner. The logical consequences, in the sense of this paper, of a system of equations or inequalities are the same as the logical consequences, in Tarski's sense, of the system together with some categorical system of postulates for the real numbers—provided that the unknowns in the equations or inequalities, before considering the consequences in Tarski's sense, are first replaced by new symbols, which play the role of primitive constants, and which do not appear elsewhere. The author is indebted to A. Church for unraveling the relation between the two ideas; the general metatheorem parallel to Theorems 1 and I, and valid for functional calculi of all orders is stated by Church in [2].

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SELECTED SELECTION THEOREMS

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1. Introduction. The theory of selections, a systematic account of which appears in the *Annals of Mathematics* [6] [7] [8], contains two theorems which are both of general interest and relatively easy to prove ([6; part of Theorem 3.2″] and [7; part of Theorem 1.2]). The purpose of this note is to make these theorems, together with their proofs and some applications, accessible to the casual reader, who might find the sheer bulk of the more detailed *Annals* investigation somewhat discouraging.

If X and Y are topological spaces, and ϕ a function from X to the subsets of Y, then a *selection* for ϕ is a continuous $f: X \to Y$ such that $f(x) \in \phi(x)$ for every x in X. All other unfamiliar concepts appearing in the following theorems will be defined at the end of this introduction.

THEOREM 1 [6]. *If X is a paracompact space, then every lower semi-continuous function ϕ from X to the non-empty, closed, convex subsets of a Banach space Y admits a selection.*

THEOREM 2 [7]. *If X is paracompact and zero-dimensional, and if Y is a complete metric space, then every lower semi-continuous function ϕ from X to the non-empty, closed subsets of Y admits a selection.*

The difference between Theorem 1 and Theorem 2, it should be noted, is that Theorem 1 restricts the *range* of ϕ, while Theorem 2 restricts its *domain.* Intermediate results, where $\dim X < n$ and where the range of ϕ is subjected to restrictions weaker than those in Theorem 1, can be found in [7; Theorem 1.2].

It is shown in [6; Theorem 3.2″] that the converse of Theorem 1 is also true; in other words, any T_1-space X which has the property stated in Theorem 1 is paracompact. This provides a new characterization of paracompactness.

We now state the three most important corollaries to Theorems 1 and 2. The first of them was first proved by R. Bartle and L. Graves [2]; the second and third are Corollaries 1.4 and 1.5 of [7].

COROLLARY 1 (Bartle-Graves). *If X and Y are Banach spaces, and if $u: Y \to X$ is continuous, linear, and onto, then there exists a continuous $f: X \to Y$ such that $uf(x) = x$ for every $x \in X$.***

COROLLARY 2. *If X is a zero-dimensional paracompact space, Y a complete metric space, and if $u: Y \to X$ is continuous, open, and onto, then there exists a continuous $f: X \to Y$ such that $uf(x) = x$ for every $x \in X$.*

* The results in this paper were obtained while the author was an A. E. C. fellow.

** One can *not* always pick f to be *linear*, since this would imply the existence of a continuous projection from Y onto the null-space of u, which need not exist [9]. One can, however, at least pick f such that $f(\alpha x) = \alpha f(x)$ for every scalar α [6; Proposition 7.2].

COROLLARY 3. *Let X be a zero-dimensional paracompact space, let C be the complex numbers, and let $g: X \to C$ be continuous. Also let w be any polynomial with complex coefficients.*† *Then there exists a continuous $f: X \to C$ such that $w(f(x)) = g(x)$ for every x in X.*‡

We conclude this introduction with a quick review of our principal concepts. Let X be a topological space. A function ϕ from X to the subsets of a topological space Y is *lower semicontinuous* if $\{x \in X \mid \phi(x) \cap U \neq \phi\}$ is open in X for every open $U \subset Y$. If $\mathcal{U}$ and $\mathcal{W}$ are coverings of X, then $\mathcal{U}$ is a *refinement* of $\mathcal{W}$ if every $V \in \mathcal{U}$ is a subset of some $W \in \mathcal{W}$. A collection $\mathcal{U}$ of subsets of X is *locally finite* if every $x \in X$ has a neighborhood which intersects only finitely many $V \in \mathcal{U}$. We call X *paracompact* if it is Hausdorff, and if every open covering of X has an open, locally finite refinement. (It is known [3] [10] that every metric space and every compact Hausdorff space is paracompact, and that every paracompact space is normal; for more details, see [4; p. 156].) Finally, X is *zero-dimensional* (dim $X = 0$) if every finite open covering of X has a disjoint, finite, open refinement. This implies (and for compact Hausdorff or separable metric spaces is implied by) the condition that X has a base consisting of sets which are both open and closed.

2. Proof of Theorem 1. Before embarking on the proof of Theorem 1, we state, and for completeness prove, a useful and standard result about "partitions of unity" on paracompact spaces [5].

PROPOSITION 1. *If $\{V_\alpha\}_{\alpha \in A}$ is a locally finite, open covering of normal space X, then there exists a family $\{p_\alpha\}_{\alpha \in A}$ of continuous functions from X to the closed unit interval with the following properties:*

(a) *p_α vanishes outside V_α,*

(b) *$\Sigma_{\alpha \in A} p_\alpha(x) = 1$ for every x in X.*

Proof. By a theorem of Dieudonné, [3; Theorem 6], there exists an open covering $\{W_\alpha\}_{\alpha \in A}$ of X such that $\overline{W}_\alpha \subset V_\alpha$ for every $x \in X$. By the normality of X, we can find continuous functions $q_\alpha: X \to [0, 1]$ such that

$$q_\alpha(x) = 1 \qquad x \in \overline{W}_\alpha,$$
$$q_\alpha(x) = 0 \qquad x \in X - V_\alpha.$$

Now define $p_\alpha: X \to [0, 1]$ by

$$p_\alpha(x) = \frac{q_\alpha(x)}{\sum_{\beta \in A} q_\beta(x)}.$$

† As the proof shows, w may actually be any open (in particular, any analytic) map from a subset of C into C whose range contains $g(X)$.

‡ This corollary can be used to prove the known result that every normal operator A on a Hilbert space has an n-th root lying in the smallest strongly closed *-algebra of operators containing A.

The family $\{p_\alpha\}_{\alpha\in A}$ satisfies all our requirements.

Turning now to the proof of Theorem 1, we begin with the following lemma, which will be used in an inductive proof of the theorem.

LEMMA 1. *If X is paracompact, Y a normed linear space, ψ a lower semi-continuous function from X to the non-empty convex subsets of Y, and $r>0$, then there exists a continuous $f:X\to Y$ such that $f(x)\in S_r(\psi(x))$ for every $x\in X$.**

Proof. For every $y\in Y$, let $U_y=\{x\in X\mid y\in S_r(\psi(x))\}$. Then

$$U_y = \{x \in X \mid \psi(x) \cap S_r(y) \neq \varnothing\},$$

and it follows from the definition of lower semi-continuity that every U_y is open in X. Hence $\{U_y\}_{y\in Y}$ is an open covering of X, and therefore has an open locally finite refinement $\{V_\alpha\}_{\alpha\in A}$.† Now pick the family $\{p_\alpha\}_{\alpha\in A}$ of continuous functions from X to the interval $[0, 1]$ as in Proposition 1. Also pick, for each $\alpha\in A$, a $y(\alpha)\in Y$ such that $V_\alpha\subset U_{y(\alpha)}$. The desired $f:X\to Y$ can now be defined by

$$f(x) = \sum_{\alpha\in A} p_\alpha(x)y(\alpha).$$

To see that this works, observe first that each $x\in X$ has a neighborhood U intersecting only finitely many V_α, and on this U, f is the sum of finitely many continuous functions. Hence every $x\in X$ has a neighborhood on which f is continuous, and therefore f is continuous on X. Moreover, for each $x\in X$, $f(x)$ is a *convex*, linear combination of finitely many $y(\alpha)$, all of which lie in the *convex* set $S_r(\psi(x))$, and hence $f(x)\in S_r(\psi(x))$. This completes the proof of the lemma.

Proof of Theorem 1 *itself.* We are going to construct a sequence of continuous functions $f_i:X\to Y$ such that, for every $x\in X$,

$$\text{(a)}\quad f_i(x) \in S_{2^{-i+2}}(f_{i-1}(x)) \qquad (i = 2, 3, \cdots)$$

$$\text{(b)}\quad f_i(x) \in S_{2^{-i}}(\phi(x)) \qquad (i = 1, 2, \cdots).$$

This will be sufficient, because then (by (a)) the sequence $\{f_i\}_{i+1}^{\infty}$ is uniformly Cauchy, and therefore converges uniformly to a continuous $f:X\to Y$, and it follows from (b) that $f(x)\in\phi(x)$ for every x in X.

We construct $\{f_i\}_{i=1}^{\infty}$ by induction. The existence of a function f_1, satisfying (b) for $i=1$, follows immediately from Lemma 1. Suppose that $f_1, \cdots, f_n$ have been constructed to satisfy (a) and (b) for $i=1, \cdots, n$, and let us construct f_{n+1} to satisfy (a) and (b) for $i=n+1$.

For each $x\in X$, let

$$\phi_{n+1}(x) = \phi(x) \cap (S_{2^{-n}}(f_n(x)));$$

* If B is a subset of a metric space Y with metric ρ, and if $r>0$, then $S_r(B)$ denotes $\{y\in Y\mid \rho(y, B)<r\}$.

† We assume that $V_\alpha\neq V_\beta$ for $\alpha\neq\beta$.

then $\phi_{n+1}(x)$ is never empty by our induction hypothesis. Let us check that ϕ_{n+1} is also lower semi-continuous. We must show that, if U is open in Y, then $V=\{x\in X\mid \phi_{n+1}(x)\cap U\neq\varnothing\}$ is open in X; to do this, we will show that every $x_0\in V$ has a neighborhood contained in V. Now for $x_0\in V$, pick $y_0\in\phi(x_0)$, and a positive $\lambda<2^{-n}$ such that $y_0\in S_\lambda(f_n(x_0))$. Let

$$W_1=\{x\in X\mid \phi(x)\cap(S_\lambda(f_n(x_0))\cap U)\neq\varnothing\};$$
$$W_2=\{x\in X\mid f_n(x)\in S_{2^{-n}-\lambda}(f_n(x_0))\}.$$

Then W_1 is open because ϕ is lower semi-continuous, W_2 is open because f_n is continuous, and $x_0\in(W_1\cap W_2)\subset V$.

Since ϕ_{n+1} is lower semi-continuous, we can apply Lemma 1 to find a continuous $f_{n+1}:X\to Y$ such that

$$f_{n+1}(x)\in S_{2^{-n-1}}(\phi_{n+1}(x))$$

for every $x\in X$. But then

$$f_{n+1}(x)\in S_{2^{-n+1}}(f_n(x)),$$

which is (a), and

$$f_{n+1}(x)\in S_{2^{-n-1}}(\phi(x)),$$

which is (b).

3. Proof of Theorem 2. In the proof of Theorem 2, we shall need the equivalence of (a) and (b) in the following elementary and known result, which we prove here for completeness. Note that for *compact* X the equivalence of (a) and (b) is obvious.

PROPOSITION 2. *The following three properties of a normal space X are equivalent:*

(a) *Every finite open covering of X has a disjoint finite open refinement (i.e.,* dim $X=0$).

(b) *Every locally finite open covering of X has a disjoint open refinement.*

(c) *If $A\subset V\subset X$, with A closed and V open, then there exists an open and closed $W\subset X$ with $A\subset W\subset V$.*

Proof (a)→(c): Let A and V be as in (c). Then $\{V, X-A\}$ is a finite open covering of X, which by assumption has a finite, disjoint open refinement $\mathcal{U}$. Let W be the union of all elements of $\mathcal{U}$ which are subsets of V. The W satisfies all our requirements.

(c)→(b): Let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite open covering of X. Again applying Dieudonné's theorem [3; Theorem 6], we can find an open covering $\{V_\alpha\}$ of X such that $\overline{V}_\alpha\subset U_\alpha$ for all α. By assumption we can now find, for each α, an open and closed W_α such that $\overline{V}_\alpha\subset W_\alpha\subset U_\alpha$. Now well-order the index set A, and for each $\alpha\in A$, let $R_\alpha=W_\alpha-\bigcup_{\beta<\alpha}W_\beta$. Now for each α, $\bigcup_{\beta<\alpha}W_\alpha$ is the union of a locally finite collection of closed sets, and is therefore closed. Hence each R_α is

open, and $\{R_\alpha\}_{\alpha\in A}$ is a disjoint, open refinement of $\{U_\alpha\}_{\alpha\in A}$.

(b)→(a): This is obvious.

Having established Proposition 2, we now turn to the proof of Theorem 2. It begins with the following lemma, which is a precise analogue of Lemma 1.

LEMMA 2. *If X is paracompact and zero-dimensional, Y a metric space, ψ a lower semi-continuous function from X to the non-empty subsets of Y, and if $r>0$, then there exists a continuous $f:X\to Y$ such that $f(x)\in S_r(\psi(x))$ for every x in X.*

Proof. For every $y\in Y$, let $U_y=\{x\in X \mid y\in S_r(\psi(x))\}$. Just as in the proof of Lemma 1, each U_y is open in X, and $\{U_y\}_{y\in Y}$ is an open covering of X. Since X is paracompact, $\{U_y\}_{y\in Y}$ has an open, locally finite refinement $\mathcal{R}$. Since X is a zero-dimensional normal space, $\mathcal{R}$ has an open *disjoint* refinement $\mathcal{W}$, by Proposition 2. For each $W\in\mathcal{W}$, pick a $y(W)\in Y$ such that $W\subset U_{y(W)}$, and define $f:X\to Y$ by

$$f(x) = y(W) \qquad \text{if } x\in W.$$

It is obvious that f satisfies all our requirements. This completes the proof of the lemma.

Theorem 2 now follows from Lemma 2 *in verbatim* the same way that Theorem 1 followed from Lemma 1.

4. Proof of the corollaries. To see how Corollary 1 follows from Theorem 1, we begin by observing that, by the open mapping theorem of Banach [1; p. 38, (1)], u is open (*i.e.*, maps open sets into open sets). Define ϕ from X to the non-empty, closed, convex subsets of Y by $\phi(x)=u^{-1}(x)$. The openness of u immediately implies (in fact, is equivalent to) the lower semi-continuity of ϕ. Since X is a Banach space, it is metric, and therefore paracompact. Hence, by Theorem 1, there exists a selection f for ϕ, and this f obviously satisfies our requirements.

The proof that Theorem 2 implies Corollary 2 proceeds in *precisely* the same fashion, except that this time it is *assumed* that u is open.

Finally, observe how Corollary 3 follows from Theorem 2. The polynomial w, like any analytic function, maps open sets into open sets (see, for instance, [11]). This implies that the function ϕ from X to the closed, non-empty subset of C, defined by $\phi(x)=w^{-1}(g(x))$, is lower semi-continuous. Hence, by Theorem 2, there exists a selection f for ϕ, and this f clearly does the trick.

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ESSENTIAL MAPPINGS*

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1. Introduction. The reader is requested to perform (at least mentally) the following experiment. Tie the ends of a piece of string together to form a loop, drive two pegs into a plane surface, and place the loop around the pegs as in Figure 1.

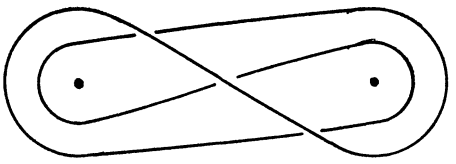


FIG. 1

Now, if one attempts to pull the string from around the pegs, keeping it on the plane and not lifting it over the pegs, it will be discovered that this is impossible. However, if either of the pegs is removed, then it is easy to pull the loop free from the other peg.

In this paper we discuss a generalization of this phenomenon. It is shown that if n is any positive integer, and n pegs are inserted into the plane surface, then it is possible to place the loop around the pegs in such a way that it is impossible to slide the loop free from the pegs, but if any one of the pegs is removed then it is possible to slide the loop free from the remaining $n-1$ pegs. Such a position of the loop for the case $n=3$ is shown in Figure 2.

* This work was completed during a period in which the author was partially supported by National Science Foundation grant NSF-G1353.

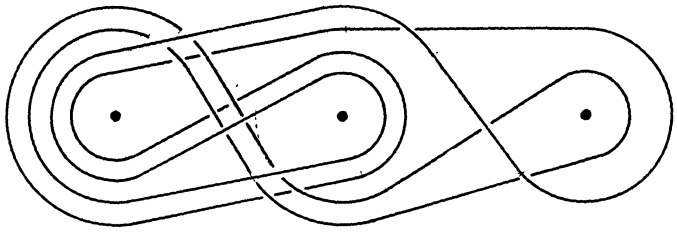


FIG. 2

2. A theorem about essential mappings. Let E^2 be Euclidean 2-space, and let I be the closed interval of real numbers from 0 to 1. If X is a subset of E^2, a *path* in X is a mapping (continuous function) on I into X. A path f is a *loop* if $f(0)=f(1)$. If $p=f(0)=f(1)$, then f is a *loop at* p.

If f and g are paths in E^2 and $f(1)=g(0)$, then fg is the path h for which

$$h(t) = f(2t), \qquad 0 \leqq t \leqq \tfrac{1}{2},$$
$$h(t) = g(2t-1), \qquad \tfrac{1}{2} \leqq t \leqq 1.$$

If f, g and h are paths for which $f(1)=g(0)$ and $g(1)=h(0)$, then fgh is defined to be the path $(fg)h$. Likewise, $fghk$ is $(fgh)k$, *etc.*

If f is a path, then f^{-1} is the path defined by $f^{-1}(t)=f(1-t)$.

If f and g are paths in X, $f(0)=g(0)$ and $f(1)=g(1)$, then f and g are *equivalent in* X if and only if there exists a continuous function H on $I\times I$ into X such that:

(i) $H(0, t)=f(t)$ for $0\leqq t\leqq 1$;
(ii) $H(1, t)=g(t)$ for $0\leqq t\leqq 1$;
(iii) $H(s, 0)=f(0)=g(0)$ for $0\leqq s\leqq 1$;
(iv) $H(s, 1)=f(1)=g(1)$ for $0\leqq s\leqq 1$.

The basic facts concerning paths, loops and equivalence of paths are contained in [1].

The *constant loop* at p is the function f for which $f(t)=p$ for each t in I. A loop g in a set X is *essential* in X if and only if g is not equivalent in X to the constant loop at $g(0)$.

We now state and prove a theorem which is suggested by the string experiment.

THEOREM 1. *If A is a non empty finite subset of E^2, then there exists a loop g which is essential in E^2-A, but which is not essential in E^2-B for every proper subset B of A.*

Proof. For each positive integer n, let a_n be the point $(2n-1, 0)$, and let A_n be the set whose members are $a_1, \cdots, a_n$. If the set A of our theorem has n members, then there exists a homeomorphism of E^2 onto itself which maps A onto the set A_n. It follows that we may assume without loss of generality that

for some n, A is the set A_n. We prove our theorem by induction on n.

We define for each positive integer n, paths u_n and v_n by the formulas

$$u_n(t) = (2n - 1 + \cos \pi t, \sin \pi t)$$
$$v_n(t) = (2n - 1 - \cos \pi t, -\sin \pi t)$$

for $0 \leqq t \leqq 1$.

For each positive integer n, let $p_n = (2n, 0)$. We define loops f_n at p_n inductively by letting

$$f_1 = u_1 v_1,$$
$$f_{n+1} = u_{n+1} f_n u_{n+1}^{-1} v_{n+1}^{-1} f_n^{-1} v_{n+1}.$$

We let R_n be the range of f_n. It is obvious that there exists a retract of $E^2 - A_n$ onto R_n, and consequently f_n is essential in $E^2 - A_n$ if and only if f_n is essential in R_n. We wish to prove by induction that f_n is essential in R_n for each positive integer n. It is well known that f_1 is essential in R_1. Let us assume that f_k is essential in R_k, and show that this implies that f_{k+1} is essential in R_{k+1}. We define a loop g at p_k by letting $g = u^{-1}{}_{k+1} f_{k+1} u_{k+1}$. It is well known that f_{k+1} is essential in R_{k+1} if and only if g is essential in R_{k+1}. However, g is equivalent in R_{k+1} to $h = f_k u_{k+1}^{-1} v_{k+1}^{-1} f_k^{-1} v_{k+1} u_{k+1}$. If we can prove that h is essential in R_{k+1}, then it will follow that g and f_{k+1} are also essential in R_{k+1}.

Let C be the circle with center a_{k+1} and radius 1. Both R_k and C are homeomorphic to connected complexes, $R_{k+1} = R_k \cup C$, and $R_k \cap C$ consists of the single point p_k. It follows from a well known theorem (see [1] p. 179) that the fundamental group of R_{k+1} is the free product of the fundamental groups of R_k and C. Since f_k, $u_{k+1}^{-1} v_{k+1}^{-1}$, f_k^{-1}, and $v_{k+1} u_{k+1}$ are essential loops in R_k, C, R_k and C respectively, it follows that h is an essential loop in R_{k+1}. We have thus proved that f_n is essential in $E^2 - A_n$ for each n.

Now let B be a proper subset of A_n. We wish to prove that f_n is not essential in $E^2 - B$. In order to accomplish this, we prove by induction the following slightly more general proposition:

If B is a set which consists of a finite number of the points $a_1, a_2, a_3, \cdots$, and $A_n - B$ is non-empty, then f_n is not essential in $E^2 - B$.

The above proposition is obvious for $n = 1$. Let us assume that it is true for $n = k$ and that B is a finite subset of $a_1, a_2, a_3, \cdots$ such that $A_{k+1} - B$ is non-empty. If $A_k - B$ is non-empty, then f_k is non-essential in $E^2 - B$, and hence f_{k+1} is equivalent to $u_{k+1} u_{k+1}^{-1} v_{k+1}^{-1} v_{k+1}$ which is equivalent to a constant loop. If $A_k - B$ is empty, then $a_{k+1} \notin B$, and it follows that $v_{k+1} u_{k+1}$ and $u_{k+1}^{-1} v_{k+1}^{-1}$ are not essential in $E^2 - B$; and hence f_{k+1} is not essential in $E^2 - B$.

3. Conclusion. Although there is an obvious correspondence between the physical problem concerning the string which we stated in section 1 and the theorem which we proved in section 2, there is not a complete analogy between

the two problems. This is seen by examining Figure 3.

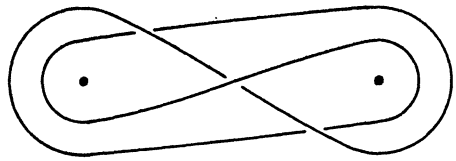


FIG. 3

The strings pictured in Figures 1 and 3 project onto the same loop in the plane, and this loop is not essential in the complement of either of the two points. However, if one of the pegs is removed, the string in Figure 1 can be slid free from the remaining peg, while the string in Fig. 3 cannot be slid free from the remaining peg. We now formulate a mathematical theorem in 3-space which more nearly approximates the string problem.

Let E^3 be Euclidean 3-space. If X is a subset of E^3, a *simple loop* in X is a mapping f of I into X which has the property that $f(t)=f(s)$ if and only if either $t=s$ or one of the numbers t, s is 0 and the other is 1. A simple loop f in X is *isotopically non-essential* in X if and only if there exists a mapping H of $I\times I$ into X such that:

(i) $H(0, t)=f(t)$ for $0\leqq t\leqq 1$;

(ii) $H(s, t)=f(0)$ if $t=0$, $t=1$, or $s=1$;

(iii) for each fixed s, $0\leqq s<1$, the function F_s defined by $F_s(t)=H(s, t)$ is a simple loop.

A simple loop in X that is not isotopically non-essential in X is said to be *isotopically essential* in X.

The following theorem is a mathematical formulation of the string problem.

THEOREM 2. *If A is the union of a finite number of parallel lines in E^3, then there exists a simple loop in E^3-A which is isotopically essential in E^3-A, but which is isotopically non-essential in E^3-B for every proper subset B of A.*

Proof. Let n be the number of lines in A. We may without loss of generality assume that A is the set of all points $(2k-1, 0, r)$ for which r is real, k is an integer, and $1\leqq k\leqq n$. Let f_n be the plane loop which was defined in the proof of Theorem 1, let $X(t)$ be the first coordinate of $f_n(t)$ and let $Y(t)$ be the second coordinate of $f_n(t)$. We define paths g and h in E^3-A by letting $g(t)=(X(t), Y(t), 2t)$ and $h(t)=(2n+\sin \pi t, 0, 1+\cos \pi t)$ for $0\leqq t\leqq 1$. Since $g(1)=h(0)$, the product path gh is defined. It is easily verified that gh is a simple loop in E^3-A, and that gh has the properties of the loop whose existence is asserted in our theorem.

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this Department has been temporarily suspended.

A CERTAIN CLASS OF SEMIGROUPS

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Recently I had occasion to consider a certain cancellation semigroup S in which the equation $ab=xba$ is solvable for x when a and b are arbitrarily given in S. It is clear that every group is such a semigroup; so also is every subsemigroup of a group which contains the commutator subgroup of the given group. On the other hand it will be pointed out here that such semigroups although more general than groups have many properties of groups. This will be done in connection with the proof of our main theorem below and the remarks following it.

THEOREM. *If S is a cancellation semigroup in which the equation $ab=xba$ is solvable for x when a and b are arbitrarily given in S then S contains a subgroup G_* and S is contained in a super group G^* such that G^*/G_* is Abelian.*

Proof. We show first that S has a unique identity e. By assumption there is an e so that $aa=eaa$; it follows that $a=ea$, $ea=e^2a$ and $e=e^2$. Then for any b in S, $be=be^2$ and $b=be$; similarly $b=eb$. Hence e is an identity for S. If f were also an identity then $f=fe=e$ and hence e is unique.

Now if a and b are arbitrary in S then by assumption $ab=xba$. We denote this x by $[a, b]$ and call it the *commutator* of a with b. This commutator is unique; for $xba=yba$ implies $x=y$.

$[a, b]b$ will be called the *conjugate of b by a* and will be denoted by b^a.

Then N will be called a *normal subsemigroup* of S if N is a subsemigroup and if, for every n in N, n^s is also in N for every s in S.

If N is a normal subsemigroup of S then there is a homomorphism σ of S under which the set of elements mapped onto σe is precisely N. That is, there is a mapping σ so that if b is in S then σb is defined to be bN, the set of elements bn where n ranges over N; thus $\sigma b=bN$ and $\sigma(ab)=(\sigma a)(\sigma b)$. As usual bN will be called a coset of S with respect to N, and our assertion is that the set of cosets constitutes a semigroup denoted by S/N which is a homomorphic image of S. For the proof of the above statements we refer the reader to [2], pp. 33–34, the proof for groups being applicable.

We next show that the subsemigroup S' generated by all the commutators of S is a group. We have $ab=[a, b]ba$ and $ba=[b, a]ab$ whence $ab=[a, b][b, a]ab$ and $e=[a, b][b, a]$. Hence every commutator y has an inverse y^{-1}. Suppose $mn\cdots p$, and $rs\cdots t$ are elements of S' where m, n, p, r, s, and t are commutators; then the equation $mn\cdots p=x(rs\cdots t)$ has for solution $mn\cdots pt^{-1}\cdots s^{-1}r^{-1}$ which is in S', and hence S' is a group. S' will be the

group G_* mentioned in the theorem.

We will refer the reader to a criterion of Doss (*cf.* [1]), essentially a formulation of a result of Ore [3], for the proof that S can be imbedded in a group. The criterion of Doss requires that certain elements be regular on the left (a is regular on the left if for every b there are r and s so that $ra=sb$). Now in S we have that $ab=xba$ and hence every element is regular on the left; it follows that S can be imbedded in a group K. We will let G^* be the subgroup of K generated by the elements of S, and observe that every element of S has an inverse in G^*.

Now let $[a, b]$ be in G_* and let c be arbitrary in S. Then $[a, b]^c = caba^{-1}b^{-1}c^{-1} = caba^{-1}c^{-1}b^{-1}bcb^{-1}c^{-1} = [ca, b][b, c]$ is in G_* and hence G_* is transformed into itself by the elements of S. Then since the normalizer of S is a group (*cf.* [2] p. 26) it is clear that G_* is normal in G^*, the group generated by the elements of S. But now modulo G_* the generators of G^* commute with one another and hence G^*/G_* is Abelian as the theorem asserts. This completes the proof.

It is clear how one would define *solvable*, *nilpotent*, *simple*, *normal series* and so on for the above class of semigroups. In general one should have no difficulty in carrying over those theorems of group theory in which the ideas of conjugacy and normal subgroups play the major role. Also one notes that both simple and finite semigroups are groups and hence the distinction between composition series by groups and composition series by semigroups appears in the infinite Abelian factors.

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THE MATRIC EQUATION $AX=B$ IN A FINITE FIELD

J. H. Hodges, Duke University

Let $q=p^n$ where p is a prime and let $GF(q)$ denote the Galois field of order q. Capital letters $A, B, \cdots$ will denote matrices with elements in $GF(q)$. $A(s, m)$ will denote a matrix of s rows and m columns and $A(s, m; \rho)$ a matrix of the same dimensions having rank ρ. $I(s, m; \rho)$ will denote the matrix of s rows and m columns having the identity of order ρ in its upper left-hand corner and zeros elsewhere. I_ρ will denote the identity of order ρ.

We seek the number $N(A, B)$ of solutions $X(m,t)$ of the equation

$$AX = B, \tag{1}$$

where $A=A(s, m; \rho)$ and $B=B(s, t; r)$ and $r \leqq \rho$. By [1, p. 281, Corollary 1] there exist non-singular P, Q, R, T such that $PAQ=I(s, m; \rho)$ and $RBT=$

$I(s,t;r)$. If we put $C=PR^{-1}=(\gamma_{ij})$, then (1) will reduce to

$$I(s, m; \rho)X = CI(s, t; r). \tag{2}$$

We will discuss the case when $\rho \leqq t$. By slight variations in the procedure, the same results are obtained when $\rho > t$. We write (2) as

$$\begin{bmatrix} I_\rho & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_{11} & X_{21} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} I(\rho, \rho; r) & 0 \\ 0 & 0 \end{bmatrix},$$

where $X_{11}=X_{11}(\rho, \rho)$, $X_{12}=X_{12}(\rho, t-\rho)$, $X_{21}=X_{21}(m-\rho, \rho)$, $X_{22}=X_{22}(m-\rho, m-\rho)$ and $C_{11}=C_{11}(\rho, \rho)$, $C_{12}=C_{12}(\rho, s-\rho)$, $C_{21}=C_{21}(s-\rho, \rho)$, $C_{22}=C_{22}(s-\rho, s-\rho)$. This implies that

$$X_{11} = C_{11}\mathrm{I}(\rho, \rho; r); \qquad X_{12} = 0; \qquad X_{21}, X_{22} \quad \text{are arbitrary} \tag{3}$$

and also a necessary condition on C, namely $C_{21}I(\rho, \rho; r)=0$. Writing this condition as

$$[C^1_{21} \quad C^2_{21}]\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = 0, \quad \text{where } C^1_{21} = C^1_{21}(s-\rho, r), \quad C^2_{21} = C^2_{21}(s-\rho, \rho-r),$$

we obtain the necessary condition $C^1_{21}=0$. Thus if $\gamma_{ij}=0$ for $\rho<i\leqq s$, $1\leqq j\leqq r$, the number of solutions of (2) is the number of choices of X_{21} and X_{22}. Thus we have proved

THEOREM 1. *If the equation* (1) *has any solutions, their number is*

$$N(A, B) = q^{t(m-\rho)}. \tag{4}$$

The condition for the existence of solutions is given by

THEOREM 2. *A necessary and sufficient condition that the equation* (1) *have solutions* $X(m, t)$ *is that there exist non-singular* P, Q, R, T *such that* $PAQ=I(s, m; \rho)$ *and* $RBT=I(s, t; r)$ *and if* $C=PR^{-1}=(\gamma_{ij})$, *then* $\gamma_{ij}=0$ *for* $\rho<i\leqq s$ *and* $1\leqq j\leqq r$.

The same method may be used to determine the number $N(A, B, C)$ of solutions $U=U(s, m)$, $V=V(t, n)$ of the matric equation

$$UA + BV = C, \tag{5}$$

where $A=A(m, n; r_1)$, $B=B(s, t; r_2)$ and $C=C(s, n; \rho)$. We are able to prove

THEOREM 3. *If the equation* (5) *has any solutions, their number is*

$$N(A, B, C) = q^{s(m-r_1)+n(t-r_2)+r_1r_2}. \tag{6}'$$

A necessary and sufficient condition, similar to that given in Theorem 2, may be obtained for the existence of solutions of (5).

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PARTICULAR INTEGRALS OF LINEAR DIFFERENTIAL EQUATIONS

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In a recent issue of this MONTHLY (March, 1955, p. 174), Fettis develops a method for obtaining a particular integral of the linear differential equation with constant coefficients

$$Y^{(n)} + a_0 Y^{(n-1)} + \cdots + a_{n-1} Y = f(x).$$

The method given by the author is known as that of the Cauchy function and is explained in Goursat-Hedrick-Dunkel, *Differential Equations*, Vol. 2, Part II, p. 108. It applies equally well to equations with variable coefficients.

It is interesting to note that the Cauchy function described by Fettis is closely related to the resolvent kernel of an integral equation of the Volterra type (Goursat, *Cours d'Analyse Mathématique*, 5th ed. Vol. 3, pp. 309–21).

In fact, let

$$D(Y) \equiv Y^{(n)} - \lambda[a_0(x)Y^{(n-1)} + a_1(x)Y^{(n-2)} + \cdots + a_{n-1}(x)Y] = 0$$

be a linear differential equation of the nth order containing the parameter λ and having variable coefficients $a_i(x)$. If the function $K(x, y)$ defined by

$$K(x, y) \equiv a_0(x) + a_1(x)(x - y) + a_2(x)\frac{(x - y)^2}{2!} + \cdots + \frac{a_{n-1}(x)(x - y)^{n-1}}{(n - 1)!}$$

is the kernel of the integral equation

$$\phi(x) - \lambda \int_0^x K(x, s)\phi(s)ds = g(x),$$

and $\Gamma(x, y; \lambda)$ is the resolvent kernel of K, then

$$\Gamma(x, y; \lambda) = K(x, y) + \lambda \int_y^x \Gamma(x, s; \lambda)K(s, y)ds.$$

Let $G(x, y; \lambda)$ be the Cauchy function of $D(y) = 0$. Then the relation referred to above is

$$\Gamma(x, y; \lambda) = \frac{1}{\lambda}\frac{\partial^n G(x, y; \lambda)}{\partial x^n}.$$

If the coefficients a_i are constant, the preceding functions have the respective forms $K(x-y)$, $\Gamma(x-y; \lambda)$ and $G(x-y; \lambda)$. Thus a particular integral is $\int_0^x G(x-s)f(s)ds$ as given by Fettis. For the special equation treated, namely

$$Y^{iv} - 2Y'' + Y = f(x); \qquad (a = 1),$$

the kernel is $2(x-y) - \frac{1}{6}(x-y)$, and $\Gamma(x, y; 1) = \frac{1}{2}(x-y)\mathrm{ch}(x-y) + \frac{3}{2}\mathrm{sh}(x-y)$. The Cauchy function is

$$G(x-y) = (x-y)\operatorname{ch}(x-y) - \operatorname{sh}(x-y).$$

It may be verified that $\Gamma(x, y; 1) = G^{(iv)}(x, y; 1)$.

Finally, similar remarks may be made for Cauchy functions of simultaneous differential equations for which we would need to consider systems of Volterra equations.

UNIMODULAR COMPLEMENTS

IRVING REINER, Institute for Advanced Study and University of Illinois

A vector with rational integral components is called *primitive* if the greatest common divisor of its components is unity. A well-known theorem* (valid when the components lie in any principal ideal ring) states that any primitive vector can be completed to a unimodular matrix, that is, given any primitive row vector $\mathfrak{a}$, there exists a matrix of determinant ± 1 whose first row is $\mathfrak{a}$.

Due to the many applications of this theorem, it may be of interest to point out that the result holds in situations more general than that of a principal ideal ring, although the standard proof* depends strongly on the hypothesis that all ideals are principal. In 1911, Steinitz† developed the theory of elementary divisors for matrices over the ring R of algebraic integers in an algebraic number field. Even though R is not in general a principal ideal ring, his results imply that if $a_1, \cdots, a_n \in R$ generate the unit ideal in R, then there exists a matrix with first row $(a_1 \cdots a_n)$ whose determinant is a unit in R. The purpose of this note is to give a simple proof of this theorem, valid for any ring R in which classical ideal theory holds (see van der Waerden, *Modern Algebra* II, p. 83. New York, 1950, Ungar), and in particular valid for the case considered by Steinitz.

The theorem clearly holds for $n=1$. When $n=2$, we note that the ideal (a_1, a_2) generated by a_1 and a_2 in R consists of all sums $x_1a_1+x_2a_2$, $x_1 \in R$, $x_2 \in R$. Hence if $(a_1, a_2) = (1)$, there exist $b_1, b_2 \in R$ for which

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 1,$$

so the result holds for $n=2$.

Now let $n>2$, and assume the result established for a primitive k-component row vector, where $k<n$. Observe that if U is unimodular, then $\mathfrak{a} = (a_1 \cdots a_n)$ is completable if and only if $\mathfrak{a}U$ is completable; for if $\mathfrak{a}$ is the first row of the unimodular matrix T, then $\mathfrak{a}U$ is the first row of the unimodular matrix TU. Since U has an inverse, the argument also goes in the other direction.

Next we show there exists $b \in R$ for which $(a_1, \cdots, a_{n-2}, a_{n-1}-ba_n) = (1)$. For let $(a_1, \cdots, a_{n-2}) = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_r^{\alpha_r}$ be the factorization of $(a_1, \cdots, a_{n-2})$

* MacDuffee, Theory of Matrices, Th. 21.1, p. 31. Berlin 1933, Springer.

† Math. Ann., Vol. 71, 1911, pp. 328–354.

into powers of distinct prime ideals. We need only choose b so that

$$a_{n-1} \not\equiv ba_n \pmod{\mathfrak{p}_i} \qquad (i = 1, \cdots, r).$$

For any $\mathfrak{p}_i$ dividing a_n, this holds for all b, since such a $\mathfrak{p}_i$ does not divide a_{n-1}. For the $\mathfrak{p}_i$ not dividing a_n, the congruence $a_n x \equiv a_{n-1} \pmod{\mathfrak{p}_i}$ has a unique solution x_i. For these $\mathfrak{p}_i$, we impose the condition

$$b \equiv x_i + 1 \pmod{\mathfrak{p}_i}.$$

These congruences can be solved for b by the Chinese Remainder Theorem, which is certainly valid under the hypotheses on R (see van der Waerden, *op. cit.*, II, p. 41).

Now we have

$$(a_1 \cdots a_{n-1} \quad a_n) \begin{pmatrix} I^{(n-2)} & & 0 \\ & 1 & 0 \\ 0 & -b & 1 \end{pmatrix} = (a_1 \cdots a_{n-2} \quad a_{n-1} - ba_n \quad a_n),$$

with the above matrix unimodular. Furthermore, by the induction hypothesis, $(a_1 \cdots a_{n-2}\ a_{n-1} - ba_n)$ is completable to a unimodular matrix V. Hence $(a_1 \cdots a_{n-2}\ a_{n-1} - ba_n\ a_n)$ is completable to

$$\begin{pmatrix} & & a_n \\ & V & 0 \\ & & \cdot \\ & & \cdot \\ 0 \cdots & 0 & 1 \end{pmatrix}.$$

This completes the proof.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE IRRATIONALITY OF $\sqrt{2}$

ROBERT GAUNTT, Purdue University, and GUSTAVE RABSON, Antioch College

(The following proof was invented by Robert James Gauntt, in 1952, while he was a freshman at Purdue. I was unable to induce him to write up his proof. G. R.)

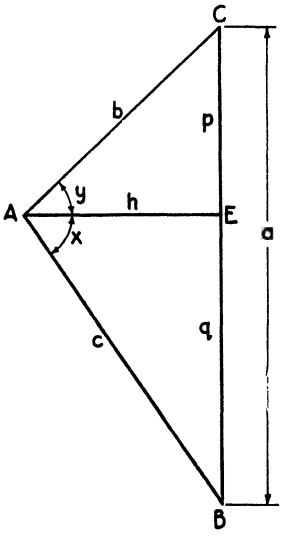
$a^2 = 2b^2$ cannot have a non-zero solution in integers because the last non-zero digit of a square, written in the base three, must be 1, whereas the last non-zero digit of twice a square is 2.

ADDITION THEOREMS FOR THE SINE AND COSINE

CARL COHEN, Cambridge Junior College

If the angles x and y are both positive and acute, the following construction leads to simple proofs of the addition theorems for the sine and cosine. We assume a knowledge of the formula for the area of a triangle in terms of two sides and the included angle, as well as the law of the cosine.

We draw angles x, y, and $x+y$ with common vertex A. Since x and y are positive acute angles, a line perpendicular to the common edge AE of x and y at E will intersect the other edges of x and y as indicated in the figure, at points which we call B and C respectively.



In this figure, the area S of ABC is:

$$S_{ABC} = \tfrac{1}{2}bc \sin (x + y). \tag{1}$$

On the other hand, by construction,

$$S_{ABC} = S_{ACE} + S_{ABE} = \tfrac{1}{2}bh \sin y + \tfrac{1}{2}ch \sin x. \tag{2}$$

Equating (1) and (2), we obtain

$$bc \sin (x + y) = bh \sin y + ch \sin x,$$

and, after dividing by bc,

$$\sin (x + y) = \frac{h}{c} \sin y + \frac{h}{b} \sin x = \sin y \cos x + \sin x \cos y.$$

Applying the law of the cosine, we write

$$a^2 = b^2 + c^2 - 2bc \cos (x + y). \tag{3}$$

Again, by construction, we have $a = p + q$, therefore

$$a^2 = p^2 + 2pq + q^2. \tag{4}$$

After applying the theorem of Pythagoras to both right triangles, we change (4) into

$$a^2 = b^2 + 2pq - 2h^2 + c^2. \tag{5}$$

Now, we equate (3) and (5) and obtain:

$$b^2 + 2pq - 2h^2 + c^2 = b^2 + c^2 - 2bc \cos (x + y).$$

We cancel b^2 and c^2 and divide both sides by $2bc$. The result is:

$$\cos (x + y) = \frac{h}{b} \cdot \frac{h}{c} - \frac{p}{b} \cdot \frac{q}{c},$$

which leads immediately to the wanted formula.

AN AVERAGING METHOD OF EXTRACTING ROOTS

J. P. BALLANTINE, University of Washington

1. Statement of the averaging method. To find the n-th root of any positive number, W, write

$$W = x^n = x_1^{n-1} x_2, \tag{1}$$

where x is the value sought, x_1 is an approximation to x, and x_2 is found by dividing W by x_1^{n-1}.

The next approximation, $\bar{x}$, is found by taking the weighted average of x_1 and x_2. The usual practice is to give x_1 and x_2 weights $n-1$ and 1, the same as their exponents in equation (1). By making the weights depend on the values of x_1 and x_2, better results may be had. It is the purpose of this paper to work out such weights, when the dependence is linear.

2. Derivation of the weights. Let

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}, \text{ where} \tag{2}$$

$$w_1 = w_{11} x_1 + w_{12} x_2, \text{ and} \tag{3}$$

$$w_2 = w_{21} x_1 + w_{22} x_2.$$

The constants w_{11}, w_{12}, w_{21}, and w_{22} will now be determined in such a way as to make $\bar{x}$ as nearly equal to x as possible.

Let

$$x_1 = x(1 + a).$$

Then

$$x_2 = x(1+a)^{1-n}$$
$$= x\left(1 + (1-n)a + \frac{-n(1-n)}{2}a^2 + \frac{n(1-n^2)}{6}a^3 \cdots\right).$$

We are, of course, supposing that a is small, so that the binomial expansion converges. Similar series are easily written for x_1^2, x_1x_2, and x_2^2.

In equation (2), the letters w_1 and w_2 are replaced by their values in equation (3), and now the letters x_1 and x_2 are replaced by their values in terms of x, a, and n. The result will be a fraction, with both numerator and denominator expressed in powers of a, namely:

$$\bar{x} = \frac{x^2(U_0 + U_1a + U_2a^2 + \cdots)}{x(V_0 + V_1a + V_2a^2 + \cdots)}, \tag{4}$$

where

$$U_0 = V_0 = w_{11} + w_{12} + w_{21} + w_{22}.$$
$$U_1 = 2w_{11} + (2-n)w_{12} + (2-n)w_{21} + (2-2n)w_{22}.$$
$$V_1 = w_{11} + w_{21} + (1-n)w_{12} + (1-n)w_{22}.$$
$$U_2 = w_{11} + \tfrac{1}{2}(n-1)(n-2)(w_{12} + w_{21}) + (n-1)(2n-1)w_{22}.$$
$$V_2 = \tfrac{1}{2}n(n-1)(w_{12} + w_{22}).$$
$$U_3 = \tfrac{1}{6}(n)(n-1)(2-n)(w_{12} + w_{21}) + \tfrac{2}{3}n(2n-1)(1-n)w_{22}.$$
$$V_3 = \tfrac{1}{6}n(1-n)(1+n)(w_{12} + w_{22}).$$

In order for x and $\bar{x}$ to be nearly equal, the parentheses in equation (4) must be nearly equal. Fortunately $U_0 = V_0$, without restriction on the weights. Though we have 4 weights, only the ratios between them count, so we may expect to impose 3 conditions, namely $U_1 = V_1$, $U_2 = V_2$, and $U_3 = V_3$. These are three homogeneous linear equations among the four weights, w_{ij}. Their solutions are:

$$w_{11} = (2n-1)(n-1), \qquad w_{12} = (n-1)(4n+1),$$
$$w_{21} = 5n - 1, \qquad w_{22} = n + 1. \tag{5}$$

3. Square roots. When $n=2$, a factor 3 drops out, and

$$w_{11} = 1, \quad w_{12} = 3, \quad w_{21} = 3, \quad \text{and} \quad w_{22} = 1. \tag{6}$$

For example, to find $\sqrt{30}$, take $x_1=5$ and $x_2=6$. Then 5 has a weight of $5+3\cdot 6=23$ and 6 has a weight of $6+3\cdot 5=21$. The weighted average is

$$\frac{23\cdot 5 + 21\cdot 6}{23 + 21} = \frac{241}{44} = 5.4772727 \cdots.$$

Actually,

$$\sqrt{30} = 5.4772255 \cdots .$$

The usual $\frac{1}{2}(5+6) = 5.5$ gives a much poorer approximation.

For the special case when $n = 2$, formula (2) can be put into a more convenient form for computation, by substituting w_1 and w_2 from equation (3) into formula (2).

$$\bar{x} = \frac{w_{11}x_1^2 + (w_{12} + w_{21})x_1x_2 + w_{22}w_2^2}{(w_{11} + w_{21})x_1 + (w_{12} + w_{22})x_2},$$

$$\bar{x} = \frac{4x^2 + (x_1 + x_2)^2}{4(x_1 + x_2)}. \tag{7}$$

For a direct derivation of formula (7) with remainder, write:

$$\begin{aligned} x - \tfrac{1}{2}(x_1 - x_2) &= e \\ 2x - (x_1 + x_2) &= 2e \\ 4x^2 - 4x(x_1 + x_2) + (x_1 + x_2)^2 &= 4e \end{aligned}$$

$$x = \frac{4x^2 + (x_1 + x_2)^2}{4(x_1 + x_2)} - \frac{e^2}{x_1 + x_2}. \tag{8}$$

For example, suppose the approximation 1.4 to $\sqrt{2}$ is known. Then $x_1 = 1.4$, $x_2 = 1.428571428 \cdots$ by division, and $x^2 = 2$.

$$\bar{x} \approx \frac{4(2) + (2.828571428)^2}{4(1.4 + 1.428571428)} = \frac{16.0008163265}{11.3142857143} = 1.414213564.$$

To approximate e of formula (8), write

$$e \approx \bar{x} - \tfrac{1}{2}(x_1 + x_2) = 1.4142135 - 1.4142857 \approx -0.000072$$

$$\frac{e^2}{x_1 + x_2} \approx \frac{0.0000000052}{2.8} = 0.000000002.$$

So that

$$x \approx \bar{x} - 0.000000002 = 1.414213562.$$

4. Cube roots. When $n = 3$, a factor 2 drops out, and

$$w_{11} = 5,\ w_{12} = 13,\ w_{21} = 7, \quad \text{and} \quad w_{22} = 2. \tag{9}$$

For example, to find $\sqrt[3]{100}$, write $5^2 \cdot 4 = 100$. Then 5 has a weight $5 \cdot 5 + 13 \cdot 4 = 77$, and 4 has a weight of $7 \cdot 5 + 2 \cdot 4 = 43$. The weighted average is

$$\bar{x} = \frac{77 \cdot 5 + 43 \cdot 3}{77 + 43} = \frac{557}{120} = 4.641666 \cdots .$$

This compares with the correct answer, $\sqrt[3]{100}=4.6415888\cdots$, and the usual average of 5 and 4 with weights 2 and 1 of $4.666\cdots$.

Referee's Note. The method given is a fourth-order iterative process for the determination of an nth root. It happens that the iteration formula is identifiable with the third of a sequence of such formulas obtainable as successive convergents of the formal continued fraction

$$\bar{x} = x_1 - \cfrac{\omega}{1 - \cfrac{(n-1)\omega/(2x_1)}{1 - \cfrac{(n+1)\omega/(6x_1)}{1 - \cdot \\ \qquad \cdot \\ \qquad\quad \cdot}}}$$

where

$$\omega = \frac{1}{n}\left(x_1 - \frac{W}{x_1^{n-1}}\right) \equiv \frac{1}{n}(x_1 - x_2),$$

with the author's notation, the two preceding formulas consequently being of the forms

$$\bar{x}^{(1)} = x_1 - \omega = \frac{1}{n}[(n-1)x_1 + x_2]$$

and

$$\bar{x}^{(2)} = x_1 - \cfrac{\omega}{1 - \cfrac{n-1}{2}\cfrac{\omega}{x_1}} = \frac{(n-1)x_1 + (n+1)x_2}{(n+1)x_1 + (n-1)x_2}x_1,$$

of second and third order, respectively.

The definition of $\bar{x}^{(1)}$ corresponds to the Newton-Raphson iteration, as applied to $f(x)=x^n-W$, while the approximation $\bar{x}^{(2)}$ corresponds to the so-called Halley iteration, as applied to the same function. For the generation of the sequence, see Frame (this MONTHLY, vol. 60, 1953, pp. 293–305).

In the special case $n=2$, the iteration (7) is equivalent to two successive applications of the Newton-Raphson iteration $\bar{x}^{(1)}$.

Whereas, for the purpose of minimizing the total labor, the usual tendency is to use a second-order process twice rather than one of the fourth order processes once, the author's calculational arrangement is a simple one and it may find some favor in practice.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1211. *Proposed by Alan Wayne, Cooper Union School of Engineering*

Prove the following theorem, discovered by Cipriano Ferraris of the *New York Riddlers Club:* "For any positive integer k, the total number of digits in the sequence 1, 2, $\cdots$, 10^k equals the number of zero digits in the sequence 1, 2, $\cdots$, 10^{k+1}."

E 1212. *Proposed by H. A. Osborn, University of California, Berkeley*

Show that $t>0$ implies $(2+\cos t)t>3\sin t$.

E 1213. *Proposed by F. D. Parker, Clarkson College of Technology*

Find the Wronskian, $W(n, k)$, of the set of functions x^n, $x^n \ln x$, $x^n \ln^2 x$, $\cdots$, $x^n \ln^k x$.

E 1214. *Proposed by Paul Payette, École Polytechnique de Montreal*

Find the envelope of the family of ellipses of constant major axis having one focus at a given point and the other focus on a given straight line.

E 1215. *Proposed by J. P. Ballantine, University of Washington*

State a necessary and sufficient condition for an ordered set of n line segments to be the consecutive sides of an n-gon possessing an inscribed circle.

SOLUTIONS

E 1171 [1956, 121]. *Correction.* The *Editorial Note* should be corrected to read " . . . , then $(n^a-1, n^b-1)=n-1$ if and only if $(a, b)=1$."

Divisors of a String of Ones

E 1181 [1955, 581]. *Proposed by F. L. Wolf, Carleton College*

Show that if an integer has all its prime factors greater than 5 then it divides some integer whose decimal representation is a string of 1's.

Solution by W. E. Briggs, University of Colorado. Since $(n, 10)=1$, n divides $10^{\phi(n)}-1=9(111\cdots 1)$. Since $(n, 3)=1$, n divides the factor consisting only of ones.

Also solved by J. A. Bond, Jr., D. H. Browne, R. L. Causey, G. B. Charlesworth, Fred Cherry, P. A. Clement, A. E. Danese, M. Delcourte, Calvin Foreman, Michael Goldberg, A. J. Goldman, D. S. Greenstein, C. A. Grimm, Nathaniel Grossman, Cornelius Groenewoud, Virginia Hanly, Juris Hartmanis, B. A. Hausmann, M. J. Hellman, Vern Hoggatt, C. W. Langley, D. C. B. Marsh, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, C. S. Ogilvy, Paul Payette, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), Azriel Rosenfeld, David Rothman, C. M. Sandwick, Sr., J. P. Scholz, R. E. Shafer, G. J. Simmons, E. P. Starke, R. P. Tapscott, C. W. Topp, Chih-yi Wang, R. J. Wisner, F. S. Zusman, and the proposer. Late solutions by J. G. Clunie, F. I. John, M. S. Klamkin, and C. F. Pinzka.

An Identity

E 1182 [1955, 581]. *Proposed by I. A. Barnett, University of Cincinnati*

If $d_k = u^k - x^k - y^k - z^k$, for $k = 1, 2, 3, \cdots$, establish the identity

$$d_{n+2} - (x + y + z)d_{n+1} + (xy + yz + zx)d_n - (xyz)d_{n-1} = u^{n-1}(u - x)(u - y)(u - z).$$

Solution by J. A. Tierney, U. S. Naval Academy. Denote the left member of the proposed identity by $f(u)$. A simple calculation shows that $f(x) = f(y) = f(z) = 0$. Hence $f(u)$ has $u-x$, $u-y$, $u-z$ as factors. In fact, $f(u) = u^{n-1}(u-x)(u-y)\cdot(u-z)$, since the four coefficients of like powers of u are equal by inspection.

Also solved by A. N. Aheart, Norman Anning, G. B. Charlesworth, A. E. Danese, J. E. Darraugh, M. Delcourte, H. M. Feldman, Edward Fleisher, Calvin Foreman, A. J. Goldman, D. S. Greenstein, C. A. Grimm, Nathaniel Grossman, M. J. Hellman, Vern Hoggatt, A. R. Hyde, P. W. M. John, D. R. Lewis, D. C. B. Marsh, W. R. McEwen, J. B. Muskat, Herbert Nadler, A. R. Nolstad, Walter Penney, B. E. Rhoades, James Robertson and Dale Woods (jointly), D. A. Robinson, Azriel Rosenfeld, David Rothman, R. E. Shafer, O. E.Stanaitis, E. P. Starke, Chih-yi Wang, Charlotte Yesselman, David Zeitlin, and the proposer. Late solutions by J. E. D'Atri, Hüseyin Demir, F. I. John, M. S. Klamkin, C. F. Pinzka, K. Subba Rao, and D. C. Russell.

Editorial Note. Denote the left member of

$$r^{n+2} - (x + y + z)r^{n+1} + (xy + yz + zx)r^n - xyzr^{n-1} \equiv r^{n-1}(r - x)(r - y(r - z)$$

by $g(r)$. The left member of the proposed identity is then given by $g(u) - g(x) - g(y) - g(z)$, or, since $g(x) = g(y) = g(z) = 0$, by $g(u)$, which establishes the proposed identity.

A Simple Series Summation

E 1183 [1955, 581]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Prove that $\sum_{n=1}^{\infty} (n-1)/n! = 1$.

I. *Solution by H. M. Gehman, University of Buffalo.* Let $\sum$ denote summation from $n=1$ to $n=k$. Then $S=\sum(n-1)/n!=\sum 1/(n-1)!-\sum 1/n!=1-(1/k!)$. As k becomes infinite, S approaches 1.

This problem occurs in the *Mathematics Student Journal* for October, 1955, p. 4, problem 46. It is attributed to the eighth annual Stanford University Competitive Examination, April 11, 1943.

II. *Solution by Edward Fleisher, Brooklyn, N. Y.* Since the series converges absolutely, we have

$$\sum_{n=1}^{\infty}(n-1)/n! = \sum_{n=1}^{\infty} 1/(n-1)! - \sum_{n=1}^{\infty} 1/n! = e-(e-1) = 1.$$

Also solved by W. A. Al-Salam, Ferrel Atkins, G. E. Bardwell, P. M. Berry, Nardler Berseck, W. E. Briggs, J. L. Brown, Jr., R. L. Causey, G. B.Charlesworth, George Cherlin, P. L. Chessin, A. E. Danese, J. E. Darraugh, M. Delcourte, Thomas Erber, H. M. Feldman, Allen Fenstermacher, Walter Fleming and David Lewis (jointly), W. L. Fields, Calvin Foreman, David Freedman, Joyce Friedman, B. A. Fusaro, A. M. Glicksman, Michael Goldberg, A. J. Goldman, D. S. Greenstein, C. A. Grimm, Cornelius Groenewoud, Nathaniel Grossman, Peter Hagis, Jr., W. G. Hanks, Frank Harary, Juris Hartmanis, M. J. Hellman, Vern Hoggatt, F. A. Homann, Raymond Huck, A. R. Hyde, L. E. Isenecker, P. W. M. John, J. D. E. Konhauser, M. A. Kirchberg, P. G. Kirmser, W. D. Lambert, C. W. Langley, Kenneth Loewen, P. K. Maloof, D. C. B. Marsh, L. C. Marshall, W. R. McEwen, G. M. Merriman, Erich Michalup, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, A. R. Nolstad, C. S. Ogilvy, Hyman Orlin, F. D. Parker, M. J. Pascual, Paul Payette, W. O. Pennell, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), W. J. Pervin, J. P. Phillips, R. B. Plymale, B. E. Rhoades, L. A. Ringenberg, James Robertson and Dale Woods (jointly), D. A. Robinson, Azriel Rosenfeld, J. W. Ross, David Rothman, C. M. Sandwick, Sr., Nina Schub, R. E. Shafer, W. A. Small, D. W. Smith, O. E. Stanaitis, E. P. Starke, F. M. Stein, D. D. Strebe, R. P. Tapscott, C. W. Topp, Chih-yi Wang, Alan Wayne, R. R. Williams, Jr., R. L. Wilson, David Zeitlin, F. S. Zusman, and the proposer. Late solutions by Hüseyin Demir, R. F. Gabriel, A. G. Grace, Jr., F. I. John, M. S. Klamkin, D. J. Peterson, C. F. Pinzka, K. Subba Rao, D. C. Russell, and Bernard Smilowitz.

Evaluation of a Limit

E 1184 [1955, 581]. *Proposed by Viktors Linis, University of Ottawa*

Let $f(x)$ be differentiable at $x=a$ and let $f(a)\neq 0$. Evaluate

$$\lim_{n\to\infty}[f(a+1/n)/f(a)]^n.$$

Solution by R. F. Church, Nicolas Johnson, and F. J. Lorenzen, Prof. R. M. Conkling's Advanced Calculus Class, University of New Hampshire. The limit sought is the same as

$$\lim_{x\to 0} [f(a+x)/f(a)]^{1/x}.$$

For x small enough, $f(a+x)$ and $f(a)$ have the same sign. Then

$$\lim_{x\to 0} \log [f(a+x)/f(a)]^{1/x} = \lim_{x\to 0} [\log f(a+x) - \log f(a)]/x.$$

Since a differentiable function of a differentiable function is differentiable, this last limit is, by definition, the derivative of $\log f(x)$ at $x=a$, or $f'(a)/f(a)$. Hence

$$\lim_{x\to 0} [f(a+x)/f(a)]^{1/x} = \exp [f'(a)/f(a)].$$

Also solved by W. A. Al-Salam, Nardler Berseck, W. E. Briggs, G. B. Charlesworth, P. L. Chessin, A. E. Danese, M. Delcourte, Thomas Erber, H. M. Feldman, Walter Fleming, David Freedman, B. A. Fusaro, Michael Goldberg, A. J. Goldman, D. S. Greenstein, Nathaniel Grossman, Peter Hagis, Jr., Virginia Hanly, Juris Hartmanis, R. E. Heaton, M. J. Hellman, Vern Hoggatt, A. R. Hyde, L. E. Isenecker, P. G. Kirmser, D. C. B. Marsh, L. C. Marshall, J. D. Miller, Morris Morduchow, Leo Moser, A. R. Nolstad, C. S. Ogilvy, M. J. Pascual, Paul Payette, Walter Penney, H. P. Rawlings, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, David Rothman, W. A. Small, O. E. Stanaitis, E. P. Starke, F. M. Stein, D. D. Strebe, R. P. Tapscott, Chih-yi Wang, J. V. Whittaker, David Zeitlin, and the proposer. Late solutions by Hüseyin Demir, A. G. Grace, Jr., F. I. John, M. S. Klamkin, and D. C. Russell.

The Proposer pointed out that this problem permits the construction of interesting examples, such as

$$\lim_{n\to\infty} \tan^n(\pi/4 + 1/n) = e^2$$

and

$$\lim_{n\to\infty} [\cos (a + 1/n)/\cos a]^n = e^{-\tan a}, \qquad \cos a \neq 0.$$

Infinite Monotone Subsequences

E 1185 [1955, 581]. *Proposed by Frederic Cunningham, Jr., University of New Hampshire.*

Prove that every infinite sequence of real numbers contains an infinite monotone subsequence.

I. *Solution by A. J. Goldman, Princeton University.* Let $\{x_n\}$ be an infinite sequence of real numbers. If, on the one hand, every infinite subsequence of $\{x_n\}$ has a (strictly) largest term, then let y_1 be the largest term of $\{x_n\}$, let y_2 be the largest term of the subsequence of $\{x_n\}$ consisting of all terms of $\{x_n\}$ *after* y_1, *etc.* Then $\{y_1, y_2, \cdots\}$ is a (strictly) monotone decreasing subsequence of $\{x_n\}$. If, on the other hand, some infinite subsequence $\{u_i\}$ of $\{x_n\}$ does *not* have a strictly largest term, then we set $z_1 = u_1$, choose z_2 to be a term *after* z_1 in $\{u_i\}$ which is no smaller than z_1, *etc.* Then $\{z_1, z_2, \cdots\}$ is an infinite monotone ascending subsequence of $\{x_n\}$.

The proof uses only order-theoretic properties of the real numbers, and is valid in any totally-ordered set.

II. *Solution by Azriel Rosenfeld, Columbia University.* If the sequence is unbounded, say to the right, the construction of a strictly ascending subsequence is immediate. Similarly, unboundedness to the left yields a strictly descending subsequence.

If the sequence is bounded, it has a cluster point (in the generalized sense in which the same point, counted infinitely often in the sequence, is a cluster point of the sequence too). Then an infinite number of terms of the sequence are contained in some (small, half-closed) interval either to the right or to the left of this point. The construction respectively of a nondecreasing or a nonincreasing subsequence is then clear.

III. *Solution by Leo Moser, University of Alberta.* This is an immediate consequence of the following theorem due to P. Erdös and G. Szekeres (*A combinatorial problem in geometry*, Composition Mathematica, vol. 2, 1935, pp. 463–470):

Theorem. From n points with monotonically increasing abscissae it is always possible to select at least $\sqrt{n}$ points with increasing abscissae and either monotonically increasing or decreasing ordinates. (If two ordinates are equal, the case may equally be regarded as increasing or decreasing.)

Also solved by A. E. Danese, D. S. Greenstein, Virginia Hanly, Juris Hartmanis, Vern Hoggatt, A. R. Hyde, L. E. Isenecker, C. W. Langley, Norman Levine, D. C. B. Marsh, Paul Payette, L. A. Ringenberg, J. W. Ross, David Rothman, J. P. Scholz, W. L. Shepherd, O. E. Stanaitis, E. P. Starke, D. D. Strebe, R. P. Tapscott, R. J. Wisner, and the proposer. Late solutions by Nathaniel Grossman, M. S. Klamkin, C. F. Pinzka, and D. C. Russell.

Editorial Note. This problem provides a direct method of proving that a Cauchy sequence has a limit. See, *e.g.*, Dick Wick Hall and Guilford L. Spencer II, *Elementary Topology* (John Wiley and Sons, 1955), pp. 30–32.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4683. *Proposed by Edgar Karst, Independence, Missouri*

Where fails the "interesting conjecture" (so called by D. H. Lehmer in a letter to the proposer, March 1952): If 2^p-1 is prime, then also $2^p-1+100$ is prime?

4684. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Prove that $a_n \to 0$ as $n \to \infty$, where

$$a_n = 1 - \frac{n-1}{1!} + \frac{(n-2)^2}{2!} - \frac{(n-3)^3}{3!} + \cdots + \frac{1}{(n-1)!}.$$

4685. *Proposed by L. J. Mordell, University of Toronto*

Prove that the n-dimensional polytope

$$|x_1| + |x_2| + \cdots + |x_n| + |x_1 + x_2 + \cdots + x_n| \leqq 2,$$

has $2^{n+1}-2$ faces of $n-1$ dimensions and contains an n-dimensional volume of $(2n)!/(n!)^3$.

4686. *Proposed by Leonard Carlitz, Duke University*

Show that

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} (2k-1) \tanh \frac{(2k-1)\pi}{2} - n^2 \right\} = -1/12, \tag{1}$$

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k \coth k\pi - \frac{n(n+1)}{2} \right\} = 1/12 - 1/4\pi. \tag{2}$$

4687. *Proposed by John Wermer, Brown University*

Let Γ be a simple closed curve in the complex plane containing the origin in its interior. Show that if Γ is not rectifiable, then we can approximate the constant 1 uniformly on Γ by functions

$$\sum_{n=-N}^{N} c_n z^n, \qquad c_0 = 0.$$

SOLUTIONS

An Indeterminate Form

4622 [1955, 45]. *Proposed by Harry Furstenberg, New York City*

Let r be any positive integer and let $i=r, r+1, \cdots, n$ and $j=1, 2, \cdots, r$. Prove

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j} \log\left(1 - \frac{i}{n+j}\right) = -r.$$

I. *Solution by Vladeta Vučković, Mathematical Institute, Belgrade, Yugoslavia.*

Direct computation and application of Stirling's formula give

$$\frac{1}{n} \sum_{j=1}^{r} \log\left(1 - \frac{r}{n+j}\right)\left(1 - \frac{r+1}{n+j}\right) \cdots \left(1 - \frac{n}{n+j}\right)$$

$$= \sum_{j=1}^{r} \log \sqrt[n]{\frac{(n-r+j)!}{(j-1)!(n+j)^{n-r+i}}} \sim \sum_{j=1}^{r} \log \frac{1}{e} = -r.$$

II. *Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota.* If we write

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j} \log\left(1 - \frac{i}{n+j}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=r}^{n} \log\left(1 - \frac{i}{n+1}\right)$$

$$+ \lim_{n\to\infty} \frac{1}{n} \sum_{i=r}^{n} \log\left(1 - \frac{i}{n+2}\right) + \cdots + \lim_{n\to\infty} \frac{1}{n} \sum_{i=r}^{n} \log\left(1 - \frac{i}{n+r}\right),$$

and make use of the definition of the definite integral, it follows at once that each of the r terms on the right side is equal to the familiar integral

$$\int_0^1 \log(1-x)\,dx = -1.$$

This completes the proof.

Also solved by A. J. Goldman, M. S. Klamkin, Julius Lieblein, A. E. Livingston, L. A. Ringenberg, Chih-yi Wang, J. V. Whittaker, and the Proposer.

A Variant of the Probability Integral

4623 [1955, 126]. *Proposed by L. L. Pennisi, University of Illinois, Navy Pier, Chicago*

Prove that

$$\lim_{x\to\infty} \frac{\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{3\cdot 5\cdot\cdots\cdot(2n+1)}}{\sum_{n=0}^{\infty} \dfrac{x^{2n}}{2\cdot 4\cdot\cdots\cdot 2n}} = \sqrt{\frac{\pi}{2}}.$$

Solution by Emil Grosswald, University of Pennsylvania.

Let $\phi(x)=f(x)/g(x)$, with $f(x)=\sum x^{2n+1}/3\cdot 5\cdot\cdots\cdot(2n+1)$, and $g(x)=\sum x^{2n}/2\cdot 4\cdot\cdots\cdot 2n$. It is easily seen that $f(x)$ and $g(x)$ are entire functions, satisfying the differential equations and initial conditions

$$f'(x) - xf(x) - 1 = 0 \qquad f(0) = 0,$$
$$g'(x) - xg(x) = 0 \qquad g(0) = 1.$$

The solutions by familiar methods are

$$f(x) = e^{x^2/2}\int_0^x e^{-t^2/2}dt, \qquad g(x) = e^{x^2/2}.$$

Consequently $\phi(x)=\int_0^x e^{-t^2/2}dt$, the classical probability integral, and the desired limit is $\sqrt{\pi/2}$.

Also solved by W. J. Blundon, B. J. Boyer, G. U. Brauer and Chih-yi Wang, Robert Breusch, F. A. Butter, Jr., G. B. Findley, N. J. Fine, Harley Flanders, A. J. Goldman, S. W. Golomb, W. O. Gordon, Cornelius Groenewoud, Emil Grosswald, Peter Hagis, Jr., Peter Henrici, A. R. Hyde, M. S. Klamkin, J. A. Larrivee, David Lovenvirth, L. I. Lowell, S. Marein-Efron, D. C. B. Marsh, C. S. Ogilvy, F. R. Olson, W. J. Pervin, G. Pólya, E. D. Rainville, L. A. Ringenberg, P. G. Rooney, D. C. Russell, F. C. Smith, M. R. Spiegel, T. W. Summers R. P. Tapscott, Arnold Walfisz, the Proposer, and one whose name does not appear. Late solution by O. E. Stanaitis.

Editorial Note. Pólya notes that the essential part of the problem appeared on the Putnam Prize examination for 1950. See [1950, 469].

A Beta-Function

4625 [1955, 127]. *Proposed by K.-F. Moppert, Basel, Switzerland*

For $a>0$, $b>-1$, prove

$$\frac{1}{a} - \binom{b}{1}\frac{1}{1+a} + \binom{b}{2}\frac{1}{2+a} - + \cdots = \frac{\Gamma(a)\Gamma(1+b)}{\Gamma(1+a+b)}.$$

I. *Solution by M. R. Spiegel, Rensselaer Polytechnic Institute.* We have

$$\int_0^x u^{a-1}(1-u)^b du = \int_0^x u^{a-1}\left[1 - \binom{b}{1}u + \binom{b}{2}u^2 - \cdots\right]du$$
$$= \frac{x^a}{a} - \binom{b}{1}\frac{x^{a+1}}{1+a} + \binom{b}{2}\frac{x^{a+2}}{2+a} - \cdots$$

if $0<x<1$ and $b>-1$. Letting $x\to 1$ and employing Abel's theorem it follows that

$$\int_0^1 u^{a-1}(1-u)^b du = \frac{\Gamma(a)\Gamma(1+b)}{\Gamma(1+a+b)} = \frac{1}{a} - \binom{b}{1}\frac{1}{1+a} + \binom{b}{2}\frac{1}{2+a} - \cdots,$$

which proves the result. If b is a positive integer or zero the series terminates and Abel's theorem is unnecessary. The uniform convergence of $(1-x)^b$ in the interval (0, 1) justifies the interchange of summation and integration.

II. *Solution by E. D. Rainville, University of Michigan.* The restrictions a not a nonpositive integer and $\mathrm{Re}(b)>-1$ are sufficient. Let ψ denote the series to be summed:

$$\psi = \sum_{n=0}^{\infty} \frac{(-1)^n\binom{b}{n}}{a+n} = \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-b)_n(a)_n}{(a+1)_n n!} = \frac{1}{a}\,{}_2F_1(-b, a; a+1; 1).$$

If $\mathrm{Re}(b)>-1$, then

$$\psi = \frac{1}{a}\,\frac{\Gamma(a+1)\Gamma(a+1+b-a)}{\Gamma(a+1-a)\Gamma(a+1+b)} = \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

Also solved by P. M. Batchelder, B. J. Boyer, Leonard Carlitz, A. E. Danese, G. B. Findley, C. L. Gape, A. J. Goldman, V. C. Harris, Peter Henrici, A. R. Hyde, M. S. Klamkin, S. Marein-Efron, F. H. Northover, C. D. Olds, P. G. Rooney, D. C. Russell, N. Shklov, F. C. Smith, T. W. Summers, and Chih-yi Wang.

Editorial Note. The problem may be found in several places, *e.g.*, Gibson, *Advanced Calculus*, p. 498, no. 23; Bromwich, *Infinite Series*, p. 194, ex. 31.

Point Set Dense in the Plane

4626 [1955, 127]. *Proposed by W. A. Michael and D. A. Page, University of Illinois*

Let A be a subset of the plane. Join each pair of distinct points of A by a line. Let A_1 consist of the points of A together with the points of intersection of distinct lines. Starting with A_1 the same construction yields a set A_2. In this way an increasing sequence of sets $A_1, A_2, \cdots$, is obtained. Let $\Omega=\cup A_n$. If A

is denumerable then so is Ω, and hence Ω is not the entire plane; however, under what conditions will Ω be dense in the plane?

Solution by the Proposers. Ω will be dense in the plane if and only if there are four points in A such that

(i) no three are collinear, and

(ii) the four points are not the vertices of a parallelogram.

For suppose A contains four such points p_1, p_2, p_3, p_4. Exactly two possibilities arise: *Case* 1. One of the points, say p_4, is interior to the triangle determined by the other three. We first show that the set E of points of Ω which lie on the line segment p_1p_2 is dense in p_1p_2. For this purpose it is enough to show that for each pair of distinct points x, y in E there exists a point z in E distinct from x and y and lying between x and y. Suppose x to be between p_1 and y. Construct lines p_1p_4 and p_2p_4, and let q_1 and q_2 be their intersections with p_2p_3 and p_3p_1 respectively. Construct lines xq_1 and yq_2, and let w be their intersection. Let z be the intersection of the lines p_3w and p_1p_2. Clearly z is the required point.

Similarly it follows that the set of points of Ω which lie on the segments p_2p_3 and p_3p_1 is dense in these segments. Ω contains points of intersection of lines joining points of these dense sets, and hence Ω must be dense in the plane.

Case 2. The four points form a convex quadrilateral. Since this quadrilateral is not a parallelogram we may extend a pair of opposite sides to a point of intersection, thus obtaining a triangle. Since the intersection of the diagonals of the quadrilateral lies inside this triangle we may now apply the argument for *Case* 1.

The necessity of conditions (i) and (ii) is easily verified.

If the plane referred to is the real projective plane, the condition(ii) may be discarded.

Also solved by Robert Breusch, Harley Flanders, J. K. Jacobsen and C. D. Gorman, and P. G. Kirmser.

Series Representing an Irrational Number

4627 [1955, 127]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

In Chrystal, *Textbook of Algebra*, vol. 2, p. 225, there is the following theorem on the representation of an irrational number:

The number represented by the series

$$\sum_{n=1}^{\infty} \frac{p_n}{r_1 r_2 \cdots r_n}$$

is irrational provided that

(1) r_n and p_n are integers such that $0 < p_n < r_n$,

(2) $r_{n+1} > r_n > 1$,

(3) the sequence $\{r_1 r_2 \cdots r_n\}$ includes all powers of the primes.

(A) Construct a counter-example, *i.e.*, a number of the stated form which is rational even though conditions (1), (2) and (3) are satisfied.

(B) Complete the list of conditions (1), (2) and (3) so that the theorem is indeed valid.

Solution by Ivan Niven, University of Oregon. (A) The following number is of the required form and is rational:

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = 1.$$

(B) The theorem is valid if the following condition is also satisfied:

(4) $p_n < r_n - 1$ for infinitely many values of n.

For if the series converged to a rational number, a/b, then by condition (3) we could choose an integer m so that b is a divisor of $r_1 r_2 \cdots r_m$. When we multiply the equation

$$\frac{a}{b} = \sum_{n=1}^{\infty} \frac{p_n}{r_1 r_2 \cdots r_n}$$

by $r_1 r_2 \cdots r_m$, and rearrange terms to get

$$a \frac{r_1 r_2 \cdots r_m}{b} - \sum_{n=1}^{m} \frac{p_n r_1 r_2 \cdots r_m}{r_1 r_2 \cdots r_n} = \sum_{i=1}^{\infty} \frac{p_{m+i}}{r_{m+1} r_{m+2} \cdots r_{m+i}},$$

we note that the left side is an integer. We get a contradiction by establishing that the series on the right converges to a value between 0 and 1. Indeed, the series on the right is positive because of condition (1), and by condition (4) the series is less than

$$\sum_{i=1}^{\infty} \frac{r_{m+i} - 1}{r_{m+1} r_{m+2} \cdots r_{m+i}},$$

but this series converges to the value 1 because the sum of the first k terms has the value

$$1 - \frac{1}{r_{m+1} r_{m+2} \cdots r_{m+k}}.$$

Thus the proof is complete and we make the following comment. Condition (2) may be replaced by the weaker inequality $r_n > 1$, and the theorem remains valid by the same proof.

Also solved by Robert Breusch, Leonard Carlitz, S. W. Golomb, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Beginning Algebra for College Students. By Lloyd L. Lowenstein, New York, John Wiley and Sons, Inc., 1953. xiii+279 pages.

This book is designed for students who have no previous preparation in algebra but who are mature enough to profit by a postulational approach to the subject. Since this approach is the distinctive feature of the book, most of the review will be given over to it.

The seven postulates on which the rest of the book is based refer to the set of natural numbers: this set is closed under addition and multiplication, which are also associative and commutative, and the distributive law holds. Each postulate is introduced by examples from arithmetic and some are followed by generous discussions which include examples of sets or operations for which the stated property does not hold. Repeated application of the postulates leads to generalizations in the form of rules, such as the laws of exponents and certain rules for factoring.

The algebra of the natural numbers is developed somewhat (parentheses, exponents, polynomials, simple equations) before introducing any new operations. Extension of the number system is motivated by inverting the operations of addition and multiplication and devising an enlarged system in which the seven postulates continue to hold. The natural numbers are introduced on page 4, zero on page 43, directed numbers on page 52, and fractions on page 88. The rational number system is defined on page 125, and the property of closure with respect to the four elementary operations is shown.

In the second half of the book, pages 137 to 248, some of the topics begun earlier receive enlarged treatment, *e.g.*, polynomials, factoring, equations, and exponents, while new topics include functions and graphs, the real number system (with a discussion of radicals), and the complex number system.

The placing of the irrational and complex numbers at the end is good in a book which is not just a review of high school algebra. Another good feature is the six-page appendix containing discussions of questions asked in the text, together with references for further study. In general the exposition throughout is clear, and the problems are well chosen.

As for the distinctive feature of the book, the postulates are carefully stated, and they are actually used to derive rules of procedure. The definitions are also carefully stated, and many are preceded by a generous preliminary discussion. (An exception is the word "term," which is not defined at all.) However, the same care is not used in extending to the gradually enlarged number system the properties stated in the postulates. It is not made clear that these properties require proof based on the original postulates and such definitions as have been made or

may be made. Thus on page 44, after the definition of zero, we find the declaration that all the rules previously built up are going to apply to the new number. Then the commutative, associative, and distributive laws are stated with zero as one element.

On the whole the book accomplishes well its main purpose, which is to present algebra as an orderly development from a few basic assumptions. After studying it, a student should be quite ready for college algebra.

W. G. CLARK
Mt. Union College

Algebra for College Students. By W. M. Whyburn and P. H. Daus. New York, Prentice Hall, Inc. 1955. 254 pp. plus tables and index.

As the authors remark in the preface, the "difficulties experienced by college mathematics students result not only from weaknesses in the manipulative skills of algebra, but also from deficiencies in geometry and arithmetic and the lack of correct habits of thinking and reasoning." These incorrect habits are often inherited by the students from the teachers in high school mathematics. The reviewer recalls a remark of a high school teacher in one of his classes, "I do not see what logic has to do with mathematics."

Texts like the Chicago syllabus or *Principles of Mathematics* by Allendoerfer and Oakley (McGraw-Hill) are specially aimed at this deficiency. The present text is a compromise between such texts and the usual college algebra texts which are often "an unenthusiastic rehash" of high school mathematics.

The book has eight chapters and an introductory review of elementary arithmetic. Chapters I–II are devoted to direct and inverse operations on numbers. The fundamental properties are given by axioms and the usual "rules" such as those concerning removal of parentheses, multiplying two negative numbers *etc.* are derived from these axioms. Real numbers are introduced by the *axiom of linear measure.* Article 16 on "Change of Base" brings out clearly the role of number 10 in writing natural numbers in arithmetic. The chapters conclude with factor theorem and Euclidean algorithm for finding the H.C. F.

Chapter III entitled "Fractions" deals with negative exponents, terminating and non-terminating decimals, approximation, variation illustrated by simple interest and simple discount, and elements of the trigonometry of an acute angle. Chapter IV treats of linear equations, graphically as well as arithmetically. The latter method introduces us to determinants and matrices. Chapters V–VI treat quadratic equations. Complex numbers are introduced and their sum and product properly defined. Simple conics come in as graphs of simple quadratics in two variables. The sine law and cosine law complete the trigonometry of a triangle. Chapter VII treats of exponents and logarithms. Irrational exponents are introduced by two explicit assumptions. The last chapter deals with "Sequences of Numbers" and "Mathematical Induction." The latter is stated in an axiom underlying a procedure used to prove or disprove "conjectured" results of a certain type.

The text is very carefully written and does not contain anything that a student will have to forget when he learns more mathematics. There are no boldtype framed "Formulas" nor any annoying schemes for working problems: Step I, Step II, *etc.* such as we find in many texts.

In conclusion, a few minor defects may be noted here: On page 9, a variable is defined as "a quantity that may change its value during the discussion." On page 113, the x and y intercepts are defined as "points"; and almost immediately we read "If the intercepts are small, an additional point is desirable."

V. D. GOKHALE
Atlanta University

Parmi les belles figures de la Géométrie dans l'espace (Géométrie du tétraèdre). By Victor Thébault. Paris, Vuibert, 1955. 11+287 pages. 2000 french francs.

Victor Thébault is known on both sides of the Atlantic as a geometer interested in the triangle and the tetrahedron. For over thirty years he has been seeking analogies between the geometry of the triangle and the geometry of the tetrahedron. His researches have been published in many journals including this MONTHLY and have been issued in the form of memoirs, articles, notes and problems. He has now gathered into one volume the principal results concerning the tetrahedron, its associated points, lines, planes and spheres.

After an introduction explaining the terminology and notation, which includes a number of distance relations, the material is divided into four chapters: Chapter I: basic configurations; Chapter II: spheres associated with a tetrahedron and a skew polygon; Chapter III: additions to the recent geometry of the tetrahedron; Chapter IV: problems proposed concerning the tetrahedron.

The topics in the first chapter include the Monge point, Euler lines, the 12-point sphere, the orthocentroidal and the Miquel spheres, the Steiner ellipsoids and some hyperboloids, associated planes and tetrahedra and some special tetrahedra. In the second chapter is the Longchamps sphere and a skew curve associated with a tetrahedron; it ends with a discussion of spheres associated with a skew polygon whose vertices lie on a sphere. The third chapter treats of the first and second Lemoine points, the spheres of Tucker, Adams, Lucas and Hagge, and the orthopole of a line. In the final chapter are 81 original problems proposed since 1913 whose solutions have led often to interesting results.

This volume should be of interest to those who are acquainted with the geometry of the triangle and the tetrahedron, and to all those who enjoy witnessing a trained mind at work in bringing into view the relations existing among lines, planes, spheres and tetrahedra associated with the configuration formed by four planes.

J. R. MUSSELMAN
Western Reserve University

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

REPRINTS OF VOLUMES OF THE MONTHLY

By arrangement with the Mathematical Association of America, Johnson Reprint Corporation of New York is now reprinting the first twenty volumes of the *American Mathematical Monthly*. These early volumes have not been available in bound form for some time, and the reprint will therefore afford libraries an opportunity to complete their sets. Cloth bound sets are priced at $275.00, paper bound sets are available at $245.00, and individual paper bound copies of Volumes 1–9 are priced at $15.00, and of Volumes 10–20, $10.00. It is anticipated that the reprinted sets will be available by May, 1956.

Orders and inquiries should be addressed to: Johnson Reprint Corporation, 125 East 23 Street, New York 10, New York.

MATHEMATICS INSTITUTE FOR TEACHERS

The second annual Mathematics Institute of the University of Oklahoma will be held June 4–15, 1956, in the air-conditioned Union Building on the main campus.

Opportunity will be given to study methods of teaching the various subjects in the high school mathematics curriculum, to get a deeper insight into the subject matter of high school mathematics with its new-type applications and to acquire source material for use with talented students. A study will also be made of some relatively simple developments in modern mathematics.

One hour of graduate credit will be given for either week of the Institute—two hours for both weeks. Send inquiries to the Institute Chairman, J. O. Hassler, Department of Mathematics, University of Oklahoma, Norman, Oklahoma.

COMMISSION ON MATHEMATICS

A Commission on Mathematics that will investigate the need for revision of the secondary school mathematics curriculum has been appointed by the College Entrance Examination Board.

New frontiers in mathematics are creating almost unlimited opportunities for growth in mathematical knowledge and its applications to physical science and engineering, to the social and biological sciences, and to business and industry. The Commission has been established by the College Board in recognition of a growing divergence between these developments and the type of mathematics taught in secondary school.

After stressing modern mathematical concepts only in graduate courses for many years, many college mathematics departments are readjusting freshman

courses to include these ideas. No equivalent trend has yet developed widely on the high school level.

The Commission on Mathematics, a group of 13 high school and college mathematics teachers, proposes to investigate the need and possibilities for revision of the high school syllabus. One or more conferences of high school and college teachers and administrators will be summoned to advise and assist the Commission. Following such meetings, and in the light of the experience and deliberations of its own members, the Commission will recommend necessary and desirable action to the College Board.

Curricular reform has often failed to result from the work of previous conferences and commissions in mathematical education because their recommendations have been only partially implemented. In view of this, the College Board is prepared to take further action. If there is general agreement as to the need and form of curricular revision, the College Board will, if necessary, foster the development of teaching materials and workshops to prepare teachers to utilize these materials.

With this charge at hand, the Commission met for the first time at Ann Arbor, Michigan, in August, 1955, and reached the following tentative conclusions:

(1) High school mathematics is in need of revision which would include the addition of new materials and the elimination of some and reorganization of other topics now being taught;

(2) In considering a topic for deletion or inclusion the Commission will bear in mind the relevance of the topic to modern mathematics, social and natural science, and engineering;

(3) Liaison with and the cooperation of all groups interested in school and mathematics curricula is essential; and

(4) The development of a single definitive syllabus is undesirable, if not impossible. The Commission therefore proposes to formulate a broad list of topics and to suggest where and how they might be introduced.

Members of the Commission include: Professor A. W. Tucker, Princeton University (Chairman); Professor C. B. Allendoerfer, University of Washington; Mr. E. C. Douglas, The Taft School, Watertown, Connecticut; Professor H. F. Fehr, Teachers College, Columbia University; Miss Martha Hildebrandt, Proviso Township High School, Maywood, Illinois; Dean A. E. Meder, Jr., Rutgers University; Professor Frederick Mosteller, Harvard University; Professor E. P. Northrop, University of Chicago; Dr. E. R. Ranucci, Weequahic High School, Newark, New Jersey; Mr. Robert Rourke, The Kent School, Kent, Connecticut; Professor G. B. Thomas, Jr., Massachusetts Institute of Technology; Professor Henry Van Engen, Iowa State Teachers College; Professor S. S. Wilks, Princeton University.

The Commission welcomes the assistance and suggestions of organizations and individuals. Those interested in corresponding with the Commission should write to Mr. Robert Kalin, Executive Assistant, Commission on Mathematics,

P.O. Box 592, Princeton, New Jersey.

NEW TELEVISION SERIES

Professor Francis A. C. Sevier of the College of South Jersey is presenting a television series on WFIL-TV. The series began on January 30 and will run through May 14. It is titled "Mathematics in Modern Life." Outlines of Professor Sevier's syllabus can be obtained by writing to him at 406 Penn Street, Camden 2, New Jersey.

WAYNE UNIVERSITY COMPUTATION LABORATORY PROGRAM

The Computation Laboratory of Wayne University is again offering an intensive training program of three weekly courses.

First Course: July 23–28. Automatic computers—their application and evaluation.

Second Course: July 30–August 4. Electronic data processing in business and government.

Third Course: August 6–11. Applications of computors to engineering, science and industry.

The final program with full information on lectures, staff, and other pertinent information will be distributed well in advance of the summer courses. For further information, write to A. W. Jacobson, Director, Computation Laboratory, Detroit 1, Michigan.

DU PONT FELLOWSHIPS AT THE UNIVERSITY OF CHICAGO

The University of Chicago announces six graduate fellowships of $1,920 for students who wish to prepare for teaching chemistry, mathematics or physics in secondary schools. E. I. du Pont de Nemours and Company has granted the University funds for these fellowships to encourage able college graduates of both sexes to enter high-school mathematics and science teaching.

The du Pont fellows will register for science and/or mathematics courses, for professional courses in the teaching of the subject, and for apprentice teaching courses. Their program of study is designed to enable them to meet the requirements for the secondary-school teaching certificate issued by most states and to advance them toward a Master of Science degree.

Applications for fellowships for 1956–1957 must be made before May 15, 1956. Further information about the du Pont Fellowships may be obtained from the Center for Teacher Education, University of Chicago, Chicago 37, Illinois.

INSTITUTE OF MATHEMATICAL SCIENCES TEMPORARY MEMBERSHIPS

The Institute of Mathematical Sciences of New York University, Professor R. Courant, Director, offers *temporary memberships* to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is active. These fields include functional analysis, ordinary and partial differential equations, mathematical physics, fluid dy-

namics, electromagnetic theory, numerical analysis and digital computing, and various specialized branches, such as group theory, topological methods of analysis, hydromagnetics, reactor theory.

The temporary members will have complete freedom to select their own activities. They may participate in the advanced graduate courses, research seminars, and research projects of the Institute and will have the opportunity of using computational facilities.

The temporary members will receive a stipend commensurate with their status. Membership will be awarded for one year, but may be renewed. Special arrangements can be made for applicants who expect to be on leave of absence from their institutions.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, New York.

INDUSTRIAL MATHEMATICS SOCIETY

Mathematics in an Industrial Economy is a new booklet issued by the Industrial Mathematics Society. It will be sent free to all who request it by writing to the Industrial Mathematics Society, 100 Farnsworth, Detroit 1, Michigan.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1956.

Boston University. July 9 to August 18: Professor Scheid, calculus of variations.

The Catholic University of America. June 27 to August 10: Mr. Tomiuk, advanced calculus I; Professor Moller, higher algebra I; Professor Wiegmann, introduction to matrix theory; Professor Rice, advanced calculus II; Professor Finan, higher algebra II; Professor Ramler, college geometry, synthetic projective geometry, differential equations.

Columbia University, Teachers College. July 9 to August 17: Professor Fehr, teaching arithmetic in elementary school, current problems in teaching secondary school mathematics; Professors Fehr and Rosskopf, research in teaching mathematics, departmental seminar in teaching mathematics; Professor Rosskopf, teaching of algebra in secondary school, history of mathematics; Dr. Sobel, field work in mathematics, teaching non-academic mathematics in the high school; Mr. Rourke, survey of higher mathematics for teachers, professionalized subject matter in advanced secondary school mathematics; Professors Fehr, Rosskopf, Sobel and Rourke, conferences and discussions on current questions in the teaching of mathematics.

DePaul University. June to August: Professor DeCicco, classical Lie theory of continuous groups, fundamental concepts of algebra; Professor Caton, introduction to the calculus of variations. June to August (evenings): Professor Caton, engineering mathematics.

Duke University. June 12 to June 17: Professor Carlitz, probability; Professor Roberts, plane geometry and trigonometry from the advanced standpoint; Professor Thomas, mathematical analysis for teachers; Professors Carlitz, Roberts and Thomas, thesis seminar.

Indiana University. June 14 to August 10: Professor Wolfe, theory and application of statistics, projective geometry; visiting Professor Snapper (Miami University), topics for mathematics teachers, higher algebra I; Professor MacKenzie, differential equations, theory of functions of a complex variable I. For the first time there will be available for qualified secondary school teachers three David A. Rothrock Summer Scholarships amounting to $200 plus basic fees.

North Carolina State College. Southern Regional Graduate Summer Session in Statistics. June 11 to July 20: Professor Hader, statistical methods I; Professor Cox, statistical methods II; Dr. Grandage, theory I; Professor Duncan (University of Florida), theory II; Professor Finkner, sample survey designs; Staff, special problems; Professor Anderson, econometric methods; Professor Monroe, advanced topics in statistical analysis II; Professor Harrell, introduction to linear programming; Professor Hildreth (Michigan State University), linear equations; Professor Heady (Iowa State College), estimation and analysis of production functions; Professor Levine, advanced calculus; Professor Smith (University of North Carolina), stochastic processes and their applications.

The Institute in Quantitative Research Methods in Agricultural Economics, sponsored by the Social Science Research Council, will meet at the College in conjunction with the Graduate Summer Session in Statistics.

Northwestern University. June 23 to August 18: Engineering mathematics I, theory of statistics, introduction to the theory of numbers, the history of mathematics II, foundations of calculus for teachers, topics in modern mathematics for teachers, nonlinear problems of the physical sciences, introduction to the theory of groups.

Syracuse University. July 2 to August 10: Professor Cole, analysis and applications I; Professor Gilbert, introduction to probability; Professor Kibbey, intermediate course in algebra II; Professor Exner, heuristic methods in elementary geometry. August 13 to September 14: Professor Hemmingsen, analysis and applications II (vector analysis); Professor Welch, higher mathematics for engineers and scientists I.

University of Buffalo. July 2 to August 10: Professor Montague, rings and ideals; Professor Gehman, theory of games.

University of California. June 18 to July 28 and July 30 to September 8. Graduate Courses in Statistics. The faculty will include Professor Cox (University of North Carolina), Professor Bates (Mount Holyoke College), Professor Blackwell and Mr. Ferguson of the Department of Statistics. The program includes two of the usual undergraduate courses in each session, adapted primarily to meet the need of students transferring from other centers who would like to undertake advanced study at the University of California during the regular

academic year. Also a graduate seminar will be conducted by Professor Blackwell. This seminar will allow for individual consultation for students working toward higher degrees.

June 18 to July 28. Graduate Course in Mathematics: Professor Jacobson (Yale University), ring theory.

University of Chicago. June 25 to August 31: A program in analysis including special courses and seminars will be offered. Visitors in residence will include A. P. Calderon, H. Helson, J. E. Littlewood, W. Rudin, R. Salem, and A. C. Schaeffer. Further information can be obtained from E. H. Spanier, Department of Mathematics, University of Chicago, Chicago 37, Illinois.

University of Colorado. June 18 to August 25: Professor Whyburn, theory of functions; Professor Hultquist, vector analysis; Dr. Crow, mathematical statistics; Professor Britton, complex variables, boundary value problems.

University of Delaware. June 18 to July 27: Professor Webber, fundamental concepts of mathematics I (designed especially for teachers).

University of Florida. June 15 to August 13: Professor Blake, introduction to mathematical thought; Professor South, mathematical statistics; Professors Lang, Blake and Smith, higher mathematics for engineers and physicists; Professor Smith, tensor analysis; Professor Phipps, foundation of geometry; Professor Cowan, Fourier series; Professor Gager, history of elementary mathematics; Staff, individual work.

University of Illinois. June 18 to August 11: Professor Suzuki, linear algebra; Professor Landin, number, length, area; Professor Bateman, measure and integration.

University of Maryland. June 25 to August 3: Professor Jackson, foundations of number theory (primarily for teachers); Professor Hall, elementary topology; Dr. Rosen, introduction to complex variable theory.

University of Michigan. June 25 to August 17: Professor Alexander, theory of probability, elementary mathematical statistics; Professor Brauer, higher algebra, seminar in algebra; Professor Carver, mathematical statistics, finite differences; Professor Coburn, operational mathematics; Professor Craig, significance tests; Professor Dushnik, advanced calculus; Dr. Gehring, advanced mathematics for engineers; Professor Gentry, synthetic projective geometry, topics in geometry; Professor Hay, vector analysis, theory of plates and shells; Professor Herzog, functions of a complex variable, partial differential equations; Professor Hildebrandt, calculus of variations; Professor Jones, teaching of algebra, history of geometry; Professor LeVeque, matrix theory; Professor Ullman, Fourier series, application of functions of a complex variable; Professor Young, foundations of mathematics, general spaces.

A Summer Institute for Teachers of Collegiate Mathematics will be held at the University during the summer session. (See this MONTHLY, March, 1956.)

University of Nebraska. June 13 to August 3: Professor Jackson, differential equations; Professor Magnus, vector analysis; Professor Leavitt, topics in algebra; Staff, reading course, thesis.

University of North Carolina. June 7 to July 14: Professor Brauer, some recent results in number theory; Professor Cameron, fundamental concepts; Professor Garner, history of mathematics; Professor Jones, non-euclidean geometry; Professor Linker, differential equations; Professor Mackie, theory of equations; Professor Winsor, introduction to higher geometry. July 16 to August 22: Professor Hill, elementary mathematical statistics; Professor Hoyle, elementary algebra from an advanced viewpoint; Professor Lasley, analytic geometry from a higher standpoint; Professor MacNerney, summability; Professor Wall, introduction to the theory of determinants and matrices.

University of Oklahoma. June 7 to August 5: Professor Bernhart, college geometry; Professor Pan, elementary differential equations; Professor LaFon, vector analysis; Professor Springer, ordinary and partial differential equations; Professor Brixey, theory of numbers; Professor Goffman, theory of probability.

University of Pennsylvania. June 25 to August 4: Professor Herstein, matrix theory and numerical applications; Professor Epstein, methods of analysis; Professor Fine, probability and statistics; Professor Gottschalk, concepts of mathematics (this course carries credit towards a master's degree in education with specialty in mathematics).

University of South Carolina. Professor Lytle, theory of equations; Professor Novak, college geometry; Professor Hedberg, theory of algebraic numbers; Professor Strebe, theory of functions of a real variable; Professor Williams, theory of functions of a complex variable.

University of Texas. June 5 to July 17: Professor Craig, vector and tensor analysis, applications of tensor analysis; Professor Moore, theory of sets; Professor Guy, potential theory (two terms). July 18 to August 31: Professor Cooper, functions of a complex variable; Professor Lubben, modern algebra; Professor Wall, infinite processes, analytic functions; Professor Guy, potential theory.

University of Washington. Professor Avann, linear algebra; Professor Cramlet, differential equations; Professor Ballantine, vector analysis; Professor McFarlan, topics in applied analysis; Professor Leipnik, advanced analytic geometry; Professor Haller, differential geometry; Professor Livingston, convolution transforms.

University of Wisconsin. June 25 to August 17: Professor Evans, vector analysis; Professor Hammer, theory and operation of computing machines; Professor Immel, introduction to mathematical statistics, survey of the foundations of arithmetic; Professor Jones (University of North Carolina), topics in geometry, topics in topology; Professor Kaplansky (University of Chicago), determinants and matrices, advanced topics in algebra; Professor Marden, applied differential equations, potential theory; Professor Mayor, projective geometry.

University of Wyoming. June 11 to July 13: Professor Varineau, theory of equations; Professor Steen, ordinary differential equations; Professor Barr, college geometry; Professor S. R. Smith, numerical analysis. July 16 to August

17: Professor Schwid, partial differential equations, Fourier series and boundary value problems, theory of functions of a complex variable; Professor Neubauer, history of mathematics; Professor W. N. Smith, mathematical theory of probability.

West Virginia University. June 6 to July 17: Professor Cunningham, Fourier series and partial differential equations; Professor Posey, advanced calculus; Professor Stewart, history of mathematics. July 18 to August 24: Mr. Cochran, modern geometry; Professor Peters, advanced calculus; Professor Vest, operational methods in partial differential equations.

PERSONAL ITEMS

Assistant Professor H. L. Alder of the University of California, Davis, has been promoted to an associate professorship.

Mr. A. I. Benson, formerly a staff member of the Los Alamos Scientific Laboratory, New Mexico, is manager of Lynn Digital Computations, General Electric Company, West Lynn, Massachusetts.

Mrs. Audrey J. Benson, previously a research assistant at the Los Alamos Scientific Laboratory, New Mexico, is now an assistant mathematician at the Rand Corporation, Lincoln Laboratory, Lexington, Massachusetts.

Mr. Jonas Beraru, formerly a mathematician with the Reeves Instrument Corporation, New York, New York, has a position as a member of the technical staff of Hughes Aircraft Company, Culver City, California.

Miss Ida Brisky has a position as a systems engineer with the Datamatic Corporation, Newton Highland, Massachusetts.

Mr. C. L. Davis, previously an instructor at the General Motors Institute, has accepted the position of Mathematician in the Mathematics and Computing Group of Curtiss-Wright Research Division, Clifton, New Jersey.

Mr. M. J. Di Carlo-Cottone, formerly a mathematician for the United States Air Force, Washington, D. C., is a member of the technical staff of Bell Telephone Laboratories, Whippany, New Jersey.

Dr. D. O. Ellis, associate mathematician for the Rand Corporation, Santa Monica, California, has accepted a position as Senior Research Scientist with the National Cash Register Company, Electronics Division, Hawthorne, California.

Mr. J. H. Fountain is employed as a research engineer by Consolidated-Vultee Aircraft Corporation, San Diego, California.

Mr. M. P. Friedman of Brooklyn College is now with the Research Division of the College of Engineering, New York University.

Mr. F. W. Gibson, formerly a mathematical analyst with Lockheed Aircraft Corporation, Burbank, California, is now a research engineer for the Radioplane Company, Van Nuys, California.

Dr. G. R. Glabe has been appointed to an assistant professorship at Sacramento State College.

Visiting Professor G. H. Handelman of Rensselaer Polytechnic Institute is in charge of a new program of research sponsored by the Air Force Office of Scientific Research, Air Research and Development Command, Baltimore.

Mr. M. A. Kirchberg, previously a teacher at Hopkins Township School, Michigan, has a position as a computer operator for Allis-Chalmers Manufacturing Company, Milwaukee, Wisconsin.

Miss Lola F. Kiser of the University of Georgia has been appointed to an assistant professorship at Birmingham-Southern College.

Professor Emeritus E. E. Moots of Cornell College has been appointed to a visiting professorship at Whittier College for the second semester of 1955–1956.

Assistant Professor Eugene Park of Clemson College has been promoted to an associate professorship.

Mr. Conrad Rennemann, Jr., formerly an aeronautical research scientist with the National Advisory Committee for Aeronautics, Langley Field, Virginia, has a position as a senior aerodynamicist with the Republic Aviation Corporation, Farmingdale, L. I., New York.

Mr. L. A. Rife of Mississippi Southern College has been promoted to an assistant professorship.

Mr. J. J. Schoderbek, previously a research assistant at the University of Michigan, Willow Run Research Center, Ypsilanti, Michigan, has a position as a research engineer for Lockheed Aircraft Corporation, Van Nuys, California.

Mr. C. E. Seabold, recently a mathematician at Wright-Patterson Air Force Base, Ohio, is employed as a senior technical programmer with the Standard Oil Company of Ohio.

Mr. H. F. Simmons, formerly a graduate assistant at Iowa State College, has been appointed to an assistant professorship at the University of Wichita.

Mrs. Lois B. Smith is teaching in the San Diego City Schools, California.

Mr. A. G. Wilford, previously a meteorologist for the Department of Transport, Toronto, Canada, is employed as a programmer by Computing Devices of Canada, Ottawa, Canada.

Dr. John Wishart, director of the Statistical Laboratory at St. Andrew's Hill, Cambridge, England, will be a visiting professor at the University of Wisconsin for the first semester of the academic year 1956–1957.

Associate Professor Herbert Wolf of Tri-State College has a position as a research mathematician at the Avco Manufacturing Company, Stratford, Connecticut.

Mr. T. I. Gilroy, fellow at the University of Wisconsin, died in July, 1955.

Miss Clara Hancock, retired instructor of Junior College, Virginia, Minnesota, died on March 23, 1955. She was a member of the Association for thirty-five years.

Mr. D. H. Leavens of Colorado Springs, Colorado, died on December 3, 1955. He was a charter member of the Association.

Professor Emeritus H. P. Manning of Brown University died on January 11, 1956. He was a charter member of the Association.

Professor Emeritus F. R. Morris of Fresno State College died on January 6, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1955

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1955. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

As was the case at the end of the preceding year, the total funds of the Association at the end of the year 1955 include balances held for the Committee on Visiting Lecturers and the Committee on the Undergraduate Program. These funds will be expended as the plans of these Committees require.

The Current Fund ended the year with a surplus of $2,065. Anticipated expenditures during 1956 indicate that a rather substantial deficit will be incurred in 1956. The Finance Committee is considering whether an increase in dues to $5.00 per year may become necessary. The Secretary-Treasurer will be glad to have the comments of members of the Association on this matter.

I. Total Funds of the Association on January 1, 1955

M & T Trust Co., Buffalo	$ 11,467.06	Current Fund	$ 495.62
Securities	104,831.00	Carus Fund	19,626.04
		Chace Fund	14,797.47
		Houck Fund	10,830.03
		Chauvenet Fund	1,363.14
		Dunkel Fund	17,865.41
		General Fund	40,348.91
			105,326.62
		Visiting Lecturers Fund	10,462.28
		Fund for Com. on Und. Program	509.16
	$116,298.06		$116,298.06

II. Current Fund

Balance, January 1, 1955	$ 495.62	Monthly	
Dues	23,004.76	Publication	$19,794.83
Initiation fees	908.00	Reprints	510.87
Subscriptions	4,688.13	Editor's Office	1,986.34
Sale of back numbers	260.23	Secretary-Treasurer's Office	
Advertisements	5,993.50	Clerical help	8,805.91
Sale of exchange periodicals	125.50	Postage and printing	2,037.70
Interest on General Fund	1,493.97	Office expenses	405.90
Income from Hardy Fund	120.00	Bank and Auditing fees	238.55
Transfer from other Funds	2,323.42	Board of Governors	1,737.83
Foreign exchange and bank charges	3.70	Meetings	684.00
		Committees	160.00
		Subventions	493.53
		Balance, December 31, 1955	$ 2,561.37

	III. Carus Fund	IV. Chace Fund	V. Houck Fund
Balance, January 1, 1955	$19,626.04	$14,797.47	$10,830.03
Sales of publications	1,456.41	518.98	7.00
Interest	729.52	547.14	399.69
Increase in value of securities	566.68	425.01	310.47
Less: charges and bank fees	237.71	146.49	51.14
Printing	—	2,302.86	—
Balance, December 31, 1955	$22,140.94	$13,839.25	$11,496.05

	VI. Chauvenet Fund	VII. Dunkel Fund	VIII. General Fund
Balance, January 1, 1955	$ 1,363.14	$17,865.41	$40,348.91
Interest	50.45	659.67	—
Increase in value of securities	39.19	512.42	1,160.49
Less: charges and bank fees	6.37	83.26	—
Balance, December 31, 1955	$ 1,446.41	$18,954.24	$41,509.40

	IX. Visiting Lecturers Fund	X. Fund for Committee on Undergraduate Program
Balance, January 1, 1955	$10,462.28	$ 509.16
National Science Foundation grant	20,500.00	—
Contributions toward travel expenses	1,568.33	—
Ford Foundation grant	—	25,000.00
Less: salaries, services, travel expense	7,535.54	2,313.68
Committee expenses	1,082.54	256.15
Administrative expenses	750.00	1,375.00
Balance, December 31, 1955	$23,162.53	$21,564.33

XI. Total Funds of the Association on December 31, 1955

Current Fund	$ 2,561.37	M & T Trust Co., Buffalo	$ 12,842.52
Carus Fund	22,140.94	Buffalo Savings Bank	20,000.00
Chace Fund	13,839.25	Securities	123,832.00
Houck Fund	11,496.05		
Chauvenet Fund	1,446.41		
Dunkel Fund	18,954.24		
General Fund	41,509.40		
	111,947.66		
Visiting Lecturers Fund	23,162.53		
Fund for Committee on Und. Program	21,564.33		
	$156,674.52		$156,674.52

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 150 persons have been elected to membership by the Board of Governors on applications duly certified.

Angela K. Aivaliotis, Student, Kent State University.

O. H. Alisbah, Ph.D. (Berlin) Professor, University of Ankara, Turkey; Visiting Professor, University of Pennsylvania and Rutgers University.

D. R. Anderson, Student, Bradley University.

Richard E. Anderson, M.S. (Michigan) Grad. Student, University of Michigan.

J. S. Ayars, Jr., M.A. (Illinois) Grad. Student, Stanford University.

F. T. Baker, Student, Yale University.

G. E. Baxter, Ph.D. (Minnesota) Instr., University of Minnesota.

Hester C. Boyd, Ed.M. (Mississippi) Grad. Asst., University of Mississippi.

W. H. Bradford, Ph.D. (Texas) Dean, Division of Liberal Arts, McNeese State College.

L. R. Bragg, Ph.D. (Wisconsin) Instr., Duke University.

Christine V. Brannan, A.M. (Missouri), M.S. (Kansas S.T.C.) Instr., University of Houston.

H. W. Brockman, M.A. (Ohio S.U.) Asst. Professor, Capital University.

Mrs. Helen L. Brooks, M.A. (Toledo) Instr., University of Toledo.

Felix Browder, Ph.D. (Princeton) Asst. Professor, Brandeis University.

A. W. Brunson, M.A. (Illinois) Asst. Professor, Fenn College.

B. P. Bussel, M.A. (Columbia) Instr., University of Massachusetts.

J. W. Bustamante, B.S. in E.E. (Okla. A & M. C.) Grad. Asst., Oklahoma Agricultural and Mechanical College.

J. L. Cable, Student, University of Buffalo.

Rev. R. I. Canavan, S.J., A.B. (Woodstock) Grad. Student, New York University.

Virginia Castelluccio, Student, Montclair State Teachers College.

R. L. Causey, M.S. (Kentucky) Applied Math., U.S. Naval Air Missile Test Center, Point Mugu, Calif.

W. A. Cava, Student, City College of New York.

Dorothy Chmiel, Student, Montclair State Teachers College.

R. E. Collins, Ph.D. (A.&M.C. of Texas) Research Engr., Humble Oil & Refining Co., Houston, Texas.

T. K. Cook, B.S. (Cincinnati) Grad. Asst., University of Cincinnati.

R. J. Corn, Student, Georgia Institute of Technology.

MRS. ESTHER B. COULTER, B.S. in Ed. (Bowling Green S.U.) Instr., University of Arizona.

H. L. CROWSON, M.S. (Florida) Grad. Asst., University of Florida.

T. W. DANIEL, M.A. (Arkansas) Instr., Mississippi State College.

G. L. DAVEY, M.S. (Stanford) Acting Instr., Stanford University.

MRS. ELEANOR G. DAWLEY, M.S. (Howard) Instr., Hampton Institute.

G. W. DAY, Student, San Jose State College.

R. S. DEZUR, M.A. (Montana S.U.) Acting Instr., State College of Washington.

R. J. DRISCOLL, M.A. (Loyola, Ill.) Math. Asst., White Sands Proving Ground, N. Mex.

L. E. DUBINS, Ph.D. (Chicago) Asst. Professor, Carnegie Institute of Technology.

C. D. DUNN, Student, Texas Christian University.

W. H. EDWARDS, M.A. (Texas) Asst. Professor, University of Houston.

E. G. EIGEL, JR., B.S. (M.I.T.) Grad. Fellow, St. Louis University.

W. K. ERGEN, Ph.D. (Vienna) Principal Physicist, Oak Ridge National Lab., Oak Ridge, Tenn.

W. A. FERGUSON, Ph.D. (Illinois) Asso. Professor, University of Illinois.

H. K. FLESCH, Ph.D. (Illinois) Instr., Cornell University.

CAPT. A. R. GALLAHER, U.S.N., B.S. (U.S. Naval Acad.) Head, Department of Mathematics, U. S. Naval Academy.

RHEBA E. GALLOWAY, B.S. (Virginia S.C.) Grad. Student, Virginia State College.

B. S. GARBOW, M.S. (Chicago) Asst. Math., Argonne National Lab., Lemont, Ill.

A. M. GARSIA, Res. Asst., Stanford University.

G. R. GIBSON, Student, University of Buffalo.

J. M. GIL, Graduate in Math. (Coimbra, Portugal) Teacher, Liceu de Santarém, Portugal.

L. O. GIROUARD, JR., B.S. in E.E. (California) Statistician, Pacific Telephone and Telegraph Co., Los Angeles, Calif.

N. W. GODSEY, Teaching Asst., Oklahoma Agricultural and Mechanical College.

JACK GOLDBERG, M.A. (N.Y.U.) Math. Teacher, Hackensack Board of Education, N. J.

E. F. GOLDFARB, M.A. (C.C.N.Y.) White Sands Proving Ground, N. Mex.

MRS. EMILY W. J. GOOD, M.S. (North Carolina) Temporary Instr., University of Alabama.

J. E. GROVES, JR., B.S. (Arkansas S.C.) Grad. Fellow, St. Louis University.

J. F. HANNAN, Ph.D. (North Carolina) Asst. Professor of Statistics, Michigan State University.

G. H. HART, S.M. (M.I.T.) Newcastle, Maine.

D. F. HAYES, Student, St. Mary's University.

R. E. HEATON, Ph.D. (Indiana) Asst. Professor, University of Richmond.

PETER HENRICI, Ph.D. (Swiss Federal I.T.) Asst. Professor, American University.

DAVID HERLACHER, M.S. (Iowa) Instr., Fenn College.

E. H. HIETBRINK, B.A. (Central C.) Grad. Student, University of Nebraska.

D. B. HOLDRIDGE, A.B. (Fresno S.C.) Teaching Asst., California Institute of Technology.

J. R. HOLDSWORTH, Student, San Jose State College.

D. F. HUNTINGTON, Student, University of Maine.

DORTHEA I. JACKSON, B.A. (Baker) Seismologist Asst., Shell Oil Co., Houston, Texas.

D. E. JONES, B.S. (Fort Hays Kansas S.C.) Grad. Asst., Kansas State College.

P. M. KAHN, Student, Stanford University.

L. A. KENNA, M.A. (Arizona S.C.) Instr., University of Arizona.

H. C. KERPELMAN, B.S. (Maryland) Grad. Student, Asst. Instr., University of Pennsylvania.

REV. PHILIP KERSHAW, O.S.B., B.S. in Ed. (Loyola, La.) Professor, St. Joseph Seminary, St. Benedict, La.

J. J. KINNEY, A.M. (Harvard) Instr., St. Lawrence University.

D. B. KIRK, M.A. (Pennsylvania) Res. Analyst, Mutual Benefit Life Ins. Co., Newark, N. J.

B. Y. KOO, M.A. (Maryland) Junior Instr., University of Maryland.

J. C. KOSANKE, Student, Baylor University.

FRANK KOZIN, M.S. (Illinois I.T.) Instr., Illinois Institute of Technology.

S. R. KRAFT, B.A. (George Washington) Grad. Student, University of Maryland.

MAURICE L'ABBÉ, Ph.D. (Princeton) Asso. Professor, University of Montreal.

H. L. LAWTON, M.S. (Florida S.U.) Math., U. S. Navy Mine Defense Lab., Panama City, Fla.

W. T. LAYBOURN, JR., Junior Engr., Philco Corp., Philadelphia, Pa.

IRWIN LEWIS, Student, Brooklyn College.

N. J. LEWIS, Student, Colorado College.

J. T. LIDDLE, JR., Student, Rutgers University.

J. P. LINE, M.S. (Michigan) Instr., University of Rochester.

H. G. LOOMIS, M.A. (Pennsylvania S.U.) Instr., Pennsylvania State University.

REV. H. J. LUGER, S.J., M.A. (Gonzaga) Instr., Gonzaga University.

M. R. LUTTRELL, B.S. (Florence S.T.C.) Equipment Engr., Western Electric Co., Chicago, Ill.

J. P. MAYBERRY, Ph.D. (Princeton) Professional Engr., Radio Corp. of America, Moorestown, N. J.

S. G. MAYFIELD, M.Ed. (Texas) Asst. Professor, Virginia Military Institute.

W. B. MILLER, M.A. (Lehigh) Asst. Professor, Moravian College.

A. M. MOOD, Ph.D. (Princeton) President, General Analysis Corp., Santa Monica, Calif.

Z. C. MOTTELER, Student, Stanford University.

R. W. NEWCOMB, B.S. in E.E. (Purdue) Grad. Student, Stanford University.

D. A. NORTON, Ph.D. (Wisconsin) Asst. Professor, University of California, Davis.

KATHLEEN M. O'DONNELL, M.A. (Northwestern) Instr., University of Kansas.

I. B. OLDHAM, Student, University of Oklahoma.

H. C. OWENS, M.A. (Alabama) Instr., University of Alabama.

D. B. PEASE, M.S. (Kansas S.T.C.) Electronic Scientist, U. S. Navy Mine Defense Lab., Panama City, Fla.

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THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

A meeting of members of the Association living in the New England Region was held at the University of New Hampshire, Durham, on November 26, 1955, for the purpose of organizing a section of the Association. Professors Howard Eves and Donald Kearns presided at the morning and afternoon sessions, respectively. There were 80 persons in attendance including 56 members of the Association.

At a business meeting with Professor A. A. Bennett as temporary chairman, a petition to the national organization for the establishment of a section was circulated and a set of by-laws was adopted. It was agreed that the name of the section should be the Northeastern Section, even though its geographical boundaries coincide with those of the New England Region. It was felt that this name was more appropriate in view of the fact that four Canadian provinces are to be included in the Section. The following officers were elected for a one-year term: Chairman, Professor Howard Eves, University of Maine; Vice-Chairman, Reverend S. J. Bezuszka, Boston College; Secretary-Treasurer, Professor R. E. Johnson, Smith College.

President E. L. Johnson of the University of New Hampshire gave a short welcoming address at the morning session. There was a tea at the President's home following the afternoon session.

The following papers were presented:

1. *A system of dimensional analysis*, by Reverend S. J. Bezuszka, Boston College.

2. *Concerning sines and cosines*, by Professor Ralph Beatley, Harvard University.

The speaker discussed the use of the law of sines to solve triangle problems normally solved by the law of cosines, and *vice versa*.

3. *The structure of a ring*, by Professor R. E. Johnson, Smith College.

There are six basic operations that can be performed on a given set of rings to yield new rings. It was shown how these operations could be used to express each semi-prime ring in terms of prime rings.

4. *Some applications of matrices*, by Professor R. A. Rosenbaum, Wesleyan University.

The incorporation into the undergraduate curriculum of diverse applications of matrices was recommended for a number of reasons, chief among them being the stimulus to a development of mathematical maturity on the part of the student. Examples were drawn from the areas of recursion relations (Fibonacci sequence, and the problem of the weighted string) and geometry (Möbius tetrads). In the latter case the symbiotic relationship of algebra and geometry was emphasized.

5. *Mathematicians at Ticonderoga*, by Professor D. J. Struik, Massachusetts Institute of Technology.

Among the combatants around or at Fort Carillon (Ticonderoga), 1757–58, were men with good mathematical training. Discussed are the cartographers J. F. V. Des Barres and S. Holland, the astronomer-engineer M. C. De Lotbinière and the mathematician-explorer L. A. De Bougainville. De Bougainville's two tome *Calcul integral* (1754, 1756) allows insight into early stages of the theory of elliptic integrals and ordinary differential equations. The work of the cartographers illustrates the importance of the three body problem for longitude determination.

6. *The use of computers in industry*, by Dr. R. F. Clippinger, Raytheon Manufacturing Company, Waltham, Massachusetts, introduced by the Secretary.

R. E. JOHNSON, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Catholic University of America, Washington, D. C., on December 3, 1955. Professor F. E. Johnston, Chairman of the Section, presided at the morning and afternoon sessions. There were 124 persons in attendance, including 96 members of the Association.

By a unanimous vote of the members present, the By-Laws of the Section were amended in such a way as to add a Treasurer to the roster of officers of the

Section. The Secretary presented a report of the Meeting of Section Officers held at Ann Arbor, Michigan, on August 30, 1955. Mr. W. H. Norris, Chairman of the Committee on High School Contests, discussed the results of the 1955 High School Contest conducted by the Section.

The following papers were presented:

1. *Evaluation of the definite integral* $\int_0^\infty u^{-1} \exp(-\phi^2 u/4 - u^{-1})du$, by Professor Anne E. Scheerer, Georgetown University, introduced by Professor Oliphant.

Differentiation of the integral with respect to the parameter ϕ produces a Bessel equation whose solution is a linear combination of Bessel functions of the first and third kinds. The known asymptotic behavior of these functions permits the coefficients to be calculated and the integral thus evaluated.

2. *Generalized complex numbers and addition formulas*, by Professors John Tyler (ret.) and J. A. Tierney, United States Naval Academy, presented by Professor Tierney.

A generalized complex number was defined and a method for writing such a number in polar form was developed. When the modulus is unity the components satisfy addition formulas which are generalizations of the hyperbolic and trigonometric addition formulas. An application to Diophantine analysis was presented.

3. *Two theorems of plane geometry suggested by certain formulae in kinematics*, by Professor J. P. Hoyt, United States Naval Academy.

If v_x and v_y are the rectangular components of the velocity of a point having plane motion, and if a_x and a_y are the rectangular components of its acceleration a, then the rectangular components of a along the tangent to the path and normal to the path are given respectively by $(v_x a_x + v_y a_y)/v$ and $(v_x a_y - v_y a_x)/v$, where v is the magnitude of the velocity. The geometric interpretation of these results leads to two interesting and perhaps new theorems in plane geometry.

4. *Numerical solution of ordinary linear differential equations*, by Dr. C. H. Frick, U. S. Naval Proving Ground.

By use of an integration formula $y_b - y_a = L(y_b', y_b'', \cdots, y_a', y_a'', \cdots)$, where L is a linear function, the numerical solution of a system of linear differential equations was reduced to the solution of a system of linear algebraic equations. Such integration formulas include: the trapezoidal rule, a truncation of the Euler-Maclaurin formula, an extremely accurate formula given in Milne's book on the numerical solution of differential equations, and extensions of the "Milne" formula. The truncated Euler-Maclaurin formula was used in an undergraduate paper by Miss Mary Moskos at Mary Washington College in 1953. One extension of the "Milne" formula was shown.

5. *An approximate solution of a system of non-linear equations*, by Mr. C. H. Murphy, Jr., Aberdeen Proving Ground.

Although the techniques of non-linear mechanics are usually applied to second order equations in one variable, they may be generalized to larger systems of equations. In exterior ballistics, for example, the yawing motion of a spinning projectile acted upon by a non-linear moment is described by two second order non-linear equations in two dependent variables. An approximate solution of these equations is obtained by a perturbation method and applied to the results of free flight tests. The great value of this technique is shown by the excellent internal agreement as well as by good agreement with direct wind tunnel measurements.

6. *Weakly compact transformations*, by Professor J. W. Brace, University of Maryland.

In order that all continuous linear transformations from a Banach space X to an arbitrary Banach space Y which have the property of transforming bounded sets into weakly compact sets have the additional property of transforming weakly compact sets into compact sets, it is necessary and sufficient that X have the following property: $\lim_{n\to\infty} x_n^*(x_n)=0$ for every sequence $\{x_n\}$ in X converging weakly to zero and every sequence $\{x_n^*\}$ in the first adjoint space of X converging *weakly* to zero. The paper also presents Banach spaces which have the desired property.

7. *The evolution of extended decimal approximations to π and e*, by Dr. J. W. Wrench, Jr., David Taylor Model Basin. (By invitation.)

The speaker traced the historical development of extended decimal approximations to π and e from the eighteenth century up to the present and sketched his own work in this field.

R. P. BAILEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, April 28, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY, University of Kentucky, Lexington, April 28, 1956.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, U. S. Naval Academy, Annapolis, Maryland, May 5, 1956.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN

MINNESOTA, Augsburg College, Minneapolis, May 5, 1956.

MISSOURI, Fontbonne College, St. Louis, April 21, 1956.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Oberlin College, Oberlin, April 14, 1956.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April 20–21, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May 12, 1956.

THE SLAUGHT PAPERS

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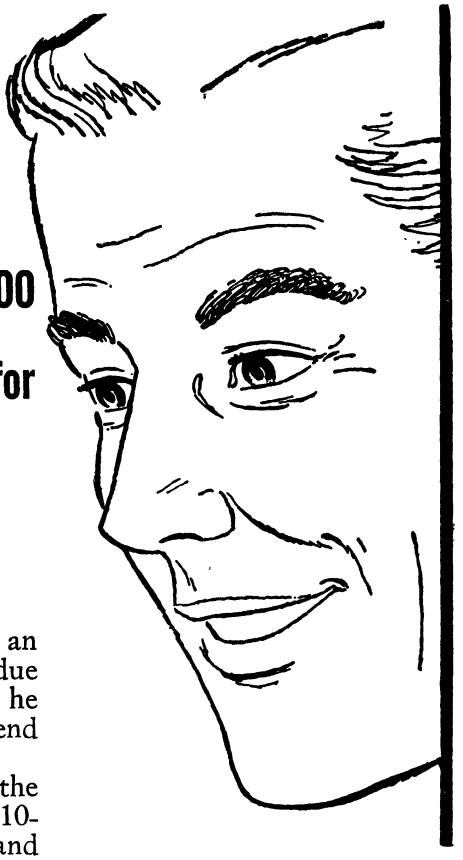
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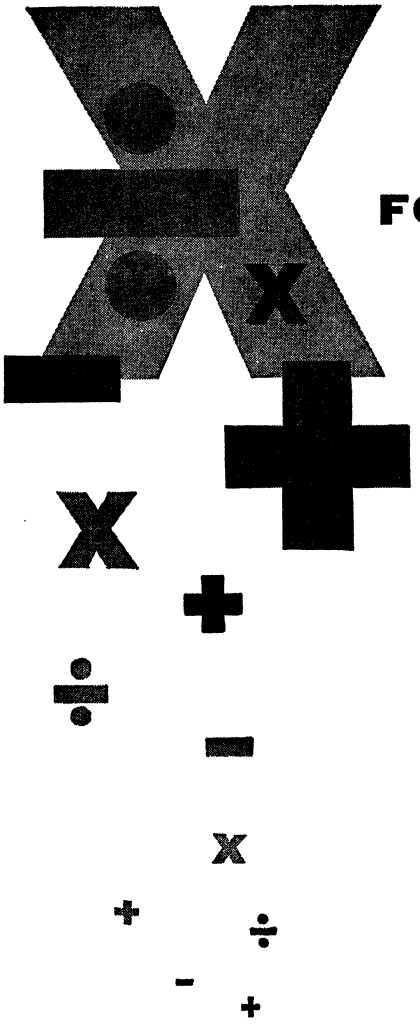
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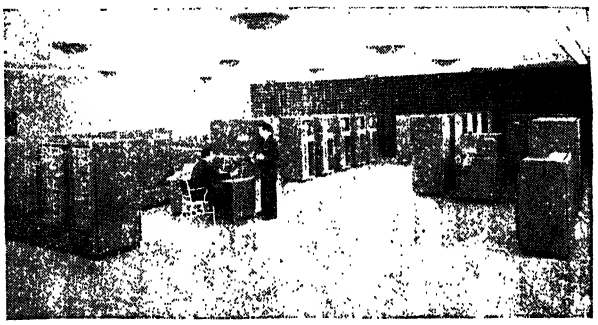
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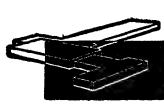
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CONSISTENCY AND COMPLETENESS—A RÉSUMÉ

FRANK DESUA, Ohio University

The modern foundational investigations of mathematics, encroaching as they do on unresolved philosophical questions, have had their polemic aspects. Controversy over ontological problems gave rise at the turn of the century to the three "schools" associated with the programs of Russell and Whitehead, of Hilbert, and of Brouwer, a trichotomy which is preserved to some extent even today. However, those results of the investigations which are perhaps the most significant for the mathematician, namely those embodied in the famous incompleteness theorems of Gödel, must be considered as well established and unimpeachable as any found in the whole of mathematics, based as they are on methods acceptable to the devotees of each of these programs.

Among the values to be gained from these investigations is the insight afforded into the nature of our mathematical preconceptions, and the revelation that even some of the most compelling of these may actually be misconceptions. Of course, once this is discovered, the void created by the dissolution of these misconceptions is apt to be immediately filled by newly formed assumptions, consciously or otherwise. The history of mathematics, as of any science, is to some extent the story of the continual replacement of one set of misconceptions by another. This is of course no cause for despair, for the newly instated assumptions very often possess the merit of being closer approximations to truth than those which they replace.

We are concerned in this paper with the detection of one such misconception

—a misconception concerning the scope and power of formal mathematical methods—and Gödel's remarkable discovery of largely unsuspected limitations inherent in such methods. This discovery is part of, and a product of, the "crisis in the foundations" of which the rumblings have not yet died away. It is not yet entirely clear what new assumptions are to supplant the old, but it is certainly clear that the old must give way.

Our paper is divided into three sections, the first of which is entitled "the Problem," which is the problem of the consistency and completeness of mathematics, first explicitly formulated by Hilbert. The last section "the Solution," concerns the surprisingly negative solution of Gödel. The second section, "Formal Systems," is a digression necessary for an appreciation of the nature of the solution. Before we attempt a statement of the problem, we concern ourselves to some extent with how it arose.

The problem. The theory of abstract sets or classes may be said to be the basic discipline of pure mathematics. Belief in the existence of sets is almost universal among mathematicians, a belief which seems to be preordained by the dictates of common sense. So compelling is the notion of set that even those who harbor doubts must at least feign belief part of the time in order to discourse intelligibly. Yet this concept, despite its seeming necessity and superficial innocence, has proved to be the Pandora's Box of mathematics.

Consider how sets are specified. Two methods are conceivable. A set may be specified *extensionally, i.e.*, by a listing of the members such as "the set consisting of the numbers 2, 3, 5, and 7" or *intensionally* through some defining property or necessary and sufficient condition for membership—technically known as a "monadic propositional function"—as "the set of all primes less than 10." The extensional specification of sets is, of course, precluded as soon as we consider infinite sets.

The intensional method is often canonized in the form of a so-called logical principle, the (unrestricted) Principle of Abstraction, which states that every property determines a set, or in technical language, every monadic propositional function has exactly one set as extension. Symbolically, this becomes:

$$(\exists S)(x)[x \in S \leftrightarrow P(x)]$$

where S is the set determined and P is the defining property.

Briefly the argument in support of this principle is this: Every condition or property determines the set of all entities possessing the property or obeying the condition (even in the case where no such entity exists, the set determined being empty). Conversely every set determines at least one property, namely the property of being a member of the set, if no other.

The principle is widely appealed to in mathematics—often in some restricted form—either explicitly or tacitly. It is implicit, for example, in the high school geometry definition of a "locus" as a point set every element of which obeys some metric condition.

In the principle of abstraction, then, we have an intuitively obvious prin-

ciple of wide applicability. Unfortunately there is one thing wrong with it. In the unrestricted form stated above, it happens to be false. It was proved to be false by Bertrand Russell in 1901.

Let's examine the process of set formation as embodied in the principle of abstraction. Out of the aggregate of all entities we choose those which possess a given property to form our set. The flaw in the principle of abstraction lies in the fact that it ignores one entity, namely the newly formed set. Conceivably the property used to form the set may neither apply nor fail to apply to this set. In this case no set is actually determined. Russell discovered a property, possessed by many entities, yet which cannot consistently be said to determine any set.

The property in question is the property which a set x may possess of "not being a member of itself" ($x \notin x$). Many sets possess the property—a set of material objects, for example—while other sets fail to possess the property—the set of all sets would, assuming the existence of such a set. Indeed in an unrestricted universe of discourse (assuming the allowability of such a universe), there must be as many sets of one kind as of the other, for the complement of any set possessing the property is a set which does not, and conversely.

It is easily seen that the above mentioned property determines no set, for suppose there were such a set S. From the defining property, $x \in S \leftrightarrow x \notin x$. Does S itself possess the property, or not? Putting S for x we have, $S \in S \leftrightarrow S \notin S$, which is a contradiction. Therefore no such set S exists, although innumerable sets x possess the property $x \notin x$.

Actually, the first proof of the non-validity of the principle of abstraction came several years earlier in a result by Burali-Forti concerning ordinal numbers. Sets of numbers of various kinds are common objects of study in mathematics. It would seem then that the set of all ordinals is perfectly well defined according to the canons of common sense mathematics. Given any entity we can decide whether it is a member of the set or not if we can determine whether it is an ordinal number or not. It came therefore as a surprise when in 1897 Burali-Forti demonstrated the non-existence of the set of all ordinals.

To recapitulate the proof, recall the following theorems of ordinal number theory: Any initial segment of the ordinals, that is, any set of ordinals containing all ordinals preceding any of its members, can be ordered according to the magnitude of its members. The resulting ordered set is well-ordered; consequently it has an ordinal number. Further this number exceeds any member of the set (in fact, it is the least number which exceeds every member of the set).

Now suppose the property of being an ordinal number determined a set S, *i.e.*, let S be the set of all ordinals. Then S would be a set to which the above theorems apply, so that it would have an ordinal number Ω. Accordingly, Ω would exceed every member of S. But if S is the set of all ordinals, Ω must be one of its members. Consequently, Ω exceeds itself, which is a contradiction.

Superficially somewhat similar difficulties arise, not only in the theory of sets, but also in the theory of statements or propositions. It would seem that

any statement which purports to be a statement of fact must be either true or false. If the statement conforms with the facts it is true, otherwise false. Thus, for example, the statement "John has quit smoking" is true only if John has actually quit smoking. (There are difficulties of an existential nature in some cases, as for example with such statements as "Pegasus has wings" or "Four sided triangles are isosceles." But such difficulties are capable of resolution through the adoption of certain conventions of interpretation, as in Russell's and Quine's theories of singular terms and descriptions.)

This view of the validity of statements, however, is subject to difficulties of another kind if no restrictions are imposed on statement formation. For if we are talking about statements themselves, *i.e.*, if we consider statements about statements, as we often do in mathematics, the statements themselves become part of the facts. Conceivably in an unrestricted discussion the facts may be so affected as to render certain such statements about statements neither true nor false.

That this is actually the case was discovered by the ancient Greeks. They constructed a statement (the Epimenides), of the form "This statement is false," which asserts its own falsehood. This statement, referring as it does to itself, can only be true if it is indeed false, and conversely.

It might be pointed out that such Epimenidean statements cannot be constructed in any properly restricted mathematical system, so that the contradiction does not arise in practice. (The Russell and Burali-Forti contradictions actually did arise in mathematical discourse, and still do occasionally.) However, as we shall see, something very closely resembling such Epimenidean statements actually can be constructed in certain formal systems, namely Gödelian formulas. This is a profound syntactical fact which we shall examine later.

A great many of these demonstrations running counter to intuitive principles have unexpectedly come to light. Cantor proved the non-existence of the set of all cardinal numbers. Berry and Richard have shown the non-existence of certain apparently well defined denumerable sets of finite numbers. Skolem has shown conversely the existence of certain number sets possessing no defining property.

We have taken the position here that these "contradictions" actually constitute demonstrations, by *reductio ad absurdum*, of the non-validity of certain intuitive principles of long standing, such as the principle of abstraction. Other positions concerning the significance of these contradictions are widely held. The authors of the contradictions themselves very often tacitly defended these principles, and consequently regarded the contradictions as "paradoxes." They had then to revise their concept of "property" or "condition," rather than "set," *etc.* Whatever position is taken however, it is clear that unbridled intuitive reasoning, even on the most basic level, can and sometimes does lead to difficulties.

With this realization we are now in a position to state what was until recently a central problem of the modern foundational investigations of mathematics, namely to devise restrictions on intuitive reasoning sufficiently strong

to preclude the possibility of inconsistency but sufficiently weak to allow mathematics the greatest possible latitude for development, at least to allow the development it has already undergone. Further the problem is to prove, if possible, by unimpeachable methods (and therein lies a danger—the principles discredited by Russell were "unimpeachable" in their day) that these restrictions are actually sufficiently strong, *i.e.*, that the resulting mathematical systems are consistent.

With the first part of this problem we shall not concern ourselves except to say that restrictions have been devised which seem to fulfill the requirements. There are Russell's type theory, Ramsey's simplification of this theory, Quine's stratification, and Zermelo's axiomatic set theory to mention a few. The *adequacy* of the resulting system for the derivation of classical mathematics is proved very directly. Either classical mathematics is actually derived within the system (as in Russell and Whitehead's *Principia Mathematica*) or it is shown that any theorem that can be derived in, say, *Principia*, can also be derived in the system under consideration. The proof of the consistency of these systems is quite another matter, and this is the part of the problem with which we are mainly concerned.

Formal systems. The systems whose consistency is investigated are not actual systems of ordinary intuitive mathematics but rather certain specializations thereof known as formal systems or formal languages. We shall investigate this concept in some detail presently, but here we want to point out differences between formal mathematics and the postulate systems of ordinary mathematics. In ordinary mathematics it is helpful in carrying out deductions to know the meanings of the technical terms (or "undefined" terms) involved. The semantics of the system is of consequence. Even when the system is semi-formalized with regard to technical terms, so that they become truly undefined, semantics plays a heavy role in the logic involved in the deductions, which is usually on a primitive verbal level.

In formal mathematics on the other hand the technical terms are analyzed to the point where all their properties which matter for the deduction of theorems have been expressed by axioms. The terms must then have a recognizable form rather than an apprehensible meaning. It is then possible to treat these terms as words essentially without meaning, for to say that these terms have meanings essential to the deductions amounts to saying that not all their properties have been expressed by axioms, *i.e.*, that the analysis is incomplete. Even the logic of the system is explicitly incorporated into the system in the form of axioms and rules of inference, and the whole is rendered in symbolic rather than verbal form. When this analysis is completed, and meanings thus left out of account, questions of form or structure become the sole consideration. Semantics has been replaced by syntax.

There are several good reasons for this preference for formal over intuitive systems in many foundational investigations, not the least of which is the fact that intuitive mathematics is a function of the mathematician as well as of the

subject matter. Whereas a topologist may embrace such principles as the axiom of choice or the continuum hypothesis without hesitancy, an algebraist may do so with compunction. Thus it becomes of interest to compare systems in which these principles are explicitly included with systems in which they are excluded. Also standards of rigor vary with the times and the mathematician, one man's proof being another man's sophistry. The intuitive mathematics of the great eighteenth century analysts, for example, is demonstrably inconsistent. Questions of rigor do not arise however in a system restricted to finitary and constructive methods of proof as is the case in many formal systems. Perhaps the chief reason for the use of formal systems rather than intuitive systems, however, is that only within formal systems can such vague semantical concepts as *axiom*, *theorem*, *proof*, *consistency*, *etc.*, be made explicit and precise. Formalization is a necessary concomitant to precision.

The study of formal systems has been criticized (and extolled!) on the grounds that it "reduces mathematics to a meaningless game." It seems plausible that such criticism is based on a misconception concerning the purpose of these studies. It would be as much to the point to criticize the study of topological spaces on the grounds that such "spaces" are incapable of propagating electromagnetic radiation. It must be kept in mind that the partial formalization of much of mathematics—of algebra, for example—has proceeded steadily since the time of the Babylonians, yet algebra certainly remains meaningful mathematics. Abstraction of meaning does not imply loss of meaning.

The extent to which intuitive mathematics can be formalized is a problem on which much light has been shed by recent foundational investigations. That at least a very considerable measure of formalization is possible for mathematical theories is evident upon a perusal of some of the attempts at such formalization—Whitehead and Russell's *Principia Mathematica*, for example. But that limitations do exist on the kinds of problems which can be stated and proved within a formal system is revealed by the researches of Gödel, Church, and others. And the delineation of such limitations is the subject of continuing research. Involved in the process of formalizing a theory may be difficulties as great as those involved in the subject matter of the theory itself.

Let's examine the way in which a formal system is actually set up. We start with a certain set of symbols which may be infinite in number. Any finite linear array of these, allowing repetitions, is called a *formula*. The formulas are divided into two disjoint sets called *well-formed formulas* (w.f.f.) and *gibberish* (only the w.f.f. are to have meaning in a semantic *interpretation* of the system). The w.f.f. are defined recursively, and in a purely syntactical or morphological manner, *i.e.*, solely in terms of their structure. Moreover they are defined effectively, *i.e.*, a finite process is specified whereby we can determine whether a given formula is well-formed or not.

A certain subset of the w.f.f. are chosen to be our initial *axioms*. (The age-old criterion of "self-evidence" of axioms is dispensed with, there being no de-

grees of perspicuity in a system from which all meaning has been abstracted.) New w.f.f. are obtained or "derived" from these axioms by the application of certain *rules of inference*, which again are recursive, syntactical, and effective. By a *proof* we mean a finite sequence of w.f.f.'s, each of which is either an axiom or is obtainable from one or more earlier formulas of the sequence by an application of a rule of inference.

The last formula of a proof is, by definition, a *theorem*. We write $\vdash A$ to denote that A is a theorem. (The symbol '$\vdash$' is called the assertion symbol.) Whether a given sequence of formulas constitutes a proof or not can be effectively determined, but whether a given formula is a theorem or not generally can not be determined by any effective uniform procedure.

An actual example of a formal system may serve to clarify some of these definitions. The following is one possible formalization of a fragment of modern logic known as the "propositional calculus":

Symbols: An infinite list of lower case letters, primed and unprimed: $a, b, c, \cdots, a', b', c', \cdots, a'', b'', c'', \cdots$
parentheses: (, and)
negation symbol (tilde): $\sim$
alternation symbol: $\vee$
Any finite array of these, for example $\vee))a'\sim$, is a formula. This particular one happens to be gibberish.

w.f.f.: Any letter (primed or unprimed) alone is w.f. If A and B are w.f., so are $\sim A$ and $(A \vee B)$. Examples: r is w.f. So are $\sim r$ and $(r \vee \sim r)$.

Axioms: An infinite list of axioms which come under the following axiom schemata:
(1) $(A \vee \sim(A \vee A))$
(2) $((\sim B \vee C) \vee B)$
(3) $(\sim(D \vee \sim E) \vee (\sim(E \vee F) \vee (F \vee D)))$
Each of these schemata represents an infinite set of axioms, namely one for each possible choice of w.f.f. in place of A, B, C, D, E, and F.

Rule of inference: From two formulas of the form G and $(\sim G \vee H)$, we infer the formula H.

This system is capable of a semantical interpretation in terms of the so-called logical connectives "not," "either $\cdots$ or $\cdots$," *etc.* The letters may stand for sentences or propositions. Thus if we let "r" denote "it is raining," then "$\sim r$" becomes "it is not raining" and "$(r \vee \sim r)$" becomes "either it is raining or it is not." As an example of a theorem and its proof we assert that $\vdash (r \vee \sim r)$.

Proof:

(i) $(\sim r \vee \sim(\sim r \vee \sim r))$
[from axiom schemata 1 replacing A by $\sim r$]

(ii) $(\sim(\sim r \vee \sim(\sim r \vee \sim r)) \vee (\sim((\sim r \vee \sim r) \vee r) \vee (r \vee \sim r)))$
[from 3 replacing D by $\sim r$, E by $(\sim r \vee \sim r)$, and F by r]

(iii) $(\sim((\sim r \vee \sim r) \vee r) \vee (r \vee \sim r))$
[using the rule of inference on steps (i) and (ii)]
(iv) $((\sim r \vee \sim r) \vee r)$
[from 2 replacing B by r, and C by $\sim r$]
(v) $(r \vee \sim r)$
[using the rule of inference on steps (iii) and (iv)]

Thus we have "proved" that it is either raining or it is not.

We note here, for later use, a binary connective of considerable importance, namely the *conditional* or *material implication*, symbolized by "$\rightarrow$," and translated by one sense of the idiom "if $\cdots$ then $\cdots$." If A and B are w.f., then $(A \rightarrow B)$ is w.f. A rule of inference equivalent to that given in the above system, but much more widely used, is *modus ponens* (or strictly speaking, its syntactical analogue, *detachment*) which allows us to obtain from two formulas of the form A and $(A \rightarrow B)$ the formula B.

Two further definitions are important for our discussion. A formal system is *consistent* if there is no w.f.f. A such that both $\vdash A$ and $\vdash \sim A$. (In what follows a stronger condition called ω-consistency is sometimes required, but we shall forego an explanation of this for the sake of brevity.) A formal system is *complete* if, for every w.f.f. A, either $\vdash A$ or the system would become inconsistent if A were introduced as an additional formal axiom (the reader may think of this as either $\vdash A$ or $\vdash \sim A$, *i.e.*, the system is complete whenever every w.f.f. is theoretically "decidable," although strictly speaking the two concepts do not coincide in all formal systems).

The solution. Our problem now becomes the problem of setting up a formal system which is *adequate* for the derivation of mathematics, and which further is *consistent* (*i.e.*, leads to no contradiction) and *complete* (*i.e.*, we would like every mathematical proposition to be at least theoretically decidable in the sense that it can either be proved or refuted). The problem of adequacy was solved by Frege, Whitehead, and Russell. (Of course, the adequacy of their systems was demonstrated for extant mathematics only. Nothing can be ascertained concerning the adequacy for all future developments of mathematics.) Several systems have been proposed which it is hoped are consistent as well as adequate. To mention a few, there is Russell and Whitehead's *Principia Mathematica*, Quine's *Mathematical Logic*, and systems by Bernays, Fraenkel, Gödel, von Neumann, Zermelo and others. Although it is hoped that these systems are consistent, none have actually been proved to be so.

The consistency problem was partially reduced in classical mathematics by the so-called "method of models." Thus Beltrami in 1868 reduced the consistency of Lobatchevsky's non-Euclidean geometry to that of Euclidean geometry. In this and other manners, the consistency of large parts of classical mathematics is ultimately reducible to that of the arithmetic of natural numbers, particularly as this theory is embodied in the Peano axioms, or in a set theory sufficiently comprehensive for the derivation of the Peano axioms.

For several inadequate partial systems the consistency problem has been

successfully solved. Post in 1921 demonstrated both the consistency and the completeness (in a somewhat different sense than that defined above) of the propositional calculus. The first order functional calculus was proved consistent in 1928 by Hilbert and Ackermann, and complete (in still another sense) in 1930 by Gödel. More recently Tarski has demonstrated the consistency and completeness of the arithmetic of real numbers, and also of elementary geometry.

With this background of success it was widely felt that the solution of the consistency problem for the arithmetic of natural numbers was essentially at hand, that it was now merely a matter of detail. The necessary tools had already been forged in Hilbert's new branch of research, metamathematics, and few doubted that a positive solution to the problem was already within reach. Surprisingly however, despite intensive efforts by Hilbert and others, and very close approaches to a solution, the problem remained refractory.

This puzzling situation was illuminated in 1931 by the appearance of certain remarkable and unexpected results of Gödel which solved in a negative fashion both the consistency and the completeness problem. These theorems, among the most remarkable in all of mathematics, revealed an unforeseen limitation in the methods of formal mathematics in some ways analogous to the limitations on physical measurements imposed by the Heisenberg Uncertainty Principle. Working within systems either presupposing the Peano axioms or possessing a syntax sufficiently rich for the derivation of these axioms—systems related to the *Principia Mathematica*—Gödel showed the impossibility of proving the consistency of such systems by so-called "finitary" methods, that is by constructive methods formalizable within the system itself. So the proof of consistency which seemed so close to attainment is actually impossible of attainment, at least in the form originally envisioned by Hilbert.

This remarkable result is but a consequence of even more fundamental results concerning the completeness of the formal systems studied by Gödel. Gödel demonstrated the incompleteness of such systems, that is, he revealed the existence of "undecidable" propositions, of which consistency is one. This result is particularly surprising in view of the fact that the formal systems with which Gödel worked are sufficiently rich in syntax for the derivation of all of classical mathematics (and presumably of much of mathematics yet to be developed). Yet despite this richness of syntax, it is possible to formulate within these systems arithmetical problems which are theoretically unsolvable within the system.

No attempt will be made here to outline Gödel's demonstrations, for even such an outline would of necessity be long, detailed, and technical. Instead we will content ourselves with a heuristic account of Gödel's reasoning which we hope will at least reveal the "point" of the proof.

Suppose we denote by "*CONSIS*" a syntactical formula whose semantic interpretation implies the consistency of the formal system (for example, *CONSIS* may mean "there is no proof for $1=0$"). The consistency problem would then be related to the problem of demonstrating $\vdash CONSIS$. One pos-

sible line of attack on the problem would be to seek a sufficient condition A such that $\vdash(A \rightarrow CONSIS)$ and then to try to demonstrate $\vdash A$. $\vdash CONSIS$ would then follow by *modus ponens*.

Now Gödel foresaw what others had failed to suspect, namely that *CONSIS* was neither provable nor refutable if the system is truly consistent. To demonstrate this he hit upon the following ingenious scheme. Suppose we seek a condition B, necessary for *CONSIS* [so that $\vdash(CONSIS \rightarrow B)$] and then show that B is true but unprovable within the system. It would then follow that *CONSIS* is undecidable, for if $\vdash CONSIS$, then B would be provable by *modus ponens*, which would be a contradiction.

The question now remains, can we construct such a formal statement, or formula B which, assuming consistency, is true but unprovable within the formal system? If we could, we will have solved in a negative fashion both the consistency problem and the completeness problem, *i.e.*, we will have shown that the methods available in a formal system adequate for extant mathematics are inadequate to demonstrate its own consistency, and thus inadequate to decide even theoretically all mathematical propositions.

Taking a hint from the Epimenides antinomy, Gödel constructed such an undecidable formula B by constructing a formula whose semantical content is that B is undecidable. Consider the Epimenidean statement: "This statement is false." By analogy, we might try "This statement can neither be proved nor disproved" for our statement B. However, one difficulty in following this Epimenidean construction in the construction of our Gödelian statement B is that the self-reference occurring in the Epimenidean statement is semantical in character. Such semantical self-reference is of course precluded in a formal system. We must seek a syntactical device which will enable statement B to refer to itself in a purely structural manner.

A semi-syntactical device is available in the English language through the use of quotation marks. The following statement, for example, is Gödelian in character:

> "yields a statement which can neither be proved nor disproved when appended to its own quotation" yields a statement which can neither be proved nor disproved when appended to its own quotation.

Of course the device of quotation is a part of the syntax of natural languages and not of mathematics, formal or otherwise, so it was necessary to seek further. Gödel accomplished his aim by a very elaborate and ingenious numerical device, assigning coordinates to the decidable formulas of the formal system, establishing an enumeration of these formulas, and constructing the undecidable formula B by an application of Cantor's diagonal process.

These results on incompleteness refer of course only to certain formal systems whose syntax is adequate for the formalization of many properties of the natural numbers used in the construction of the "undecidable" proposition B (of the Chinese remainder theorem, and of the possibility of mapping integers onto finite sequences of integers, for example). It might be thought that the

defect of the incompleteness of such systems could be remedied by the addition of further effective lists of axioms which would enable one to derive the propositions which were previously undecidable. But very surprisingly even this expedient, for all its apparent reasonableness, fails utterly. For Gödel has further shown that not only are the formal systems incomplete but also they are *incompletable* in the sense that new undecidable propositions can always be constructed within the system enlarged by the addition of these new axioms.

A final word concerning the significance of these syntactical results for ordinary intuitive mathematics. Gödel has shown that the formal systems known to be adequate for the derivation of mathematics are unsafe in the sense that their consistency cannot be demonstrated by finitary methods formalizable within the system, whereas any system known to be safe in this sense is inadequate. We are left in the position of the student of elementary mathematics who "knows what a logarithm is, but just can't put it into words." We know, or at least strongly suspect, mathematics to be consistent. But we just cannot incorporate this knowledge into a formal system.

The methods used in Gödel's demonstrations were constructive and finitary in character, and consequently acceptable even to the most cautious of the critics of mathematical methods. In view of Gödel's results, if a proof of consistency is to be forthcoming it must employ methods which are not formalizable within the system. Such proofs have been devised. Thus Gentzen has proved the consistency of number theory using transfinite induction up to a sufficiently large ordinal—a method of proof not formalizable within the system whose consistency it proves. While such methods are perhaps acceptable to most mathematicians, the acceptance is by no means universal. Indeed there are those to whom no proof will be acceptable except one of a type explicity ruled out by Gödel's discoveries. In view of the lack of a universally acceptable proof, belief in the consistency of mathematics becomes then somewhat a matter of faith rather than reason.

Suppose we loosely define a *religion* as any discipline whose foundations rest on an element of faith, irrespective of any element of reason which may be present. Quantum mechanics for example would be a religion under this definition. But mathematics would hold the unique position of being the only branch of theology possessing a rigorous demonstration of the fact that it should be so classified.

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A CHAIN OF CIRCLES ASSOCIATED WITH THE n-LINE

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1. Introduction. The Clifford-de Longchamps-Morley-Bath series of theorems in the geometry of the n-line are well known and were stated in an earlier paper.* We now add another chain of circles to the system. In section 2 an analytic method is set up and applied to the Morley circles. In sections 3 and 4 the de Longchamps points and the Clifford points and circles are discussed. In sections 5 and 6 a chain of circles which we shall call the Miquel circles and a chain of points which we shall call the Hermes points are defined. In section 7 the new chain of circles is reached and its chief properties proved. In the final sections the special cases of the 5-line and the 6-line are considered.

2. The Morley circles.† We take n lines in the plane, no three concurrent and no two parallel, and consider the "normal" equation of l_f: $x\cos\alpha_f+y\sin\alpha_f=p_f$. If $z=x+iy$ and $\bar{z}=x-iy$, this can be put into the form $z+\mathbf{f}^2\bar{z}=k_f\mathbf{f}$, where $\mathbf{f}$ is an abbreviation for $\cos\alpha_f+i\sin\alpha_f$, and k_f is twice p_f. Throughout this paper the boldface numerals $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{n}$ are used in this sense. For example $\mathbf{1}^3=\cos 3\alpha_1+i\sin 3\alpha_1$.

Thus l_1 is $z+\mathbf{1}^2\bar{z}=k_1\mathbf{1}$ and l_2 is $z+\mathbf{2}^2\bar{z}=k_2\mathbf{2}$. Then A_{12}, the intersection of l_1 and l_2 is found to be $z=-\mathbf{12}\{k_1\mathbf{2}-k_2\mathbf{1}\}/(\mathbf{12})$, where $(\mathbf{12})$ is an abbreviation for $\mathbf{1}^2-\mathbf{2}^2$. We shall use $(\mathbf{123})$ as an abbreviation for $(\mathbf{12})(\mathbf{13})(\mathbf{23})$, $(\mathbf{2346})$ for $(\mathbf{23})(\mathbf{24})(\mathbf{26})(\mathbf{34})(\mathbf{36})(\mathbf{46})$, and so on. The numerals will always be placed in ascending order.

If, as in this case, z is a function of k's and boldface numerals, $\bar{z}$ can be found by replacing $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{n}$ by $1/\mathbf{1}, 1/\mathbf{2}, \cdots, 1/\mathbf{n}$, since $\cos\alpha_f-i\sin\alpha_f=1/(\cos\alpha_f+i\sin\alpha_f)$. Thus, for A_{12}, $\bar{z}=\{k_1\mathbf{1}-k_2\mathbf{2}\}/(\mathbf{12})$.

The square of the distance between two points (x_1, y_1) and (x_2, y_2) is given by $(z_1-z_2)(\bar{z}_1-\bar{z}_2)$. Hence the equation of the perpendicular bisector of $A_{12}A_{23}$ is

$$\left[z+\mathbf{12}\frac{k_1\mathbf{2}-k_2\mathbf{1}}{(\mathbf{12})}\right]\left[\bar{z}-\frac{k_1\mathbf{1}-k_2\mathbf{2}}{(\mathbf{12})}\right]=\left[z+\mathbf{23}\frac{k_2\mathbf{3}-k_3\mathbf{2}}{(\mathbf{23})}\right]\left[\bar{z}-\frac{k_2\mathbf{2}-k_3\mathbf{3}}{(\mathbf{23})}\right].$$

This reduces to

$$z-\mathbf{2}^2\bar{z}=\frac{-2}{(\mathbf{12})(\mathbf{23})}\left[k_1\mathbf{12}(\mathbf{23})-k_2\{\mathbf{2}^4-\mathbf{1}^2\mathbf{3}^2\}-k_3\mathbf{23}(\mathbf{12})\right].$$

Similarly the perpendicular bisector of $A_{13}A_{23}$ is

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* This MONTHLY, vol. 61, 1954, p. 161.

† These circles were first announced by de Longchamps, Nouv. Corres. de Math., 1877, p. 306. Because of an important paper by F. Morley, Trans. Amer. Math. Soc., 1900, p. 99, they have been called Morley circles. R. Goormaghtigh in a paper in this MONTHLY, vol. 54, 1947, p. 327, calls them the centric circles. Other references are found in H. F. Baker, Introduction to Pure Geom., note 1, p. 330, Cambridge, 1943.

$$z - 3^2\bar{z} = \frac{-3}{(13)(23)}\left[k_1 13(23) + k_2 23(13) + k_3\{3^4 - 1^2 2^2\}\right].$$

Eliminating $\bar{z}$, we find that C_{123}, the center of the circle circumscribing $A_{12}A_{13}A_{23}$ is

$$123\left\{\frac{k_1 23(23) - k_2 13(13) + k_3 12(12)}{(123)}\right\}.$$

The radius of this circle is

$$123\left\{\frac{k_1 1(23) - k_2 2(13) + k_3 3(12)}{(123)}\right\}.$$

The equation of the circle is

$$(123)z\bar{z} - z\{k_1 1^3(23) - k_2 2^3(13) + k_3 3^3(12)\} - \bar{z}123\{k_1 23(23) - k_2 13(13) + k_3 12(12)\} = 123\{k_1 k_2 3(12) - k_1 k_3 2(13) + k_2 k_3 1(23)\}.$$

We shall abbreviate these expressions so that C_{123} is $123\sum k_1 23(23)/(123)$, r_{123} is $123\sum k_1 1(23)/(123)$, and the circle is

$$(123)z\bar{z} - z\sum k_1 1^3(23) - \bar{z}123\sum k_1 23(23) = 123\sum k_1 k_2 3(12).$$

In these summations the numerals in the subscripts of the k's and in the boldface numbers are to be rearranged in every possible way except that they must always be in ascending order among the k's and the boldface numbers respectively; the sign of a term is plus or minus according as the number of transpositions needed to put the numerals back into the original order is even or odd.

We shall use $\mathbf{n}!$ for $\mathbf{12\cdots n}$, $\mathbf{n}!/\mathbf{f}$ for $\mathbf{n}!$ with $\mathbf{f}$ omitted, $\mathbf{n}!/\mathbf{fg}$ for $\mathbf{n}!$ with $\mathbf{f}$ and $\mathbf{g}$ omitted, $\mathbf{n}!/\mathbf{f}!$ for $\overline{\mathbf{f+1}}\ \overline{\mathbf{f+2}}\cdots\mathbf{n}$ and so on. The same abbreviations will be used for integers.

It may be noted that $(\mathbf{n}!/12)(\mathbf{n}!) = (\mathbf{n}!/1)(\mathbf{n}!/2)(12)$. We now set up a chain of points as follows. $C_{n!}$ is defined as $(-1)^{n-1}\mathbf{n}!\sum k_1\mathbf{n}!/1(\mathbf{n}!/1)/(\mathbf{n}!)$, $C_{n!/f}$ as $(-1)^n\mathbf{n}!/\mathbf{f}\sum k_1\mathbf{n}!/1\mathbf{f}(\mathbf{n}!/1\mathbf{f})/(\mathbf{n}!/\mathbf{f})$.

If we take the first of these to be z_1 and the second z_2, then

$$\bar{z}_1 = \frac{\sum k_1 1^{2n-3}(\mathbf{n}!/1)}{(\mathbf{n}!)} \quad \text{and} \quad \bar{z}_2 = \frac{\sum k_1 1^{2n-5}(\mathbf{n}!/1\mathbf{f})}{(\mathbf{n}!/\mathbf{f})}.$$

Then $z_1 - z_2 = (-1)^{n-1}\{\mathbf{n}!/\mathbf{f}\}^2\sum k_1 1(\mathbf{n}!/1)/(\mathbf{n}!)$, $\bar{z}_1 - \bar{z}_2 = \mathbf{f}^2\sum k_1 1^{2n-5}(\mathbf{n}!/1)/(\mathbf{n}!)$. Then the square of the distance between $C_{n!}$ and $C_{n!/f}$ is $(-1)^{n-1}\{\mathbf{n}!\}^2/(\mathbf{n}!)^2 \cdot \sum k_1 1(\mathbf{n}!/1)\cdot\sum k_1 1^{2n-5}(\mathbf{n}!/1)$.

Since this is independent of $\mathbf{f}$, it follows that $C_{n!}$ is the center of a circle passing through the points $C_{n!/1}$, $C_{n!/2}$, $\cdots$, $C_{n!/n}$. It follows that $\mathcal{C}_{1234}$ passes through C_{123}, C_{124}, C_{134}, C_{234} and is therefore the Morley circle of the 4-line; $\mathcal{C}_{12345}$ passes

through C_{1234}, C_{1235}, C_{1245}, C_{1345}, C_{2345} and is therefore the Morley circle of the 5-line; and so on indefinitely.

We also note that the square of the radius of the circle $\mathfrak{C}_{n!/f}$ is

$$(-1)^n \frac{\{n!/f\}^2}{(n!/f)^2} \sum k_1 1(n!/1f) \cdot \sum k_1 1^{2n-7}(n!/1f).$$

3. The de Longchamps points.* Consider the circles $\mathfrak{C}_{123}$ and $\mathfrak{C}_{124}$:

$$(123)z\bar{z} - z \sum k_1 1^3(23) - \bar{z}123 \sum k_1 23(23) = 123 \sum k_1 k_2 3(12)$$
$$(124)z\bar{z} - z \sum k_1 1^3(24) - \bar{z}124 \sum k_1 24(24) = 124 \sum k_1 k_2 4(12).$$

Eliminating $\bar{z}$, we have

$$\frac{(123)z - 123 \sum k_1 23(23)}{(124)z - 124 \sum k_1 24(24)} = \frac{z \sum k_1 1^3(23) + 123 \sum k_1 k_2 3(12)}{z \sum k_1 1^3(24) + 124 \sum k_1 k_2 4(12)}.$$

This reduces to

$$[z(12) + 12\{k_1 2 - k_2 1\}][z \sum k_1 1^3(234) - 1234 \sum k_1 k_2 34(12)(34)] = 0.$$

This means that these circles intersect at A_{12} and also at a point which we shall call K_{1234} which is given by

$$\frac{1234 \sum k_1 k_2 34(12)(34)}{\sum k_1 1^3(234)}.$$

Evidently $\mathfrak{C}_{134}$ and $\mathfrak{C}_{234}$ also pass through this point. This is the de Longchamps point of the 4-line.

We now set up a chain of points $K_{n!}$, defined as

$$\frac{(-1)^n n! \sum k_1 k_2 n!/12(n!/12)\{1^{2n-6} - 2^{2n-6}\}}{\sum k_1 1^{2n-5}(n!/1)}.$$

If we use for z_1 and $\bar{z}_1$ the values given in the last section for $C_{n!,f}$ and for z_2 the value given above for $K_{n!}$, then

$$\bar{z}_2 = -\frac{\sum k_1 k_2 12(n!/12)\{1^{2n-6} - 2^{2n-6}\}}{\sum k_1 1(n!/1)}.$$

Then

$$z_1 - z_2 = \frac{(-1)^n\{n!/f\}^2 \cdot \sum k_1 1(n!/1) \cdot \sum k_1 1^{2n-7}(n!/1f)}{(n!/f) \sum k_1 1^{2n-5}(n!/1)};$$

$$\bar{z}_1 - \bar{z}_2 = \frac{\sum k_1 1^{2n-5}(n!/1) \cdot \sum k_1 1(n!/1f)}{(n!/f) \sum k_1 1(n!/1)}.$$

Hence the square of the distance between these two points is

* de Longchamps, *loc. cit.* Goormaghtigh calls these points the centric foci of the n-line.

$$\frac{(-1)^n\{n!/f\}^2}{(n!/f)^2}\cdot\sum k_1 1(n!/1f)\cdot\sum k_1 1^{2n-7}(n!/1f).$$

Comparing this with the result given at the end of section 2, we see that $K_{n!}$ lies on the Morley circles $\mathfrak{C}_{n!,1}$, $\mathfrak{C}_{n!,2}$, $\cdots$, $\mathfrak{C}_{n!,n}$. Thus $\mathfrak{C}_{1234}$, $\mathfrak{C}_{1235}$, $\mathfrak{C}_{1245}$, $\mathfrak{C}_{1345}$, $\mathfrak{C}_{2345}$, all pass through K_{12345}; and so on indefinitely.

4. Clifford points and circles.† We shall not be concerned greatly with these. However, if we take the point F_{12345} to be

$$\frac{-12345\sum k_1k_2345(12)(345)}{\sum k_1 1^3(2345)},$$

we may readily prove by the methods used above that F_{12345} is the center of a circle passing through K_{1234}, K_{1235}, K_{1245}, K_{1345}, K_{2345}. This is the Clifford circle of the 5-line. The square of its radius is

$$\frac{-\{12345\}^2[\sum k_1k_212(12)(345)]^2}{[k_1 1^3(2345)]^2}.$$

It may readily be verified that this circle also passes through K_{12345}, which is Bath's theorem.‡

Similarly it may be shown that F_{123456}, which is given by

$$\frac{-123456\sum k_1k_2k_3456(123)(456)}{\sum k_1k_2 1^3 2^3(12)(3456)},$$

lies on this circle and also on the Clifford circles $\mathfrak{F}_{12346}$, $\mathfrak{F}_{12356}$, $\mathfrak{F}_{12456}$, $\mathfrak{F}_{13456}$, $\mathfrak{F}_{23456}$.

This can be generalized to prove the existence of the Clifford circle of the $\overline{2p+1}$-line and the Clifford point of the $\overline{2p}$-line.

5. Miquel circles. We call the circle concentric with the Morley circle of the n-line and passing through its de Longchamps point, the Miquel circle§ of the n-line. Taking $C_{n!}$ to be z_1 and $K_{n!}$ to be z_2, we find that

$$z_1 - z_2 = \frac{(-1)^{n-1}\{n!\}^2\cdot\sum k_1 1^{2n-7}(n!/1)\cdot\sum k_1 1(n!/1)}{(n!)\sum k_1 1^{2n-5}(n!/1)}$$

$$\bar{z}_1 - \bar{z}_2 = \frac{\sum k_1 1^3(n!/1)\cdot\sum k_1 1^{2n-5}(n!/1)}{(n!)\sum k_1 1(n!/1)}.$$

Hence the square of the radius of the Miquel circle is

$$\frac{(-1)^{n-1}\{n!\}^2\sum k_1 1^3(n!/1)\cdot\sum k_1 1^{2n-7}(n!/1)}{(n!)^2}.$$

† Clifford, Mess. Math., 1871.

‡ Bath, Proc. Camb. Phil. Soc., 1939, p. 518.

§ Miquel, Jour. de Math., 1838, p. 485, announced important properties of the 4- and 5-lines. He did not make explicit reference to these circles.

Note that when $n=4$, the Miquel circle coincides with the Morley circle.

6. Hermes points. Consider the circles $\mathfrak{C}_{1234}$ and $\mathfrak{C}_{123}$.

$$(1234)z\bar{z}-z\sum k_1 1^5(234)+\bar{z}1234\sum k_1 234(234)=-1234\sum k_1k_2 34(34)\{1^4-2^4\}$$
$$(123)z\bar{z}-z\sum k_1 1^3(23)-\bar{z}123\sum k_1 23(23)=123\sum k_1k_2 3\{1^2-2^2\}.$$

Eliminating $\bar{z}$,

$$\frac{(1234)z+1234\sum k_1234(234)}{(123)z-123\sum k_123(23)}=\frac{z\sum k_1 1^5(234)-1234\sum k_1k_2 34(34)\{1^4-2^4\}}{z\sum k_1 1^3(23)+123\sum k_1k_2 3\{1^2-2^2\}}.$$

This reduces to

$$[z\sum k_1 1^3(234)-1234\sum k_1k_2 34(34)(12)]$$
$$[z4^2(123)-123\sum k_123(23)\{1^2+4^2\}]=0.$$

The Miquel circle of the 4-line thus intersects the Morley circle of the 3-line obtained by omitting l_4 in the de Longchamps point of the 4-line and also in the point given by

$$\frac{123\sum k_123(23)\{1^2+4^2\}}{4^2(123)}.$$

We shall call this one of the Hermes points of the 4-line and denote it by $H_{4:123}$. There are four Hermes points of the 4-line, $H_{4:123}$, $H_{3:124}$, $H_{2:134}$, $H_{1:234}$.*

We shall now consider the point $H_{f:\,n!/f}$ which we shall take to be

$$\frac{(-1)^n n!/f\sum k_1 n!/1f(n!/1f)\{1^2+f^2\}}{f^2(n!/f)}.$$

We wish to show that this represents a Hermes point of the n-line, which we define as the intersection other than $K_{n!}$ of the Morley circle of the $\overline{n-1}$-line $\mathfrak{C}_{n!/f}$, and the Miquel circle of the n-line.

We take z_1 to be the value just given. Then

$$\bar{z}_1=\sum k_1 1^{2n-7}(n!/1f)\{1^2+f^2\}/(n!/f).$$

For z_2 and $\bar{z}_2$ we take the values for $C_{n!/f}$ given in section 2. Then

$$z_1-z_2=\frac{(-1)^n\{n!/f\}^2\sum k_1(n!/1f)}{f^2(n!/f)};\quad \bar{z}_1-\bar{z}_2=\frac{f^2\sum k_1 1^{2n-7}(n!/1f)}{(n!/f)}.$$

Then the square of the distance between $H_{f:\,n!/f}$ and $C_{n!/f}$ is

$$\frac{(-1)^n\{n!/f\}^2\sum k_1(n!/1f)\cdot\sum k_1 1^{2n-7}(n!/1f)}{(n!/f)^2}.$$

* Hermes, Nouv. Ann., 1859, p. 359.

But this is the square of the radius of $\mathfrak{C}_{n!/f}$. Hence $H_{f:n!/f}$ lies on this circle.

Again, let z_1 and $\bar{z}_1$ have the same values as above, but give z_2 and $\bar{z}_2$ the values given in section 2 for $C_{n!}$. Now

$$z_1 - z_2 = \frac{(-1)^n\{n!/f\}^2 \sum k_1 1^3(n!/1)}{f^2(n!)}; \qquad \bar{z}_1 - \bar{z}_2 = \frac{-f^4 \sum 1^{2n-7}(n!/1)}{(n!)}.$$

Then the square of the distance between $H_{f:n!/f}$ and $C_{n!}$ is

$$\frac{(-1)^{n-1}\{n!\}^2 \sum k_1 1^3(n!/1) \cdot \sum k_1 1^{2n-7}(n!/1)}{(n!)^2}.$$

But this is the value obtained in section 5 for the square of the radius of the Miquel circle with center $C_{n!}$. Hence $H_{f:n!/f}$ gives the n Hermes points of the n-line by taking $f=1, 2, \cdots, n$.

7. The new chain of circles. We are now ready to set up the new chain of circles which it is the main purpose of this paper to announce.

These circles have $H_{f:n!/f}$ for center and pass through $K_{n!/f}$.

We must first find the square of the radius of this circle. Here z_1 and $\bar{z}_1$ have the values given in the last section for $H_{f:n!/f}$ and

$$z_2 = \frac{(-1)^{n-1}n!/f \sum k_1k_2 n!/12f(n!/12f)\{1^{2n-8} - 2^{2n-8}\}}{\sum k_1 1^{2n-7}(n!/1f)}$$

$$\bar{z}_2 = \frac{-\sum k_1k_2 12(n!/12f)\{1^{2n-8} - 2^{2n-8}\}}{\sum k_1 1(n!/1f)}.$$

Then

$$z_1 - z_2 = \frac{(-1)^n\{n!/f\}^2 \sum k_1 1(n!/1f) \cdot \sum k_1 1^{2n-9}(n!/1f)\{1^2 + f^2\}}{f^2(n!/f) \sum k_1 1^{2n-7}(n!/1f)};$$

and

$$\bar{z}_1 - \bar{z}_2 = \frac{\sum k_1 1^{2n-7}(n!/1f) \cdot \sum k_1 1(n!/1f)\{1^2 + f^2\}}{(n!/f) \sum k_1 1(n!/1f)}.$$

Then the square of the radius is

$$\frac{(-1)^n\{n!/f\}^2 \sum k_1 1(n!/1f)\{1^2 + f^2\} \cdot \sum k_1 1^{2n-9}(n!/1f)\{(1^2 + f^2)\}}{f^2(n!/f)^2}.$$

We shall now show that this circle also passes through the points $H_{f:n!/fg}$, where g is any value from 1 to n other than f.

Here z_1 and $\bar{z}_1$ have the values used above, while

$$z_2 = \frac{(-1)^{n-1}n!/fg \sum k_1 n!/1fg(n!/1fg)\{1^2 + f^2\}}{f^2(n!/fg)}$$

$$\bar{z}_2 = \frac{\sum k_1 1^{2n-9}(\mathbf{n}!/1\mathbf{fg})\{1^2 + \mathbf{f}^2\}}{(\mathbf{n}!/\mathbf{fg})}.$$

$$z_1 - z_2 = \frac{(-1)^n\{\mathbf{n}!/\mathbf{fg}\}^2 \sum k_1 1(\mathbf{n}!/1\mathbf{f})\{1^2 + \mathbf{f}^2\}}{\mathbf{f}^2(\mathbf{n}!/\mathbf{f})};$$

$$\bar{z}_1 - \bar{z}_2 = \frac{\mathbf{g}^2 \sum k_1 1^{2n-9}(\mathbf{n}!/1\mathbf{f})\{1^2 + \mathbf{f}^2\}}{(\mathbf{n}!/\mathbf{f})}.$$

And the square of the distance between $H_{f:\,n!/f}$ and $H_{f:\,n!/fg}$ turns out to be the same as that given above. Hence we have proved

THEOREM 1. *Each of the n Hermes points of an n-line, such as $H_{f:\,n!/f}$, f being any integer from 1 to n (where the Miquel circle whose center is $C_{n!}$ is intersected by the Morley circle $\mathfrak{C}_{n!/f}$), is the center of a circle which passes through the de Longchamps point $K_{n!/f}$ and also through $\overline{n-1}$ Hermes points $H_{f:\,n!/1f}$, $H_{f:\,n!/2f}$, $\cdots$, $H_{f:\,n!/fn}$.*

Using these methods we may show that the equation of the circle with $H_{f:\,n!/f}$ as center and passing through $K_{n!/f}$ is

$$(\mathbf{n}!/\mathbf{f})\mathbf{f}^2 z\bar{z} - z\mathbf{f}^2 \sum k_1 1^{2n-7}(\mathbf{n}!/1\mathbf{f})\{1^2+\mathbf{f}^2\} - \bar{z}(-1)^n \mathbf{n}!/\mathbf{f} \sum k_1 \mathbf{n}!/1\mathbf{f}(\mathbf{n}!/1\mathbf{f})\{1^2+\mathbf{f}^2\}$$
$$= (-1)^n \mathbf{n}!/\mathbf{f} \sum k_1 k_2 \mathbf{n}!/12\mathbf{f}(\mathbf{n}!/12\mathbf{f})\{1^{2n-8} - 2^{2n-8}\}\{1^2+\mathbf{f}^2\}\{2^2+\mathbf{f}^2\}.$$

In the same way, the circle with $H_{g:\,n!/g}$ as center passing through $K_{n!/g}$ may be written down by replacing **f** by **g** in the above.

Multiplying the first of these equations by $\mathbf{g}^2(1\mathbf{f})(2\mathbf{f}) \cdots (\mathbf{fn})$ and the second by $\mathbf{f}^2(1\mathbf{g})(2\mathbf{g}) \cdots (\mathbf{gn})$ and subtracting, we obtain the equation of the radical axis of these two circles to be

$$z\mathbf{f}^2\mathbf{g}^2 \sum k_1 1^{2n-7}(\mathbf{n}!/1) - \bar{z}(-1)^n\{\mathbf{n}!/\mathbf{fg}\}^2 \sum k_1 1^3(\mathbf{n}!/1)$$
$$= (-1)^{n-1}\mathbf{n}!/\mathbf{fg} \sum k_1 k_2 \mathbf{n}!/12\mathbf{fg}(\mathbf{n}!/12)\{1^{2n-8} - 2^{2n-8}\}\{1^4 2^4 - \mathbf{f}^4\mathbf{g}^4\}.$$

Similarly, the radical axis of the circles of this system with centers $H_{f:\,n!/f}$ and $H_{h:\,n!/h}$ can be written down by substituting **h** for **g** in the above expression.

Eliminating $\bar{z}$ between these two equations, we find that

$$z = \frac{(-1)^n \mathbf{n}! \sum k_1 k_2 \mathbf{n}!/12(\mathbf{n}!/12)\{1^{2n-8} - 2^{2n-8}\}}{\sum k_1 1^{2n-7}(\mathbf{n}!/1)}.$$

Since this is independent of **f**, **g** and **h**, it follows that

THEOREM 2. *The n circles of Theorem 1 form a system which has a common radical center.*

8. The 5-line. In the special case $n = 5$, the radical center becomes

$$\frac{-12345\sum k_1k_2345(345)\{1^2-2^2\}}{\sum k_11^3(2345)}.$$

But this is the value of F_{12345} given in section 4.

THEOREM 3. *The radical center of the five new circles of the* 5*-line is the center of the Clifford circle of the* 5*-line.*

Moreover it may be proved that

THEOREM 4. *The Clifford circle of the* 5*-line is orthogonal to each of the five new circles of the* 5*-line.*

It may also be proved laboriously that:

THEOREM 5. *The inverse points of the de Longchamps point of the* 5*-line with respect to each of its five new circles all lie on the Clifford circle of the* 5*-line.*

The equation of the straight line $A_{12}C_{124}$ is easily shown to be

$$4^2z - 1^2 2^2\bar{z} = -\frac{12[k_12\{1^2+4^2\} - k_21\{2^2+4^2\}]}{(12)}.$$

This may be proved to pass through $H_{4:123}$. Similarly $A_{13}C_{134}$ and $A_{23}C_{234}$ pass through $H_{4:123}$. This theorem of Hermes is the reason for giving the name of Hermes points to this set of points.

COROLLARY. $A_{12}C_{123}$ *passes through* $H_{3:124}$ and $H_{3:125}$.
$A_{12}C_{124}$ *passes through* $H_{4:123}$ *and* $H_{4:125}$.
$A_{12}C_{125}$ *passes through* $H_{5:123}$ *and* $H_{5:124}$.

There are three such lines passing through each of the ten vertices of the 5-line.

Again the equation of the straight line $H_{4:123}\,H_{5:123}$ is

$$4^2 5^2 z + 1^2 2^2 3^2\bar{z} - \frac{123\sum k_123(23)\{1^2+4^2\}\{1^2+5^2\}}{(123)};$$

similarly the equation of the straight line $H_{3:124}H_{5:124}$ is

$$3^2 5^2 z + 1^2 2^2 4^2\bar{z} = \frac{124\sum k_124(24)\{1^2+3^2\}\{1^2+5^2\}}{(124)}.$$

These two lines intersect at $-1234\sum k_1\ 234(234)\{1^2+5^2\}/(1234)$, which is $H_{5:1234}$.

It may be shown that $H_{5:134}H_{2:134}$ and $H_{5:234}H_{1:234}$ also pass through $H_{5:1234}$. This may be generalized as follows.

THEOREM 6. *Through* $H_{f:5\,1/f}$ *pass four straight lines joining* $H_{f:5\,1/fg}$ *and* $H_{g:5\,1/fg}$, *where* f *has any of the values* 1, 2, 3, 4 *or* 5, *and* g *any of these values other than that taken for* f.

COROLLARY. *$H_{f:\,5^1/f_\theta}$ and $H_{\theta:\,5^1/f_\theta}$ lie on the line of centers of the new circles with centers $H_{f:\,5^1/f}$ and $H_{\theta:\,5^1/\theta}$.*

9. The 6-line. In the special case $n=6$, the radical center becomes

$$\frac{123456\sum k_1k_23456(3456)\{1^4-2^4\}}{\sum k_11^5(23456)}.$$

It is easily proved by the methods used above that this point lies on each of the six new circles of the 6-line and also on the Morley circle of the 6-line.

THEOREM 7. *The six new circles of the 6-line intersect at a common point, which also lies on the Morley circle of the 6-line.*

Again, $H_{1:2345}H_{2:1345}$ is

$$1^22^2z+3^24^25^2\bar{z}=\frac{345\sum k_345(45)\{1^2+3^2\}\{2^2+3^2\}}{(345)};$$

and $H_{1:2346}H_{2:1346}$ is

$$1^22^2z+3^24^26^2\bar{z}=\frac{346\sum k_346(46)\{1^2+3^2\}\{2^2+3^2\}}{(346)}.$$

These intersect at

$$\frac{-3456\sum k_3456(456)\{1^2+3^2\}\{2^2+3^2\}}{1^22^2(3456)}.$$

Evidently $H_{1:2356}H_{2:1356}$ and $H_{1:2456}H_{2:1456}$ also pass through this point. But this point may also be proved to lie on the new circles with $H_{1:23456}$ and $H_{2:13456}$ as centers. Hence

THEOREM 8. *In a 6-line the two new circles with centers $H_{1:23456}$ and $H_{2:13456}$ meet in a point other than that common to all six circles and through this point pass also the four lines $H_{1:2345}H_{2:1345}$, $H_{1:2346}H_{2:1346}$, $H_{1:2356}H_{2:1356}$, $H_{1:2456}H_{2:1456}$.*

This may also be generalized.

It was proved in my previous article* that $A_{12}K_{1245}$, $A_{13}K_{1345}$, $A_{23}K_{2345}$ intersect at a point which we shall call $X_{123,45}$, which is on the Clifford circle of the 5-line. There are ten such points on this circle.

Similarly, $A_{12}K_{1246}$, $A_{13}K_{1346}$, $A_{23}K_{2346}$ intersect at $X_{123,46}$; and $A_{12}K_{1256}$, $A_{13}K_{1356}$, $A_{23}K_{2356}$ intersect at $X_{123,56}$.

Now in the Clifford circle with center F_{12345} which passes through F, the Clifford point of the 6-line,

$$\angle K_{1234}FX_{123,45}=\angle K_{1234}K_{1245}X_{123,45}=\angle K_{1234}K_{1245}A_{12}.$$

And in the Clifford circle with center F_{12346},

* *Loc. cit.*

$$\angle K_{1234}FX_{123,46} = \angle K_{1234}K_{1246}X_{123,46} = \angle K_{1234}K_{1246}A_{12}.$$

But A_{12}, K_{1234}, K_{1245}, K_{1246} all lie on the circle $\mathfrak{C}_{124}$. Hence $\angle K_{1234}FX_{123,45} = \angle K_{1234}FX_{123,46}$. Therefore $X_{123,45}X_{123,46}$ passes through F. Similarly $X_{123,56}$ may be proved to lie on this line. We thus obtain

THEOREM 9. *The sixty points on the* 6 *Clifford circles of the* 5*-lines obtained by omitting in turn each of the sides of the* 6*-line, lie three by three on twenty straight lines all of which pass through the Clifford point of the* 6*-line.*

The analytic proof of this theorem is very laborious. Lachlan, in the preface to his *Modern Pure Geometry** said: "Although Analysis may be more powerful as an instrument of research, . . . it might well be taken as an axiom, based upon experience, that every geometrical theorem admits of a simple and direct proof—by the principles of Pure Geometry." If the writer had possessed enough acumen to develop his other theorems by such methods, a good deal of space might have been saved.

* London, 1893.

THE SOLUTION OF A SPECIAL HEAT AND DIFFUSION EQUATION

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1. Introduction. The solution of the problem discussed here has been used to obtain results of some importance to geophysics [1–3]. Although the derivation has little mathematical novelty, it has never been published, and may prove interesting, if only as a slightly out-of-the-ordinary example for differential equations courses.

The equation to be considered arises in connection with the problem of finding temperatures and heat flows due to radioactivity within a sphere. It is shown in [2] that, assuming spherical symmetry, constancy of the thermal conductivity, *etc.*, the required solution may be obtained by the superposition of a number of solutions of the same general type, namely, solutions of the equations

$$\rho c \frac{\partial \theta}{\partial t} = K\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r}\right) + \begin{cases} 0, & 0 \leqq r \leqq r_1, \\ H_0 e^{-\lambda t}, & r_1 \leqq r \leqq a, \end{cases} \tag{1}$$

with boundary conditions

$$\begin{aligned} &\theta(a, t) = 0, \\ &\theta(0, t) < \infty, \\ &\theta(r, 0) = 0. \end{aligned} \tag{2}$$

Call the temperatures in the regions $0 \leqq r \leqq r_1$ and $r_1 \leqq r \leqq a$ θ_1 and θ_2 respectively, and let

$$u_1 = r\theta_1, \tag{3}$$

$$u_2 = r\theta_2. \tag{4}$$

Write k for $K/\rho c$ and J for $H_0/\rho c$. Then the problem is to solve the equations

$$\frac{\partial u_1}{\partial t} = k\frac{\partial^2 u_1}{\partial r^2}, \tag{5}$$

$$\frac{\partial u_2}{\partial t} = k\frac{\partial^2 u_2}{\partial r^2} + Jre^{-\lambda t} \tag{6}$$

subject to the conditions

$$\begin{aligned} u_1(0, t) &= 0, \qquad u_1(r, 0) = 0, \\ u_1(r_1, t) &= u_2(r_1, t), \\ \frac{\partial}{\partial r}u_1(r_1, t) &= \frac{\partial}{\partial r}u_2(r_1, t), \\ u_2(a, t) &= 0, \qquad u_2(r, 0) = 0. \end{aligned}$$

The most obvious method of solution is to apply the Laplace Transform, and this will be given here. A second method, which gives the final solution in a slightly neater form, will be discussed a little later.

2. Solution using the Laplace transform.* The subsidiary equations obtained by applying the Laplace transformation to (5), (6), and (7) are

$$p\bar{u}_1 = k\frac{d^2\bar{u}_1}{dr^2}, \tag{8}$$

$$p\bar{u}_2 = k\frac{d^2\bar{u}_2}{dr^2} + \frac{Jr}{p+\lambda}, \tag{9}$$

where

$$\begin{aligned} \bar{u}_1(0) &= 0, \qquad \bar{u}_2(a) = 0, \\ \bar{u}_1(r_1) &= \bar{u}_2(r_1), \\ \frac{d}{dr}\bar{u}_1(r_1) &= \frac{d}{dr}\bar{u}_2(r_1). \end{aligned} \tag{10}$$

The general solution of (8) is

$$\bar{u}_1 = c_1e^{qr} + c_2e^{-qr}, \tag{11}$$

* The notation used is the same as that in Carslaw and Jaeger: Conduction of Heat in Solids, Oxford, 1947.

where q stands for $\sqrt{p/k}$. The boundary condition $\bar{u}_1(0)=0$ will be satisfied if this is taken in the form

$$\bar{u}_1 = A \sinh qr. \tag{12}$$

Similarly, the general solution of (9) is

$$\bar{u}_2 = B \sinh qr + C \cosh qr + \frac{rJ}{p(p+\lambda)} \cdot \tag{13}$$

In order for the remaining boundary conditions (10) to be satisfied, the following equations must hold:

$$B \sinh qa + C \cosh qa + \frac{aJ}{p(p+\lambda)} = 0, \tag{14}$$

$$B \sinh qr_1 + C \cosh qr_1 + \frac{r_1 J}{p(p+\lambda)} = A \sinh qr_1, \tag{15}$$

$$Bq \cosh qr_1 + Cq \sinh qr_1 + \frac{J}{p(p+\lambda)} = Aq \cosh qr_1. \tag{16}$$

Hence,

$$\begin{aligned} A &= -\frac{J}{p(p+\lambda)\sinh qa}\left(a - r_1 \cosh q(a-r_1) - \frac{\sinh q(a-r_1)}{q}\right), \\ B &= -\frac{J}{p(p+\lambda)\sinh qa}\left(a - r_1 \cosh qa \cosh qr_1 + \frac{\cosh qa \sinh qr_1}{q}\right) \\ C &= \frac{J}{p(p+\lambda)}\left(\frac{\sinh qr_1}{q} - r_1 \cosh qr_1\right). \end{aligned} \tag{17}$$

Thus the solutions of (8) and (9) satisfying (10) are

$$\bar{u}_1 = \frac{J \sinh qr}{p(p+\lambda)\sinh qa}\left(-a + r_1 \cosh q(a-r_1) + \frac{\sinh q(a-r_1)}{q}\right), \tag{18}$$

$$\begin{aligned} \bar{u}_2 = \frac{J}{p(p+\lambda)}\Bigg[r - \frac{\sinh qr}{\sinh qa}\left(a - r_1 \cosh qa \cosh qr_1 + \frac{\sinh qr_1 \cosh qa}{q}\right) \\ + \cosh qr\left(\frac{\sinh qr_1}{q} - r_1 \cosh qr_1\right)\Bigg]. \end{aligned} \tag{19}$$

The solutions for u_1 and u_2 may now be obtained by inverting the above solutions, using the Fourier-Mellin theorem. The calculation of residues involved is tedious, but elementary, and need not be given here. The poles of the functions are at the points $p=0$, $p=-\lambda$, and $p=-m^2\pi^2k/a^2$, for $m=0, 1, 2, \cdots$.

Thus the solution obtained is in each case in the form of a single first term plus an infinite series, *viz.*:

$$\frac{u_1}{J} = \frac{ae^{-\lambda t}}{\lambda \sin \eta a}\left[\sin \eta r\left(1 - \frac{r_1}{a}\cos \eta(a - r_1) - \frac{1}{\eta a}\sin \eta(a - r_1)\right)\right] \tag{20}$$

$$- \frac{2a^3}{\pi k}\sum_{m=1}^{\infty} e^{-(m^2\pi^2 k/a^2)t}\cdot \sin\frac{m\pi r}{a}\cdot\left[\frac{(-1)^m - \frac{r_1}{a}\cos\frac{m\pi r_1}{a} + \frac{1}{m\pi}\sin\frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)}\right],$$

$$\frac{u_2}{J} = \frac{ae^{-\lambda t}}{\lambda \sin \eta a}\left[\sin \eta r - \frac{r}{a}\sin \eta a + \left(\frac{r_1}{a}\cos \eta r_1 - \frac{1}{\eta a}\sin \eta r_1\right)\sin \eta(a - r)\right] \tag{21}$$

$$- \frac{2a^3}{\pi k}\sum_{m=1}^{\infty} e^{-(m^2\pi^2 k/a^2)t}\cdot \sin\frac{m\pi r}{a}\cdot\left[\frac{(-1)^m - \frac{r_1}{a}\cos\frac{m\pi r_1}{a} + \frac{1}{m\pi}\sin\frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)}\right]$$

where η stands for $\sqrt{\lambda/k}$.

The solution for θ is therefore*

$$\frac{\theta}{J} = \frac{ae^{-\lambda t}}{r\lambda \sin \eta a}\cdot\begin{cases}\sin \eta r\left(1 - \frac{r_1}{a}\cos \eta(a - r_1) - \frac{1}{\eta a}\sin \eta(a - r_1)\right) & \text{for } 0 \leqq r \leqq r_1 \\ \sin \eta r - \frac{r}{a}\sin \eta a + \left(\frac{r_1}{a}\cos \eta r_1 - \frac{1}{\eta a}\sin \eta r_1\right)\sin \eta(a - r) & \text{for } r_1 \leqq r \leqq a\end{cases} \tag{22}$$

$$- \frac{2a^3}{\pi kr}\sum_{m=1}^{\infty} e^{-(m^2\pi^2 k/a^2)t}\cdot \sin\frac{m\pi r}{a}\cdot\left[\frac{(-1)^m - \frac{r_1}{a}\cos\frac{m\pi r_1}{a} + \frac{1}{m\pi}\sin\frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)}\right].$$

3. Second method of solution. The second method of solution† is simply to try a solution of equations (5) and (6) of the form

$$u_1' = R_1(r)e^{-\lambda t}, \tag{23}$$

$$u_2' = R_2(r)e^{-\lambda t}. \tag{24}$$

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* This solution was first obtained by G. Comenetz [1]. That for the simpler problem with $r_1 = 0$ has been worked out independently by H. C. Urey, and is given in his book, The Planets, p. 52. Both can be obtained quite easily from a very general solution by A. N. Lowan, Phys. Rev., vol. 44, 1933, p. 769.

† This was kindly suggested to me by Professor G. F. Duff.

Then R_1 and R_2 satisfy

$$\frac{d^2R_1}{dr^2} + \eta^2 R_1 = 0, \tag{25}$$

$$\frac{d^2R_2}{dr^2} + \eta^2 R_2 + Jr = 0, \tag{26}$$

where

$$\begin{gathered} R_1(0) = 0, \qquad R_1(a) = 0, \\ R_1(r_1) = R_2(r_1), \\ \frac{dR_1(r_1)}{dr} = \frac{dR_2(r_1)}{dr}. \end{gathered} \tag{27}$$

These equations can be solved in much the same manner as equations (8) and (9), giving

$$R_1 = \frac{Ja \sin \eta r}{\lambda \sin \eta a}\left(1 - \frac{r_1}{a}\cos \eta(a - r_1) - \frac{1}{\eta a}\sin \eta(a - r_1)\right) \tag{28}$$

$$R_2 = \frac{Ja}{\lambda \sin \eta a}\left(\sin \eta r - \frac{r}{a}\sin \eta a + \left(\frac{r_1}{a}\cos \eta r_1 - \frac{1}{\eta a}\sin \eta r_1\right)\sin \eta(a - r)\right). \tag{29}$$

The corresponding solutions u_1' and u_2' satisfy (5) and (6) and all of the boundary conditions (7) save the two

$$u_1(r, 0) = 0, \qquad u_2(r, 0) = 0.$$

However, this is easily corrected, for if

$$u = u' + U, \tag{30}$$

where

$$u' = \begin{cases} u_1' & 0 \leqq r \leqq r_1 \\ u_2' & r_1 \leqq r \leqq a, \end{cases} \tag{31}$$

then in order for u to satisfy (5), (6), and (7) completely, U must satisfy

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial r^2}, \tag{32}$$

where

$$\begin{gathered} U(a, t) = 0, \qquad U(0, t) = 0, \\ U(r, 0) = -u'(r, 0). \end{gathered} \tag{33}$$

This of course is a standard heat conduction problem, and the solution is

$$U = \sum_{m=1}^{\infty} A_m e^{-(m^2\pi^2 k/a^2)t} \sin \frac{m\pi r}{a}, \tag{34}$$

where

$$A_m = \frac{2}{a} \int_0^a U(r, 0) \sin \frac{m\pi r}{a}\, dr. \tag{35}$$

The calculation of the A_m is fairly easy, and the final solution obtained is the same as (22), where now the summation term corresponds to U and the first term to u'.

However, this method of solution points to a neater and better way of writing (22), for from (33)

$$u'(r, 0) = -U(r, 0).$$

But

$$\begin{aligned} u'(r, t) &= u'(r, 0)e^{-\lambda t} \\ &= -U(r, 0)e^{-\lambda t} \\ &= \frac{2a^3}{\pi k r} \sum_{m=1}^{\infty} e^{-\lambda t} \cdot \sin \frac{m\pi r}{a} \cdot \left[\frac{(-1)^m - \frac{r_1}{a} \cos \frac{m\pi r}{a} + \frac{1}{m\pi} \sin \frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)} \right]. \end{aligned} \tag{36}$$

Hence the final solution can be written

$$\begin{aligned} \frac{\theta}{J} = \frac{2a^3}{\pi r} \sum_{m=1}^{\infty} &(e^{-\lambda t} - e^{-(m^2\pi^2 k/a^2)t} \cdot \sin \frac{m\pi r}{a} \\ &\cdot \left[\frac{(-1)^m - \frac{r_1}{a} \cos \frac{m\pi r_1}{a} + \frac{1}{m\pi} \sin \frac{m\pi r_1}{a}}{m(a^2\lambda - m^2\pi^2 k)} \right] \end{aligned} \tag{37}$$

(remembering $\eta^2 = \lambda/k$).

4. Remarks. The heat flow can at once be found from (37), *viz.*

$$\begin{aligned} F(r, t) &= -K \frac{\partial \theta}{\partial r} \\ &= \frac{2a^3 H_0 k}{\pi r} \sum_{m=1}^{\infty} (e^{-\lambda t} - e^{-(m^2\pi^2 k/a^2)t}) \cdot \left(\frac{1}{r} \sin \frac{m\pi r}{a} - \frac{m\pi}{a} \cos \frac{m\pi r}{a} \right) \\ &\quad \cdot \left[\frac{(-1)^m - \frac{r_1}{a} \cos \frac{m\pi r_1}{a} + \frac{1}{m\pi} \sin \frac{m\pi r_1}{a}}{m(a^2\lambda - m^2\pi^2 k)} \right]. \end{aligned} \tag{38}$$

The average temperature is also of interest. This is given by

$$\bar{\theta} = \frac{3}{4\pi a^3}\int_0^a 4\pi r^2\theta\,dr$$

$$= \frac{6a^2J}{\pi^2}\sum_{m=1}^{\infty}(e^{-\lambda t} - e^{-(m^2\pi^2k/a^2)t})(-1)^{m+1}$$

$$\cdot\left[\frac{(-1)^m - \frac{r_1}{a}\cos\frac{m\pi r_2}{a} + \frac{1}{m\pi}\sin\frac{m\pi r_1}{a}}{m(a^2\lambda - m^2\pi^2k)}\right].$$

The behaviour of these solutions is quite complicated. Some results for the special case of the Earth are given in the references; [3] contains a discussion of the reasons for using this solution, and the method of applying it to fairly complicated distributions of radioactivity.

5. Application to the diffusion of argon. An interesting application of the above solution is to the problem of the diffusion of a gas being produced within a solid by radioactive decay.* The differential equation which applies is (using the notation for the case of the gas argon-40 being produced by the decay of potassium-40)

$$\frac{\partial A_c^{40}}{\partial t} = D\nabla^2 A_c^{40} + \lambda_k K_0^{40} e^{-\lambda t} \tag{40}$$

where A_c^{40} denotes the concentration of argon-40, K_0^{40} the initial concentration of potassium-40, and λ and λ_k are constants.

The solution required is that for the *average* argon content at time t (denoted by A^{40}) of a spherical solid with the potassium distributed homogeneously throughout, and with zero argon content at $t=0$ and zero surface concentration of argon (*i.e.*, similar boundary conditions to those of section 1). This solution is given by setting $r_1=0$ in (39), and substituting A^{40} for $\bar{\theta}$, D for k, and $\lambda_k K_0^{40}$ for J. Thus,

$$\frac{A^{40}}{\lambda_k K_0^{40}} = \frac{6a^2}{\pi^2}\sum_{m=1}^{\infty}\frac{e^{-\lambda t} - e^{-(m^2\pi^2D/a^2)t}}{m^2(m^2D\pi^2 - a^2\lambda)}, \tag{41}$$

or, writing K^{40} for $K_0^{40}e^{-\lambda t}$, c for λ_k/λ, and s for $a\sqrt{\lambda/D}$,

$$\frac{1}{c}\frac{A^{40}}{K^{40}} = \frac{6}{\pi^2}\sum_{m=1}^{\infty}\frac{1 - e^{-[(m^2\pi^2/s^2)-1]\lambda t}}{m^2\left(m^2\frac{\pi^2}{s^2} - 1\right)}. \tag{42}$$

The diagram shows the behaviour of this function for various values of s. (The

* This was pointed out to me by Dr. H. A. Shillibeer, who developed the results independently in connection with his researches on the potassium-argon method of dating minerals. The diagram and most of this section are taken from [4], with his kind permission.

λ used is the decay constant for potassium-40, $.55\times10^{-9}$ years^{-1}) Evidently there are two distinct types of curves. This can be seen from the formula (42), for

$$\lim_{t\to\infty}\left(\frac{1}{c}\frac{A^{40}}{K^{40}}\right)=\begin{cases}\dfrac{6}{\pi^2}\displaystyle\sum_{m=1}^{\infty}\frac{1}{m^2\left(\dfrac{m^2\pi^2}{s^2}-1\right)}=\dfrac{3}{s^2}-\dfrac{3}{s}\cot s-1 & \text{if } s<\pi,\\ +\infty & \text{if } s>\pi.\end{cases}\tag{43}$$

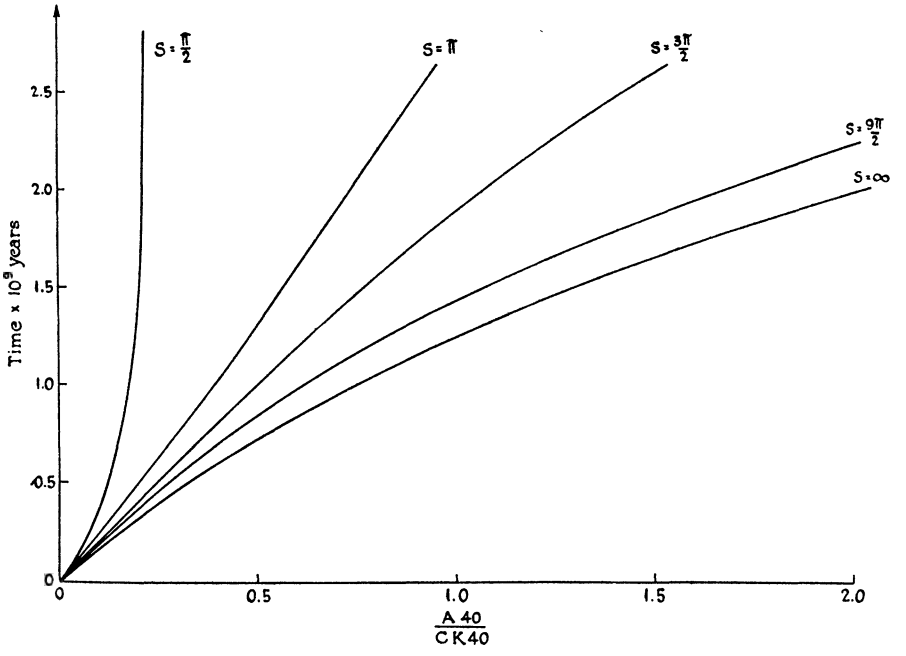
The limiting case $s=\pi$ is given by

$$\frac{1}{c}\frac{A^{40}}{K^{40}}=\frac{6\lambda t}{\pi^2}+\frac{6}{\pi^2}\sum_{m=2}^{\infty}\frac{1-e^{-(m^2-1)\lambda t}}{m^2(m^2-1)},\tag{44}$$

which, as the graph shows, is asymptotic from below to a straight line, *viz.*

$$\frac{1}{c}\frac{A^{40}}{K^{40}}=\left(\frac{21}{2\pi^2}-1\right)+\frac{6\lambda}{\pi^2}t.\tag{45}$$

It may be noted that if s were $<\pi$ in physical fact, it would be almost impossible to use measurements of A^{40}/K^{40} to determine t, the age. However, experimental results obtained for argon seem to show s is large for most mineral samples, so that this age-determination method is practicable.



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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

EVALUATION OF PRODUCT SEQUENCES BY MATRIX METHODS*

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1. Introduction. Let

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k \tag{1}$$

be a regular transformation by means of which a sequence $s_0, s_1, \cdots$, is evaluable to σ if the series in (1) converges for each n and $\sigma_n \to \sigma$ as $n \to \infty$. The hypothesis that (1) is regular means that $\sigma_n \to s$ whenever $s_n \to s$. We shall use the well known conditions

$$\sum_{k=0}^{\infty} |a_{nk}| \leqq M \qquad n = 0, 1, 2, \cdots \tag{2}$$

$$\lim_{n \to \infty} a_{nk} = 0 \qquad k = 0, 1, 2, \cdots \tag{3}$$

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1, \tag{4}$$

which are necessary and sufficient for regularity of (1). An exposition of these matters is given, for example, by Hardy [1].

We shall say that the transformation (1) and the matrix $A = (a_{nk})$ have the product property P if A is real, regular, and such that if s_n and s_n' are real bounded sequences evaluable to σ and σ' respectively, then the product sequence $s_n s_n'$ is evaluable to the product $\sigma\sigma'$.

* A similar problem is considered by Mazur and Orlicz: On linear methods of summation, Studia Math., vol. 14, 1954, pp. 129–160.

2. Sequences of zeros and ones. The following theorem shows that the methods of Cesàro, Abel, and Borel and many other standard methods for evaluation of sequences fail to have property P. Moreover it shows that if A has property P, then relatively few sequences of zeros and ones are evaluable A.

THEOREM 1. *If A has property P and if $s_0, s_1, \cdots$, is a sequence of zeros and ones evaluable to σ, then σ is* 0 *or* 1.

To prove this theorem, we note that if A has property P and if s_n is a sequence of zeros and ones evaluable A to σ, then the sequence $s_n s_n$ (which is the same as the sequence s_n) is evaluable A to σ^2, so that $\sigma=\sigma^2$ and σ is 0 or 1.

We now employ some results of which an exposition has been given by Hill [2]. To each sequence $s_0, s_1, s_2, \cdots$ of zeros and ones there corresponds a point t of the interval $0 \leqq t \leqq 1$ having the binary expansion $s_0 s_1 s_2 \cdots$. If A is regular, then the set of sequences of zeros and ones evaluable A corresponds to a set of values of t having Lebesgue measure 0 or 1, and, moreover, if the measure is 1 then the set of zeros and ones evaluable A to 1/2 corresponds to a set of values of t having measure 1. It follows from this and Theorem 1 that if A has property P, then the set of sequences of zeros and ones evaluable A corresponds to a set of values of t having measure 0.

3. A criterion for property *P*.

THEOREM 2. *A real regular matrix A has property P if and only if s_n^2 is evaluable A to σ^2 whenever s_n is a real bounded sequence evaluable A to σ.*

Let a real regular matrix have property Q if s_n^2 is evaluable A to σ^2 whenever s_n is a real bounded sequence evaluable A to σ. Since we may take $s_n' = s_n$, it is obvious that if A has property P then it also has property Q. To prove the converse, let A have property Q and let s_n and s_n' be real bounded sequences evaluable A to σ and σ' respectively. The real bounded sequences $\{s_n+s_n'\}$ and $\{s_n-s_n'\}$ are then evaluable A to $\sigma+\sigma'$ and $\sigma-\sigma'$ and hence the sequences $\{(s_n+s_n')^2\}$ and $\{(s_n-s_n')^2\}$ are evaluable A to $(\sigma+\sigma')^2$ and $(\sigma-\sigma')^2$. Since $4s_n s_n' = (s_n+s_n')^2-(s_n-s_n')^2$ and $4\sigma\sigma' = (\sigma+\sigma')^2-(\sigma-\sigma')^2$, it follows that $s_n s_n'$ is evaluable A to $\sigma\sigma'$ and Theorem 2 is proved. A variant of Theorem 2 which may be useful is the following.

THEOREM 3. *A real regular matrix A has property P if and only if s_n^2 is evaluable A to* 0 *whenever s_n is evaluable A to* 0.

This is a consequence of the fact that if s_n^2 is evaluable A to 0 whenever s_n is evaluable A to 0 and if s_n' is evaluable A to σ' then $\{s_n'-\sigma\}$ is evaluable A to 0, so $\{(s_n'-\sigma)^2\}$ is evaluable A to 0, and linearity of A implies that $(s_n')^2$ is evaluable A to $(\sigma')^2$.

4. Consequences of property *P*.

THEOREM 4. *If a real bounded sequence s_n is evaluable to σ by a matrix having property P, then*

$$\liminf_{n\to\infty} |s_n| \leqq |\sigma| \leqq \limsup_{n\to\infty} |s_n|. \tag{5}$$

To prove Theorem 4, it will be sufficient to prove that

$$\limsup_{n\to\infty} |s_n^2 - x| \geqq |\sigma^2 - x|, \qquad x \geqq 0, \tag{6}$$

because putting $x=0$ in (6) gives the last inequality in (5), and choosing x so great that $x>\sigma^2$ and $x>s_n^2$ for each n gives the first inequality in (5). To prove (6), suppose that there is some real x for which it is false. Then $x\neq\sigma^2$ and there is a number θ for which

$$\limsup_{n\to\infty} \left|\frac{s_n^2 - x}{\sigma^2 - x}\right| < \theta < 1. \tag{7}$$

With M satisfying (2), choose a positive integer m such that $\theta^m M<1$. Let

$$t_n = \frac{s_n^2 - x}{\sigma^2 - x}. \tag{8}$$

The hypothesis of Theorem 4 implies that s_n^2 is evaluable A to σ^2 and hence that s_n^2-x is evaluable A to σ^2-x so that t_n is evaluable A to 1 and t_n^m is evaluable A to 1. Thus

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} t_k^m = 1. \tag{9}$$

Because of (7) and (8), we can choose k_0 such that $|t_k| <\theta$ when $k\geqq k_0$. Use of (9) and (3) then gives

$$\lim_{n\to\infty} \sum_{k=k_0}^{\infty} a_{nk} t_k^m = 1. \tag{10}$$

But use of (2) gives, for each $n=0, 1, 2, \cdots$,

$$\left| \sum_{k=k_0}^{\infty} a_{nk} t_k^m \right| \leqq \theta^m \sum_{k=k_0}^{\infty} |a_{nk}| \leqq \theta^m M < \frac{1}{2}. \tag{11}$$

The contradiction in (10) and (11) establishes (6) and completes the proof of Theorem 4.

THEOREM 5. *If a real bounded sequence s_n is evaluable to σ by a matrix A having property P, then σ is a cluster value (or limit point) of the sequence s_n.*

The hypothesis implies that the sequence $s_0-\sigma$, $s_1-\sigma$, $\cdots$ is evaluable A to 0 and hence Theorem 4 implies that

$$\liminf_{n\to\infty} |s_n - \sigma| \leqq 0 \leqq \limsup_{n\to\infty} |s_n - \sigma|.$$

The result follows.

5. Conclusion. For the following remarks, the author is indebted to R. P. Agnew. The transformation defined by $\sigma_0=s_0$, $\sigma_1=s_1$, and

$$(12)\qquad \begin{cases} \sigma_{2n} = -2s_{3n-1} + 3s_{3n} & n > 0, \\ \sigma_{2n+1} = -4s_{3n-1} + 5s_{3n} & n > 0, \end{cases}$$

has property P. This is an easy consequence of the fact that $\sigma_n \to L$ if and only if $s_{3n} \to L$ and $s_{3n-1} \to L$, the values of the numbers s_{3n-2} being completely irrelevant. It may be possible to show that each transformation having property P must have a form resembling (12) in essential respects.

The following seems to be an entirely reasonable conjecture. A given matrix and transformation A have property P if and only if there is an increasing sequence $n_1, n_2, n_3, \cdots$ of non-negative integers which depends only upon A and which is such that an arbitrary sequence $s_0, s_1, s_2, \cdots$ is evaluable to L if and only if the special subsequence $s_{n1}, s_{n2}, s_{n3}, \cdots$ converges to L.

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RATIONAL UNIVALENT FUNCTIONS

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Let $\mathfrak{F}$ be the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent in the unit circle $E(|z|<1)$. The subclasses of $\mathfrak{F}$ whose coefficients a_n belong to a quadratic field have been studied by Friedman [1] and Bernardi [2]. Recently Linis [3] gave a short proof of a theorem of Friedman which states that if the a_n are rational integers then $f(z)$ is a rational function and is one of the nine functions:

$$z,\ z/(1 \pm z),\quad z/(1 \pm z^2),\quad z/(1 \pm z)^2,\quad z/(1 \pm z + z^2).$$

He also gave some rational functions whose expansion had Gaussian integers as coefficients.

The purpose of this note is to extend the method of proof given by Linis to quadratic fields with negative discriminant and to give criteria for selecting which combinations of possible coefficients yield rational functions which are regular and univalent in E.

For completeness we will sketch Linis' proof. Let a_n be a rational integer. Let $z/f(z)=\sum_{n=0}^{\infty} b_n z^n$. By Prawitz's theorem [4] $\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leqq 1$. Clearly $b_0=1$, $b_1=-a_2$, and since $f(z)$ is univalent in E, $|a_2| \leqq 2$ so that $|b_1| \leqq 2$. From Prawitz's theorem it follows that $|b_n| \leqq 1$, $n \geqq 2$. The coefficients b_n can be computed by the determinant $(-1)^n b_n=\Delta(C_{ij})$ where $C_{ij}=0$, $j>i+1$; $C_{ij}=1$, $j=i+1$; $C_{ij}=a_{j-i+2}$, $i, j=1, \cdots, n$. Hence the b_n are rational integers. The

possible values of b_n are: $b_0=1$; $b_1=0, \pm 1, \pm 2$; $b_2=0, \pm 1$; $b_n=0$, $n \geqq 3$. Certain combinations of these values yield the nine above-mentioned functions.

Let $R(\sqrt{m})$ be a quadratic field where m is a square-free rational integer. If $m \equiv 2$ or $3 \pmod 4$ the integers of $R(\sqrt{m})$ are given by $a+b\sqrt{m}$, where a and b range over all rational integers. If $m \equiv 1 \pmod 4$ the integers of $R(\sqrt{m})$ are given by $\frac{1}{2}(a+b\sqrt{m})$ where a and b are both even or both odd rational integers. We will be interested only in the cases in which m is one and in which m is negative. The units of $R(\sqrt{m})$ are those numbers whose norm is ± 1. The only units of $R(\sqrt{m})$ where m is negative are ± 1 unless $m=-1, -3$, in which case the additional units are $\pm i$ and $\pm\omega, \pm\bar{\omega}$, respectively, where $\omega=\frac{1}{2}(-1+\sqrt{3}i)$ and $\bar{\omega}$ is the complex conjugate of ω. (For the arithmetic of algebraic number fields see [5], Chap. VII).

Let $a_n \in R(\sqrt{m})$, for a given m, the b_n are given by the previously mentioned determinant, so that $b_n \in (R\sqrt{m})$. Since $R(\sqrt{m})$ has no elements whose modulus is a number between zero and one, it follows from Prawitz's theorem that b_n, $n \geqq 3$, in the expansion of $z/f(z)$ are all zero.

The possible values of b_n are as follows:

$$m = -1; \quad b_0 = 1; \quad b_1 = 0, \pm 1, \pm 2, \pm i, \pm 2i, \pm(1 \pm i); \quad b_2 = 0, \pm 1, \pm i.$$

$$m = -2; \quad b_0 = 1; \quad b_1 = 0, \pm 1, \pm 2, \pm\sqrt{2}i, 1 \pm 2i, \quad b_2 = 0, \pm 1.$$

$$m = -3; \quad b_0 = 1; \quad b_1 = 0, \pm 1, \pm 2, \pm\omega, \pm\bar{\omega}, \pm 2\omega, \pm 2\bar{\omega}, \pm(1-\omega), \pm(1-\bar{\omega}), \pm(\omega-\bar{\omega}); \quad b_2 = 0, \pm 1, \pm\omega, \pm\bar{\omega}.$$

$m<-3$; $b_0=1$, $b_1=0, \pm 1, \pm 2$; $b_2=0, \pm 1$, the same as when $m=1$.

Before enumerating the b_n which yield regular univalent rational functions, let us state two lemmas.

LEMMA 1. *The function $f(z)=z/(1+b_1z)$ is regular and univalent in E if and only if $|b_1| \leqq 1$.*

LEMMA 2. *The function $f(z)=z/(1+b_1z+b_2z^2)$ with $|b_2|=1$ is regular and univalent in E if and only if $b_1/(b_2)^{1/2}$ is real and in modulus less than or equal to 2.*

The proof of Lemma 1 is clear. The univalency in Lemma 2 follows immediately. Since $|b_2|=1$, $f(z)$ will be regular in E if and only if the zeros of the denominator lie on the periphery of E. Let $z=s/(b_2)^{1/2}$, then the denominator is $s^2+[b_1/(b_2)^{1/2}]s+1$. Its zeros are on $|s|=1$ and hence are complex conjugates. Using these properties and the quadratic formula we see that the zeros will be of unit modulus if and only if b_1^2/b_2 is positive and less than or equal to 4 which implies that $b_1/(b_2)^{1/2}$ is real and in modulus less than or equal to 2.

We note that every field $R(\sqrt{m})$ includes the field $R(\sqrt{1})$, hence we will enumerate those b_n not concerned with $R(\sqrt{1})$.

Making use of the following facts: $(\omega)^{1/2}=-\bar{\omega}$, $(-\omega)^{1/2}=i\bar{\omega}$, $(\bar{\omega})^{1/2}=\omega$, $(-\bar{\omega})^{1/2}=-i\omega$, and the properties of the cube roots of unity, we can easily construct a table of values of the b_n which satisfy the two lemmas.

b_2	0	1	-1	i	$-i$	ω	$-\omega$	$\bar{\omega}$	$-\bar{\omega}$
b_1, $m=-1$	$\pm i$	none	$\pm i$, $\pm 2i$	0, $\pm(1+i)$	0, $\pm(1-i)$				
b_1, $m=-2$	none	none	$\pm\sqrt{2}i$	none	none				
b_1, $m=-3$	$\pm\omega$, $\pm\bar{\omega}$	none	$\pm(\omega-\bar{\omega})$	none	none	0, $\pm\bar{\omega}$, $\pm 2\bar{\omega}$	0, $\pm(1-\omega)$	0, $\pm\omega$, $\pm 2\omega$	0, $\pm(1-\bar{\omega})$
b_1, $m<-3$	Same as when $m=1$.								

This proves the following theorem first given by Bernardi [2] by other methods.

THEOREM. *The subclass of $\mathfrak{F}$ having algebraic integral coefficients in $R(\sqrt{m})$, $m=1$ or a square-free rational negative integer, consists of forty five rational functions of the form $f(z)=z/(1+b_1z+b_2z^2)$, where b_1 and b_2 are given by the above table.*

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A CONJECTURE CONCERNING NEIGHBORING TETRAHEDRA

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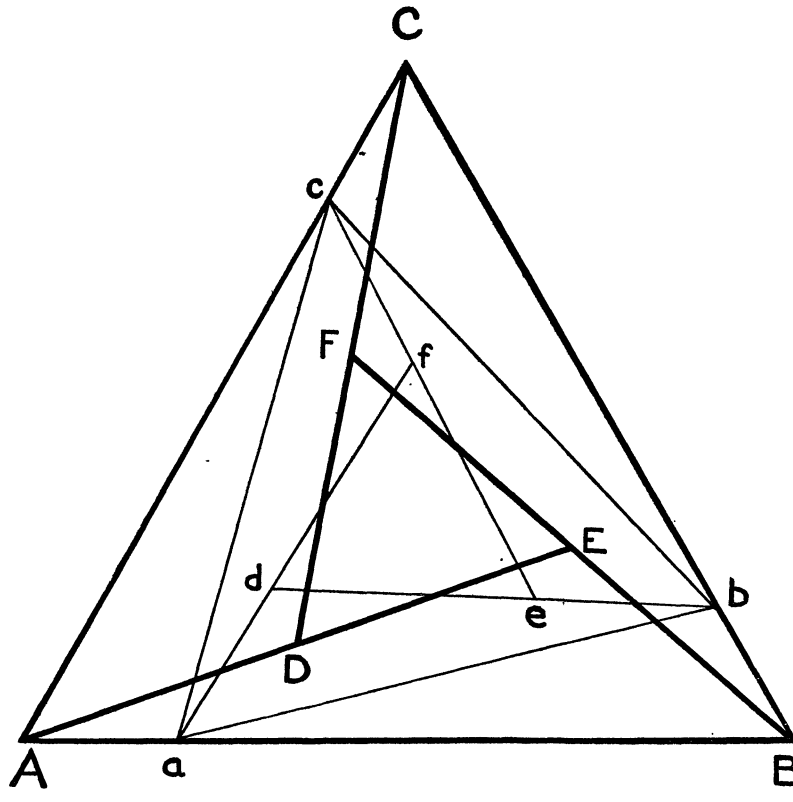
Two bodies in space are said to be neighboring if they have no interior points in common and there is a surface belonging to both their frontiers. A (finite or infinite) number of bodies are said to be neighboring if every pair of them are neighboring.

It is known that it is possible to have infinitely many neighboring bodies—even infinitely many neighboring convex bodies.* This suggests the problem of defining, by means of some property, a class of bodies such that the maximum possible number of neighboring bodies of this class is finite, and of determining this maximum number.

Consider the class of tetrahedra. It is not difficult to verify that at most 4 neighboring tetrahedra can stand with their bases on one and the same plane (and lie, with the exception of these bases, on one and the same side of this plane). Consequently, the maximum possible number of neighboring tetrahedra

* For the literature on this result, as well as an interesting discussion of some related problems, see Heinrich Tietze, Gelöste und ungelöste mathematische Probleme aus alter und neuer Zeit, Munich, Biederstein, 1949.

is certainly not greater than 17, and we conjecture that it is 8. That it is at least 8 is shown by the following example.



In the accompanying figure, all the lines are supposed to be coplanar. Let Q and q be points on opposite sides of the plane ABC. Then the following 8 tetrahedra are obviously neighboring:

$$QABE, \quad QBCF, \quad QCAD, \quad QDEF,$$
$$qabd, \quad qbce, \quad qcaf, \quad qdef.$$

ON ANALYTIC FUNCTIONS HAVING A POSITIVE REAL PART IN THE UNIT CIRCLE

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A theorem of basic importance in the theory of univalent functions is the following:

Theorem. *Let* $f(z)=1+\sum_{n=1}^{\infty} a_n z^n$ *be regular and have* $\mathrm{Re}(f(z))^* \geqq 0$ *in* $|z|<1$. *Then* $|a_n| \leqq 2$, $n=1, 2, \cdots$.

* By Re($f(z)$) we understand the real part of $f(z)$.

The function $f(z)=(1+z)(1-z)^{-1}$ shows that this bound on the coefficients is sharp.

Proofs of the above theorem have been given by Carathéodory [1], Toeplitz [4], Nehari [3] and others. It is the object of this note to give another proof which is elementary in character and would seem to suggest from its structure the theorem to be proven. For this purpose we state the following lemma whose proof depends only on simple manipulations of the Cauchy integral formula. Such a proof may be found in Goursat [2].

LEMMA. *Let* $f(z)=U(x, y)+iV(x, y)$ *be regular in* $|z|\leqq R$. *Let* $z=re^{i\theta}$ *and* $\zeta=Re^{i\psi}$. *Then for* $|z|<R$ *we may write*

$$f(z) = iV(0,0) + \frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\psi})\frac{\xi+z}{\xi-z}\,d\psi. \tag{1}$$

The proof of the above theorem now proceeds as follows. We have

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n,$$

where $a_n=f^{(n)}(0)/n!$. But by (1) above

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{\pi}\int_0^{2\pi} U(Re^{i\psi})\xi^{-n}d\psi$$

where we take $R<1$. Also since $U(0, 0)=1$ the mean value theorem for harmonic functions yields

$$\frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\psi})d\psi = 1.$$

But since $U(x, y)\geqq 0$ it follows that

$$R^n|a_n| \leqq 2\left\{\frac{1}{2\pi}\int_0^{2\pi}|U(Re^{i\psi})|\,d\psi\right\} = 2\left\{\frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\psi})d\psi\right\} = 2\cdot 1 = 2$$

for all $R<1$ and therefore $|a_n|\leqq 2$, $n=1, 2, \cdots$.

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ON THE nTH DERIVATIVE OF COMPOSITE FUNCTIONS

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Let f be the function whose value at x is $f(x)$. Let $\mathbf{D}^n f$ denote the nth derivative of f, that is, the function assuming for any x the value $\mathbf{D}^n f(x)$. If f and g are two functions, then

$$(1_n) \qquad \mathbf{D}^n(f+g) = \mathbf{D}^n f + \mathbf{D}^n g$$

and according to Leibniz

$$(2_n) \qquad \mathbf{D}^n(f\cdot g) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\mathbf{D}^{n-r}f\cdot\mathbf{D}^r g.$$

Let f and g be functions. Throughout this note the juxtaposition of two functions will denote the substitution of the second into the first (hence, the dot will not be omitted in products of functions). For example, $(fg)(x) = (f g(x))$ and $(\mathbf{D}f)g$ is the result of substituting g into $\mathbf{D}f$. The derivative of the function fg is then given by:

$$(3_1) \qquad \mathbf{D}(fg) = (\mathbf{D}f)g\cdot\mathbf{D}g.$$

Formulae for $\mathbf{D}^n(fg)$ have been found by Faà di Bruno and Schlömilch.† The purpose of this note is the establishment of two other formulae for $\mathbf{D}^n(fg)$. The first is

$$(3_n) \qquad \mathbf{D}^n(fg) = \sum_{r=1}^{n} (\mathbf{D}^r f)g\cdot \sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!}\, g^{r-s}\cdot\mathbf{D}^n(g^s).$$

Formula (3_n) gives an explicit form for the operators $\mathbf{Y}_{n,r}$ in the relation

$$(4_n) \qquad \mathbf{D}^n(fg) = \sum_{r=1}^{n} (\mathbf{D}^r f)g\cdot\mathbf{Y}_{n,r}(g)$$

which was used by Schlömilch. It is known that the $\mathbf{Y}_{n,r}$ satisfy the recursion formula:§

$$(5) \qquad \mathbf{Y}_{n+1,r}(g) = \mathbf{D}g\cdot\mathbf{Y}_{n,r-1}(g) + \mathbf{D}\mathbf{Y}_{n,r}(g),$$

where

$$(6) \qquad \mathbf{Y}_{n,0}(g) = 0 = \mathbf{Y}_{n,n+1}(g).$$

Let f be the function assuming at x the value $f(x) = x^k/k!$, where $0 \leqq k \leqq n$. Substituting this f in (4_n) yields a second recursion formula

† See, for example, Charles Jordan, Calculus of Finite Differences, p. 33, Sopron, Hungary, 1939.

§ Riordan, Derivatives of composite functions, Bull. Am. Math. Soc., vol. 52, pp. 664–667, 1946. Also C. Jordan.

$$(7_{n,k}) \qquad \mathbf{Y}_{n,k}(g) = \frac{1}{k!}\mathbf{D}^n(g^k) - \sum_{r=0}^{k-1} \frac{g^{k-r}}{(k-r)!} \cdot \mathbf{Y}_{n,r}(g).$$

To prove (3_n), one must show that

$$(8_{n,r}) \qquad \mathbf{Y}_{n,r}(g) = \sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-s} \cdot \mathbf{D}^n(g^s).$$

The proof is by induction. By (3), $\mathbf{Y}_{1,1}(g) = \mathbf{D}g$. By induction, in view of (6) and (5), $\mathbf{Y}_{n,1}(g) = \mathbf{D}^n g$, which agrees with $(8_{n,1})$. Assume $(8_{n,k})$ for $k = 0, 1, \cdots, r-1$. To prove $(8_{n,r})$, in view of $(7_{n,r})$, one has to show

$$\sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-s} \cdot \mathbf{D}^n(g^s) = \frac{1}{r!}\mathbf{D}^n(g^r) - \sum_{k=0}^{r-1} \frac{g^{r-k}}{(r-k)!} \cdot \sum_{s=0}^{k} \frac{(-1)^{k-s}}{s!(k-s)!} g^{k-s} \cdot \mathbf{D}^n(g^s).$$

The term for $s = r$ on the left equals $\mathbf{D}^n(g^r)/r!$ on the right. For a fixed s, $0 \leqq s \leqq r-1$, the coefficient of $g^{r-s} \cdot \mathbf{D}^n(g^s)$ on the right is

$$-\sum_{k=s}^{r-1} \frac{(-1)^{k-s}}{(r-k)!s!(k-s)!}.$$

Let $k = s+p$, then this becomes

$$-\frac{1}{s!} \sum_{p=0}^{r-s-1} \frac{(-1)^p}{(r-s-p)!p!},$$

which equals

$$-[(1-1)^{r-s} - (-1)^{r-s}]/[s!(r-s)!] = \frac{(-1)^{r-s}}{s!(r-s)!}.$$

But this is also the coefficient of $g^{r-s} \cdot \mathbf{D}^n(g^s)$ on the left. This proves $(8_{n,r})$.

The second formula for $\mathbf{D}^n(fg)$ is

$$(9_n) \qquad \mathbf{D}^n(fg) = \sum_{r=1}^{n} (\mathbf{D}^r f)g \cdot \frac{1}{r!} \sum \frac{n!}{p_1! \cdots p_r!} (\mathbf{D}^{p_1} g) \cdot \cdots \cdot (\mathbf{D}^{p_r} g)$$

where the second summation is taken over the integral solutions of $p_1 + \cdots + p_r = n$ and $p_i > 0$, for $i = 1, \cdots, r$. The proof of (9_n) is based on the following.

LEMMA: $\mathbf{Y}_{n,r}(g)$ *consists of those terms in the expansion of* $\mathbf{D}^n(g^r)/r!$ *which do not contain* g *as factor.*

By (3), $\mathbf{Y}_{1,1}(g) = \mathbf{D}g$. Assume that $\mathbf{Y}_{n,r}(g)$, $r = 0, 1, \cdots, n$ contains no terms with g as factor. It follows by (5) that $\mathbf{Y}_{n+1,r}(g)$ contains no terms with g as factor. But by $(8_{n,r})$,

$$\mathbf{Y}_{n,r}(g) = g\cdot \sum_{s=0}^{r-1} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-1-s}\cdot \mathbf{D}^n(g^s) + \frac{1}{r!}\mathbf{D}^n(g^r).$$

Hence the first term on the right, containing g as factor, must cancel with the corresponding terms in the expansion of $\mathbf{D}^n(g^r)/r!$.

The proof of (9_n) is completed by applying the multinomial analogue of (2_n) to $\mathbf{D}^n(g^r)$.

$$\frac{1}{r!}\mathbf{D}^n(g^r) = \frac{1}{r!}\sum \frac{n!}{p_1!\cdots p_r!}\mathbf{D}^{p_1}g\cdot\cdots\cdot\mathbf{D}^{p_r}g,$$

where the summation extends over the non-negative integral solutions of $p_1+\cdots+p_r=n$. A factor g appears whenever one of the derivatives is of order 0. The stipulation $p_i>0$ eliminates these terms, yielding, along with the lemma, formula (9_n).

In conclusion, we note the following consequences of the above formulae. In view of (5) it follows by induction that $\mathbf{Y}_{n,n}(g) = [\mathbf{D}g]^n$ for any $n\geqq 1$. Hence, $(8_{n,n})$ yields the following identity

$$[\mathbf{D}g]^n = \sum_{s=0}^{n} \frac{(-1)^{n-s}}{s!(n-s)!} g^{n-s}\cdot \mathbf{D}^n(g^s) \text{ for any } g. \tag{10}$$

It may also be shown that

$$\mathbf{Y}_{n,n-1}(g) = \frac{n(n-1)}{2}\cdot\mathbf{D}^2g\cdot[\mathbf{D}g]^{n-2}.$$

Hence:

$$\mathbf{D}^2\cdot g[\mathbf{D}g]^{n-1} = \frac{2(n-1)!}{(n+1)!}\sum_{s=0}^{n}\frac{(-1)^{n-s}}{(n-s)!s!} g^{n-s}\cdot\mathbf{D}^{n+1}(g^s) \text{ for any } g. \tag{11}$$

Also, comparing (9_n) with the corresponding formula of Faà di Bruno, we obtain the identity

$$\sum \frac{\left(\frac{\mathbf{D}g}{1!}\right)^{q_1}\cdot\left(\frac{\mathbf{D}^2g}{2!}\right)^{q_2}\cdot\cdots\cdot\left(\frac{\mathbf{D}^ng}{n!}\right)^{q_n}}{q_1!\cdot q_2!\cdot\cdots\cdot q_n!} = \frac{1}{r!}\sum\frac{(\mathbf{D}^{p_1}g)\cdot\cdots\cdot(\mathbf{D}^{p_r}g)}{p_1!\cdot\cdots\cdot p_r!} \tag{12}$$

where the summation is over

$$q_1+q_2+\cdots+q_n = r$$

and

$$q_1+2q_n+\cdots+nq_n = n$$

where the summation is over

$$p_1+p_2+\cdots+p_r = n$$

and

$$p_i > 0.$$

Many other identities may be found by particular choices of g in (10), (11), and (12).

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A FUNDAMENTAL THEOREM OF CALCULUS

M. K. FORT, JR., The University of Georgia

The following theorem is frequently called the fundamental theorem of the differential calculus.

THEOREM A. *If F_1 and F_2 are functions which have the same derivative at each point of an interval I, then there exists a constant c such that $F_1(x)-F_2(x)=c$ for each x in I.*

The proof of Theorem A follows easily from:

THEOREM B. *If F is a function whose derivative is 0 at each point of an interval I, then there exists a constant c such that $F(x)=c$ for each x in I.*

The usual proof of Theorem B is based on the Mean Value Theorem, the proof of the Mean Value Theorem is based on Rolle's Theorem, and proof of Rolle's Theorem is based on the Maximum Value Theorem for continuous functions. Since a proof of this last theorem is often omitted from the first course in calculus, a complete proof of Theorem B is usually not given. Recently, Professor Harry D. Ruderman pointed out to the author that it would be desirable to have a direct proof of Theorem B which would avoid the use of the Mean Value Theorem. In this note we give such a proof.

We start with the fact that every bounded, monotone sequence of real numbers converges. This fact is stated as an axiom in many calculus texts. It is then easy to prove that if $J_1 \supset J_2 \supset J_3 \supset \cdots$ is a nested sequence of closed intervals which is obtained by the "bisection process", then there exists a unique number p which is a member of each J_n. Moreover, if x_n is a member of J_n for each n, then the sequence $x_1, x_2, x_3, \cdots$ converges to p. We now give a proof of Theorem B which is based on these facts about nested sequences of closed intervals.

It is sufficient to prove that if a and b are points of I then $F(a)=F(b)$. We may assume that $a<b$.

If $J=[u, v]$ is a closed interval which is contained in I, we define $S(J)=\left|(F(u)-F(v))/(u-v)\right|$. The following lemma is obvious and its proof is omitted.

LEMMA. *If $u<t<v$, $J'=[u, t]$, $J''=[t, v]$ and $J=[u, v]$, then either $S(J') \geqq S(J)$ or $S(J'') \geqq S(J)$.*

We make use of the Lemma and the "bisection process" to obtain a sequence $J_1 \supset J_2 \supset J_3 \supset \cdots$ of closed intervals such that $J_1=[a, b]$ and $S(J_{n+1}) \geqq S(J_n)$

for each n. There exists a unique point p which is a member of J_n for each n. We now use the Lemma to obtain for each n a non-degenerate subinterval K_n of J_n such that one end point of K_n is p, the other end point x_n of K_n is also an end point of J_n, and $S(K_n) \geqq S(J_n)$. The sequence $x_1, x_2, x_3, \cdots$, converges to p, and since $F'(p)=0$ we obtain $\lim_{n\to\infty} (F(x_n)-F(p))/(x_n-p)=0$. Thus $\lim_{n\to\infty} S(K_n)=0$. Since $S(K_n) \geqq S(J_n) \geqq S(J_1) \geqq 0$ for each n, we see that $S(J_1)=0$. This implies that $F(a)=F(b)$, which concludes the proof of Theorem B.

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS*

W. G. LEAVITT, University of Nebraska

Let $\{y_i\}$ $(i=1, \cdots, n)$ be a set of independent variables, and suppose that a system of m linear homogeneous differential equations with constant coefficients is to be solved. The usual approach to this problem is to introduce derivatives of the $\{y_i\}$ as auxiliary variables, reducing the system to a (larger) set of equations all of the first order. In the following, an alternative approach is given, which makes use of the "elementary divisor theory" for matrices of polynomials.

Using matrix notation, the system of m equations may be written

$$T(D)Y = 0, \tag{1}$$

where $T(D)$ is an m by n matrix whose elements are polynomials in the differential operator D, and Y is a column vector whose components are the y_i.

A natural approach to the solution of (1), analogous to that of the theory of a single linear equation, is to assume a solution of form

$$Y = (K_0 + K_1 x + \cdots + K_s x^s)e^{ax}, \tag{2}$$

where the K_i are constant vectors and a is a scalar constant. Substituting in (1) and equating coefficients of x to zero, we obtain a series of equations

$$\begin{aligned}
&T(a)K_s = 0 \\
&T(a)K_{s-1} + sT'(a)K_s = 0 \\
&T(a)K_{s-2} + (s-1)T'(a)K_{s-1} + \frac{s(s-1)}{2!}T''(a)K_s = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&T(a)K_0 + T'(a)K_1 + T''(a)K_2 + \cdots + T^{(s)}(a)K_s = 0.
\end{aligned} \tag{3}$$

It is clear from the first equation that a necessary condition for the existence of a solution is that $T(a)$ must be singular; that is, the equation $T(a)X=0$ must be solvable. (If $T(a)$ is square, for example, this means that a must be a root of the characteristic equation $|T(x)|=0$.)

* Presented (in part) to the Mathematical Association of America, Nebraska Section, May 2, 1953.

In some cases† it is convenient to solve these equations iteratively for $K_s, K_{s-1}, \cdots$. However this procedure leads in general to certain difficulties such as arise, for example, when the first equation of (3) has several solutions not all of which make possible a solution of the second equation, and so on. Furthermore, it is not clear from this approach just what value of s should be used, nor is it evident that all solutions must necessarily be of form (2).

To analyze this problem more carefully we take advantage of the isomorphism* between the polynomials in D and the set of all polynomials in an indeterminate λ. This makes possible the use of a well-known theorem which states that there exist square matrices $P(D)$ and $Q(D)$ each having unit determinant such that $U=PTQ$ has form

$$U(D) = \begin{pmatrix} u_1 & & & & & \\ & u_2 & & & 0 & \\ & & \ddots & & & \\ & & & u_h & & \\ & & & & 0 & \\ & 0 & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \tag{4}$$

in which the rank h is the same as that of $T(D)$ and the $\{u_i\}$ (the "invariant factors") are so arranged that u_1 divides u_2, u_2 divides u_3, and so on.§ The matrices $P(D)$ and $Q(D)$ may, in fact, be obtained as products of elementary matrices. Thus substitution in (1) of $Y=QX$ and multiplication by P leads to the equation

$$UX = 0. \tag{5}$$

The first result is that if $h<n$ there will be $n-h$ solutions of (5) involving arbitrary functions. These will be of form

$$X_i = \psi_i(x)\delta_i \qquad (i = h+1, \cdots, n) \tag{6}$$

with ψ_i arbitrary and δ_i the vector having 1 in the ith position, zero elsewhere. Using the usual properties of the differential operator, it is easily verified that (1) has corresponding solutions

$$Y_i = \psi_i Q_{i0} + \psi_i' Q_{i0}' + \frac{\psi_i'' Q_{i0}''}{2!} + \cdots \qquad (i = k+1, \cdots, n), \tag{7}$$

† See Frazer, Duncan and Collar, Elementary Matrices, Cambridge University Press, 1952, p. 172.

* For simplicity, we will henceforth use this isomorphism without explicit reference. For example, when we say that $g(D)$ is a factor of $f(D)$ we will mean that $g(\lambda)$ divides $f(\lambda)$.

§ See, for example, C. C. MacDuffee, Vectors and Matrices, Carus Mathematical Monograph No. 7, Math. Assoc. of Am., 1943, p. 143. (The generalization to non-square matrices is obvious.)

where $Q_{i0}^{(k)}$ indicates the ith column of the kth derivative of $Q(\lambda)$ evaluated at $\lambda=0$.

Now suppose that one of the invariant factors, say u_i, has a factor $(D-a)^c$ (an "elementary divisor"). Then clearly (5) has solutions

$$X_j = t^j e^{at}\delta_i \qquad (j = 0, 1, \cdots, c-1). \tag{8}$$

Again using well-known properties of the differential, the corresponding solutions of (1) are

$$Y_j = (t^j Q_{ia} + jt^{j-1}Q_{ia}' + \frac{j(j-1)}{2!}t^{j-2}Q_{ia}'' + \cdots + Q_{ia}^{(j)})e^{at} \tag{9}$$

where $Q_{ia}^{(j)}$ is the ith column of the jth derivative of $Q(\lambda)$ evaluated at $\lambda=a$. Note that this is a solution of type (2) with $s=j$.

Now it is clear from an examination of the form of $U(D)$, (4), that the only solutions of (5) (other than those given in (6)) are of this type. Then since every solution of (1) corresponds to a solution of (3), it follows that the set of all solutions of type (9), together with (7), give all solutions of (1). Hence a knowledge of the elementary divisors of $T(D)$, and of the transformation matrix $Q(D)$ permits us to write down immediately the general solution of the system (1).

ON FINDING DIAMETERS OF CONIC SECTIONS

E. A. HEDBERG, University of South Carolina

Nearly all current textbooks in analytic geometry give special formulas for diameters of conic sections in which the chords have a given slope m, listing each type of conic separately and giving only the standard forms, where the center or vertex is at the origin. That all these formulas are related seems, from this writer's experience, not universally known. I state the general case as a theorem.

THEOREM. *Let* $f(x, y)\equiv Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ *be the general equation of a conic section. The equation of the diameter which bisects the chords of slope* m *may be found by differentiating* $f(x, y)=0$ *with respect to* x *and replacing* dy/dx *by* m.

The statement of the theorem implies the existence of chords with slope m. It will be pointed out in the proof of the theorem that the condition that chords exist is that $A+Bm+Cm^2\neq 0$. Attention is also called to the fact that the theorem applies to the degenerate conics.

One may establish the theorem by direct substitution of $y=mx+k$ in $f(x, y)=0$, setting up the formulas for midpoints of the chords, eliminating the parameter k from these expressions, and comparing the result with the theorem stated above. However, a proof which avoids such manipulations would seem desirable. The following proof involves a minimum of actual computation.

The fact that the locus of the midpoints of the chords with a given slope m is a straight line may be verified without determining the actual locus. For if one substitutes $mx+k$ for y in the expanded form of $f(x, y)$, in order to determine the abscissas of the points of intersection of the line $y=mx+k$ with the conic $f(x, y)=0$, it is at once apparent that k does not enter $A+Bm+Cm^2$, the coefficient of x^2, and that it enters the coefficient of x to the first degree only. If $A+Bm+Cm^2\neq 0$ the quadratic equation has two roots, x_1 and x_2, and the abscissa, $X=\frac{1}{2}(x_1+x_2)$, of the midpoint of the chord involves k to the first degree only. The ordinate, Y, of the midpoint, found from $Y=mX+k$, is linear in k, hence it is plain that the equation obtained by eliminating k from the expressions for X and Y is linear in X and Y. This straight line will henceforth be referred to as the diameter.

The coefficients which appear in the equation of the diameter are rational functions of m and the coefficients of $f(x, y)$. The only denominators which appear are constant multiples of $A+Bm+Cm^2$, hence we may express the equation of the diameter using polynomial coefficients by clearing of fractions. The method used in obtaining the equation is obviously independent of the type of conic under consideration, the only limitation being that $A+Bm+Cm^2$ is not zero.

If $A+Bm+Cm^2=0$ then there is at most one finite point of intersection of the line $y=mx+k$ with $f(x, y)=0$, whatever value of k is used. That is, in this case, there are no chords, in the ordinary sense, for the corresponding value of m. The parabola, for instance, has no chords parallel to its axis, the hyperbola none parallel to an asymptote. In the following, therefore, it will be assumed that $A+Bm+Cm^2\neq 0$.

We now find the equation of the diameter, using the expanded form of $f(x, y)$ without making any explicit substitutions. If $mx+k$ were substituted for y in $f(x, y)=0$, and the condition for equal roots applied, which is also the condition that $y=mx+k$ be tangent to $f(x, y)=0$, the result would be $(d/dx)f(x, mx+k)=0$. However,

$$\frac{d}{dx}f(x, mx+k) \equiv \frac{\partial}{\partial x}f(x, y)\Big|_{y=mx+k} + m\frac{\partial}{\partial y}f(x, y)\Big|_{y=mx+k}.$$

That is, the points $(x, mx+k)$ at which the lines of the system $y=mx+k$ are tangent to the conic $f(x, y)=0$ lie on the straight line

$$\frac{\partial}{\partial x}f(x, y) + m\frac{\partial}{\partial y}f(x, y) = 0.$$

Now for an ellipse there are always two real points of tangency. The diameter plainly passes through these points, hence the above process leads to the equation of the diameter. But we have seen that the coefficients in the equation of the diameter are independent of the type of conic involved. Thus, in every case, the differentiation process leads to the equation of the diameter.

An application of the above process to finding the equations of the medians of a triangle may be of interest. The following represents the solution in the general case. Let $L_1 \equiv A_1x+B_1y+C_1=0$, $L_2 \equiv A_2x+B_2y+C_2=0$, $L_3 \equiv A_3x+B_3y+C_3=0$ be the equations of the sides of a triangle. Let us assume that $B_3 \neq 0$. To find the equation of the median upon the side $L_3=0$ write the equation of the diameter of the degenerate conic $L_1L_2=0$ which bisects the chords of slope $-A_3/B_3$. Applying the general rule of the theorem above, one arrives at the result:

$$L_1\left[A_2+B_2\left(\frac{-A_3}{B_3}\right)\right]+L_2\left[A_1+B_1\left(\frac{-A_3}{B_3}\right)\right]=0;$$

or

$$L_1(A_2B_3-A_3B_2)+L_2(A_1B_3-A_3B_1)=0.$$

In determinant form this equation becomes:

$$\begin{vmatrix} L_1 & -L_2 & 0 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = 0.$$

It may be readily verified that the last equation represents the median upon the side $L_3=0$ even when $B_3=0$.

A COMMENT ON SNYDER'S CORRECTION FORMULA*

EVELYN FRANK, University of Illinois

The correction formula (2) of [5] goes back to Lagrange [4]. It is also found, for example, in [6], pp. 222–223, and has recently been repeated a number of times, for example, in [1] and [2]. In [3] it is extended to n equations in n unknowns.

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5. Ralph W. Snyder, One more correction formula, this MONTHLY, vol. 62, 1955, pp. 722–725.

6. Fr. A. Willers, Practical Analysis, Dover Publications, Inc., New York, 1948.

* See [5].

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1216. *Proposed by N. A. Court, University of Oklahoma*

The area of the triangle formed by the midpoints of three (not necessarily concurrent) cevians drawn through the three vertices of a given triangle is equal to one fourth of the area of the triangle determined by the feet of those cevians.

E 1217. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Evaluate

$$\prod_{d|n} d^{\phi(n/d)+\phi(d)}.$$

E 1218. *Proposed by Robert Spira, Berkeley, Calif.*

Consider the two propositions:

I. If $(a, b)=1$, then $ax+b$ assumes infinitely many prime values.

II. If $(a, b)=1$, then $ax+b$ assumes at least one prime value.

I is Dirichlet's theorem. Clearly I implies II. Show that II implies I.

E 1219. *Proposed by D. A. Robinson, University of Wisconsin*

The sequence $\{s_n\}$, where $s_1=\sqrt{2}$ and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$, has a finite limit. Find this limit in closed form.

E 1220. *Proposed by Judith Blankfield, University of Illinois*

It is well known that if

$$r_1 > r_2 > r_3 > \cdots \to 0,$$

then

$$\left| \sum_{t=n+1}^{\infty} (-1)^t r_t \right| \leqq r_n.$$

Prove the following generalization which reduces to the above case when $\theta=\pi$.

$$\left| \sum_{t=n+1}^{\infty} r_t e^{it\theta} \right| \leqq r_n/\sin(\theta/2), \qquad 0 < \theta < 2\pi.$$

SOLUTIONS

Total Number of Permutations of n Things

E 1186 [1955, 655]. *Proposed by David Freedman, Student, McGill University*

Show that the total number of permutations of n things is $[n!e]$, where $[x]$ denotes the greatest integer in x.

Solution by P. B. Johnson, Occidental College. The total number P of permutations of n things is the sum of the number of permutations taken n, $n-1$, $n-2$, $\cdots$, 1, 0 at a time. Hence

$$P = \sum_{r=0}^{n} n!/r! = n!e - \sum_{r=n+1}^{\infty} n!/r!.$$

But

$$0 < \sum_{r=n+1}^{\infty} n!/r! < \sum_{s=1}^{\infty} 1/(n+1)^s = 1/n \leqq 1,$$

whence the result.

Also solved by A. N. Aheart, Ferrel Atkins, S. K. Berberian, D. G. Brennan, G. B. Charlesworth, John Christopher, A. E. Danese, R. E. Daniels and J. D. E. Konhauser (jointly), Marcel Delcourte, J. M. Elkin, Michael Goldberg, D. S. Greenstein, Nathaniel Grossman, M. J. Hellman, Vern Hoggatt, P. W. M. John, E. S. Keeping, D. C. B. Marsh, F. E. Milne, T. F. Mulcrone, C. S. Ogilvy, Hiram Paley, Walter Penney, C. F. Pinzka, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., J. P. Scholz, W. A. Small, Bernard Smilowitz, O. E. Stanaitis, Art Steger, Chih-yi Wang, Alan Wayne, David Zeitlin, and the proposer. Late solutions by A. C. Goddard, A. R. Hyde, F. I. John, and M. S. Klamkin.

A Limit Problem

E 1187 [1955, 655]. *Proposed by James Goode, Student, Georgia Institute of Technology*

Find

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k.$$

I. *Solution by G. B. Charlesworth, Hofstra College.* By areas

$$\int_{n}^{2n+1} dx/x < \sum_{k=n}^{2n} 1/k < \int_{n-1}^{2n} dx/x,$$

whence

$$\ln (2n+1)/n < \sum_{k=n}^{2n} 1/k < \ln 2n/(n-1).$$

Therefore

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k = \ln 2.$$

II. *Solution by A. E. Danese, Mount Morris, N. Y.* We have

$$\sum_{k=n}^{2n} 1/k = \sum_{k=1}^{2n} 1/k - \sum_{k=1}^{n-1} 1/k$$

$$= \left(\sum_{k=1}^{2n} 1/k - \ln 2n\right) - \left(\sum_{k=1}^{n} 1/k - \ln n\right) + 1/n + \ln 2.$$

Since the limit of each parenthesis is Euler's constant γ, we have

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k = \ln 2.$$

III. *Solution by O. E. Stanaitis, St. Olaf College.* From the definition of a definite integral

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k = \lim_{n\to\infty} (1/n) \sum_{k=0}^{n} 1/(1 + k/n) = \int_0^1 dx/(1 + x) = \ln 2.$$

IV. *Solution by Albert Wilansky, Lehigh University.* One may easily show by induction that

$$\sum_{k=n+1}^{2n} 1/k = \sum_{k=1}^{2n} (-1)^{k-1}/k,$$

whence the required limit is ln 2.

Also solved by D. S. Adorno, W. A. Al-Salam, J. E. Darraugh, Marcel Delcourte, Hüseyin Demir, I. A. Dodes, E. S. Eby, J. M. Elkin, Calvin Foreman, A. M. Glicksman, Michael Goldberg, D. S. Greenstein, R. E. Greenwood, Cornelius Groenewoud, Nathaniel Grossman, M. J. Hellman, A. S. Hendler, T. W. Hildebrandt, Vern Hoggatt, W. H. Hooker, F. A. Homann, P. W. M. John, P. B. Johnson, L. M. Jordan, E. S. Keeping, M. A. Kirchberg, J. D. E. Konhauser, H. D. Lipsich, Kenneth Loewen, D. C. B. Marsh, Eugene McLachlan, Elliott Mendelson, J. D. Miller, T. A. Newton, C. S. Ogilvy, Ingram Olkin, Hiram Paley, F. D. Parker, Walter Penney, W. J. Pervin, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, C. M. Sandwick, Sr., L. P. Scholz, D. D. Strebe, Chih-yi Wang, R. M. Warten, Alan Wayne, David Zeitlin, and the proposer. Late solutions by A. R. Hyde, F. I. John, M. S. Klamkin, and C. F. Pinzka.

An Orthocentric Tetrahedron

E 1189 [1955, 655]. *Proposed by N. A. Court, University of Oklahoma*

If two pairs of spheres with non-coplanar centers are such that each sphere

of one pair is orthogonal to the two spheres of the other pair, then the tetrahedron formed by the centers of similitude of the two pairs of spheres is orthocentric.

Solution by P. D. Thomas, Air Force Armament Center, Eglin Air Force Base, Florida

A consequence of the orthogonality conditions stated is that the two coaxial pencils determined by the two pairs of spheres as given are orthogonal. That is, every sphere of either pencil is orthogonal to every sphere of the other, and of course the two lines of centers are orthogonal.

Now the sphere of similitude of each given pair (the sphere having for diameter the segment determined by the centers of similitude of the pair) belongs to the coaxial pencil determined by that pair, and the two respective spheres of similitude are orthogonal.

But the ends of two skew rectangular diameters of two orthogonal spheres are vertices of an orthocentric tetrahedron. Hence the proposition.

See N. A. Court, *Modern Pure Solid Geometry*, pp. 153, 168, 179, 185, 189, 269.

Also solved by Roscoe Woods and the proposer.

A Lucky Larry

E 1190 [1955, 655]. *Proposed by R. A. Miller, University of Mississippi*

A student wrote the following in solving the problem, "Find the value of $(3+x)^{2/3}$ when $x=3/8$:"

$$(3 + x)^{2/3} = (3 + 3/8)^{2/3} = 2 + 1/4 = 9/4.$$

Investigate the conditions under which

$$(a + b/c)^{m/n} = (a + b/c)(m/n),$$

where a, b, c, m, n are positive integers.

Solution by E. P. Starke, Rutgers University. First we note that two different positive integral k ($k>1$) powers must differ by more than k, since

$$r^k - s^k = (r - s)(r^{k-1} + r^{k-2}s + \cdots + s^{k-1}),$$

and $r-s \geqq 1$ and the second expression in parentheses exceeds k.

Consider

$$u^{m/n} = u(m/n),$$

where u is positive and rational, and m, n are relatively prime positive integers. If $m>n$, we have $u^{m-n}=(m/n)^n$. Since $m-n$ and n are relatively prime, both m and n must be perfect $(m-n)$th powers. But they differ by $m-n$. By our earlier remark this is impossible unless $m-n=1$. Hence $m=n+1$ and

$$u = a + b/c = (1 + 1/n)^n$$

with n arbitrary.

The case $m=n$ is trivial. If $m<n$, we have $(n/m)^n=u^{n-m}$ and, by the same argument, $n=m+1$ and

$$u = a + b/c = (1 + 1/m)^{m+1}.$$

The case cited by the Proposer results from $n=3$, $m=2$.

Also solved by G. B. Charlesworth, Marcel Delcourte, I. A. Dodes, Michael Goldberg, Cornelius Groenewoud, Virginia Hanly, D. C. B. Marsh, O. C. Nussmann, C. S. Ogilvy, Walter Penney, Susan Pyeatt, Azriel Rosenfeld, G. J. Simmons, O. E. Stanaitis, Chih-yi Wang, R. M. Warten, H. R. Wesson, and Hazel Schoonmaker Wilson. Late solutions by A. R. Hyde, F. I. John, M. S. Klamkin, and M. A. Rashid.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4688. *Proposed by A. H. Clifford, Tulane University*

What positive integers n have the property that every positive integer less than n is expressible as the sum of distinct divisors of n?

4689. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $f(x)$ be any function such that $f'''(x)\geqq 0$, $f(n)\sim f(n+1)$. Prove that

$$\sum_{n=0}^{\infty}(-1)^n t^{f(n)}$$

tends to $\frac{1}{2}$ as $t\to 1^-$.

4690. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Being given a tetrahedron $ABCD$ and the tetrahedron $A_1B_1C_1D_1$ obtained by passing planes through A, B, C, D parallel to the opposite faces of $ABCD$, show that

$$PA^2 + PB^2 + PC^2 - 2PD^2 - PD_1^2$$

is a constant independent of the position of point P. Extend this property to a skew polygon of n vertices.

4691. *Proposed by R. E. Johnson, Smith College*

A weakly prime ideal of a ring R is any ideal I having the property that either $aR \subset I$ or $Ra \subset I$ implies that a is in I. Give an elementwise characterization of the unique minimal weakly prime ideal of R.

4692. *Proposed by A. E. Currier, United States Naval Academy*

Given $f(x)$ defined by

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n x^{2^n},$$

prove that

$$(1) \qquad -\frac{3+8\sqrt{6}}{750} < \liminf_{x \to 1-} f(x) < -\frac{1}{750},$$

$$(2) \qquad \frac{1}{750} < \limsup_{x \to 1-} f(x) < \frac{3+8\sqrt{6}}{750}.$$

SOLUTIONS

Digits of the Fibonacci Numbers

4609 [1954, 646]. *Proposed by Vern Hoggatt, San Jose State College, California*

Let $q_k(F_n)$ be the kth digit (from the right) of F_n, the nth Fibonacci number ($F_0=0$, $F_1=1$, $F_{n+2}=F_{n+1}+F_n$). Let s be defined such that $q_k(F_n)=0$ for $n=0, 1, 2, \cdots, s$ but $q_k(F_{s+1}) \neq 0$. (1) Show that the sequence $q_k(F_n)$ has at least another run of $s+1$ consecutive zeros. (2) Prove or disprove: the sequence $q_k(F_n)$ has never more than $s+1$ consecutive zeros.

Editorial Note. No communication regarding this problem has been sent in, but some closely related questions are answered in the Proposer's paper, A type of periodicity for Fibonacci Numbers, *Mathematics Magazine*, vol. 28, 1955, pp. 139–142.

Monic Irreducible Polynomials

4624 [1955, 126]. *Proposed by H. S. Shapiro, New York University*

Let J_p denote the field of integers modulo p, where p is a prime, and $J_p[x]$ the ring of polynomials with coefficients in J_p. Then the number of monic irreducible polynomials of degree n in $J_p[x]$ is equal to

$$\frac{1}{n} \sum_{d|n} \mu(d) p^{n/d}$$

where μ is the Möbius function.

Solution by B. M. Stewart, Michigan State University. A polynomial of degree s which is irreducible in $J_p(x)$ divides $x^{p^n}-x$ if and only if s divides n. (Ex. 1, p. 180, MacDuffee, *Introduction to Abstract Algebra.*) Furthermore, $x^{p^n}-x$ has no repeated factors. Hence if $N(p, s)$ indicates the number of distinct monic irreducible polynomials of degree s in $J_p(x)$, then

$$p^n = \sum sN(p, s)$$

summed over the positive divisors s of n.

This is a perfect situation for applying the inversion formula which involves the Möbius function μ. The result is

$$nN(p, n) = \sum \mu(d)p^{n/d}$$

summed over the positive divisors d of n.

Also solved by P. T. Bateman, Robert Breusch, L. Carlitz, L. E. Clarke, Harley Flanders, S. W. Golomb, D. C. B. Marsh, Arnold Walfisz, Koichi Yamamoto, and the Proposer.

Editorial Note. The result may be found in many places including Dedekind, *Journ. reine u. angew. Math.*, vol. 54 (1857), pp. 19–22; L. E. Dickson, *Linear Groups* (1901), p. 18; H. Hasse, *Zahlentheorie* (1949) p. 32—references supplied by readers.

Legendre Polynomials

4628 [1955, 186]. *Proposed by L. Carlitz, Duke University*

Let p be an odd prime and let $P_m(x)$ denote the Legendre polynomial of degree m, where $p=2m+1$. Show that

$$P_m(3) \equiv P_m(-3) \equiv \begin{cases} 0 \pmod{p} & (m \text{ odd}) \\ 2a \pmod{p} & (m \text{ even}), \end{cases}$$

where a is the unique odd integer determined by

$$p = a^2 + b^2, \qquad a \equiv b + 1 \pmod 4.$$

Solution by the Proposer. Since $P_m(-x)=(-1)^m P_m(x)$, it suffices to discuss $P_m(3)$.

We shall make use of the following well known formula for the Legendre polynomial:

$$P_m(1 - 2x) = F(-m, m + 1; 1; x), \tag{1}$$

where $F(a, b; c; x)$ denotes the hypergeometric function. (For proof, see Whittaker and Watson, *Modern Analysis*, 4th edition, p. 311).

Now it is clear from (1) that

$$P_m(1 - 2x) \equiv \sum_{k=0}^{m} \binom{m}{k}^2 x^k \pmod{p}.$$

For $x = -1$ this becomes

$$P_m(3) \equiv \sum_{k=0}^{m} (-1)^k \binom{m}{k}^2 \pmod{p}. \tag{2}$$

For m odd the sum in the right member of (2) evidently vanishes since the kth term and the $(m-k)$th term cancel. Hence $P_m(3) \equiv 0 \pmod{p}$ in this case.

When m is even, let $m = 2n$; then we have

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}. \tag{3}$$

This may be proved by comparing coefficients of x^{2n} in both members of the identity

$$(x - 1)^{2n}(x + 1)^{2n} = (x^2 - 1)^{2n}.$$

It follows from (2) and (3) that

$$P_{2n}(3) \equiv (-1)^n \binom{2n}{n} \pmod{p = 4n + 1}.$$

But by a formula of Gauss (Theorie residuorum biquadraticorum, *Werke*, vol. 2, p. 90)

$$\frac{1}{2^{2n}} \binom{2n}{n} \equiv 2a \pmod{p},$$

where a is the unique odd integer determined by $p = a^2 + b^2$, $a \equiv b + 1 \pmod 4$. Since $2^{2n} \equiv (2/p) = (-1)^{(p^2-1)/8} = (-1)^n$, where $(2/p)$ is the Legendre symbol, it follows that $P_{2n}(3) \equiv 2a \pmod{p}$.

The following additional remark has some interest. If $p = 2m + 1 \equiv 3 \pmod 4$, then

$$\frac{1}{p} P_m(3) \equiv -\sum_{r=1}^{m} (-1)^r \binom{m}{r}^2 \left(1 + \frac{1}{2} + \cdots \frac{1}{r}\right) \pmod{p}.$$

The problem was suggested by its connection with elliptic functions and particularly with the lemniscate function (see A. Hurwitz, *Mathematische Annalen*, vol. 51 (1899), pp. 196–226; *Mathematische Werke*, Basel, 1933, vol. 2, pp. 342–373). The following general result may be stated. Assume that the elliptic function sn x admits of complex multiplication so that the period quotient belongs to an imaginary quadratic field of discriminant d. Let k^2 denote the corresponding singular modulus and let p be an odd prime such that the Legendre symbol $(d/p) = -1$. Then $P_m(1 - 2k^2) \equiv 0 \pmod{p}$. See *Mathematische*

Annalen, vol. 127 (1954), pp. 162–169 and *Monatshefte für Mathematik*, vol. 58 (1954), pp. 77–90. This result does not, however, determine the residue of $P_m(1-2k^2) \pmod{p}$ when $(d/p)=+1$.

Subring of a Radical Ring

4629 [1955, 186]. *Proposed by M. Henriksen and S. Perlis, Purdue University*

A commutative ring R is called a radical ring (in the sense of Jacobsen) if every element has a quasi-inverse; *i.e.*, for every a in R, there is an x in R such that $a+x+ax=0$. In particular, if every element of R is nilpotent, then R is a radical ring. Need every subring of a radical ring be a radical ring?

I. *Solution by A. J. Goldman, Princeton University.* We show first that an arbitrary commutative ring R (radical or not) without unity can be imbedded in a radical ring R''. Imbed R in a commutative ring R' with unity (it is well known that this can be done) and let F be the quotient field of R'. Then R is imbedded in the subring R'' of F consisting of all elements of the form $a/(1+b)$, where a and b range through R; it is clear that R'' is actually a ring. R'' is radical because $a/(1+b)$ has the quasi-inverse $-a/(1+a+b)$.

To answer the question in the negative, it now suffices to exhibit a commutative ring R without unity which is not radical. The even integers form such a ring, since the element 2 has no quasi-inverse.

II. *Solution by P. M. Cohn, the University, Manchester, England.* Let R be the ring of formal power series in x over a field F, with zero constant term. Then R is a radical ring since any f in R has the quasi-inverse $-f+f^2-f^3+\cdots$. The polynomials in R form a subring S which is not a radical ring since it contains x but no quasi-inverse of x. For if

$$f = a_1x^n + \cdots + a_nx \qquad (a_1 \neq 0)$$

were a polynomial satisfying $x+f+xf=0$, then, by equating the coefficient of x^{n+1} to zero we obtain $a_1=0$, which is a contradiction. Thus no non-zero element of S is a quasi-inverse of x; also zero is not.

Also solved by W. R. Abel and C. D. Gorman, E. H. Batho, S. W. Golomb, C. R. Phelps, Gustave Rabson, R. B. Reisel, D. W. Sasser, W. R. Scott, M. F. Smiley, M. L. Tomber, Jerrold Yos, and the Proposer.

A Continued Fraction Related to the Bessel's Functions

4631 [1955, 187]. *Proposed by F. H. Northover, Memorial University of Newfoundland*

Evaluate the unending continued fraction

$$1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\cdots.$$

Solution by D. J. Newman, Republic Aviation Corporation, Farmingdale,

N. Y. Applying the method of continued fractions to the differential equation $y=xy''$ (See Ince, *Differential Equations*, p. 178) we obtain

$$x\frac{y'}{y} = 1 + \frac{x}{2+}\,\frac{x}{3+}\,\frac{x}{4+}\cdots$$

for some solution y. (Namely, letting $x\to 0$, it is that solution for which $y(0)=0$, $y'(0)=1$). All that remains then is to solve the equation

$$y = xy'', \qquad y(0) = 0, \qquad y'(0) = 1$$

and our answer will be $y'(1)/y(1)$.

The differential equation is of the Bessel type and the required solution is then $y=x^{1/2}I_1(2x^{1/2})$, whence $y'=I_0(2x^{1/2})$. This gives as our final answer $I_0(2)/I_1(2)$, or in series form

$$\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\Big/\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}.$$

A numerical value, 1.433127, is easy to obtain either from the series or from convergents to the continued fraction.

Also solved by Leonard Carlitz, M. Delcourte, V. C. Harris, P. G. Kirmser, D. H. Lehmer, Erich Michalup, C. D. Olds, C. F. Pinzka, C. C. Sams, R. E. Shafer, M. R. Spiegel, Chih-yi Wang, Alan Wayne, and the Proposer.

Editorial Note. The proposed continued fraction is a special case of many in the literature. Solvers referred to Chrystal, *Algebra*, vol. 2, p. 494, and pp. 520–522; Erdelyi, *et al.*, *Higher Transcendental Functions*, vol. 2, formula 7.11(23); H. S. Wall, *Continued Fractions*, p. 349; Perron, *Die Lehre von der Kettenbrüchen*, p. 299 and pp. 479–480.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Transform Calculus with an Introduction to Complex Variables by E. J. Scott. Harper and Brothers, New York, 1955. 8+330 pages. $7.50.

In the introduction the author indicates that this book is directed toward advanced undergraduate and graduate students, mainly in physics, chemistry, and engineering. For this audience it is excellent, providing them with many techniques and specific results of much importance in their work. Mathematical rigor is attempted within reason, but the author's emphasis is more on vigor than on rigor. He does not attempt the most general known statement of a

theorem if its proof would take him through circuitous pedantics. Thus he confines himself to piecewise continuity in his statement of the Fourier integral theorem (which he does not prove). Some theorems are stated imperfectly, and the English is not always perfect. Some discussions might be written more satisfactorily from the logician's viewpoint, but it would be difficult to do this without spoiling their pedagogic value. There are some misprints.

Chapter I gives the highlights of the theory of functions of a complex variable. Chapter II introduces the Fourier and Laplace transforms and their inverses. Some highlights of the theory of the Laplace transform are given in Chapter III, and are applied to ordinary differential equations with constant coefficients in Chapter IV. Chapter IV contains a number of examples of practical interest from electric circuit theory and elasticity.

In Chapter V matrix concepts are introduced and applied to systems of linear differential equations of the type that occur in multiple degree of freedom electrical and mechanical systems. These are solved using the Laplace transform. A number of examples involving actual applications to electrical and mechanical networks are given.

In Chapter VI, the Laplace transform is applied to the solution of difference equations; in particular several applications are made to mechanical and electrical problems.

Chapter VII is a long chapter in which the Laplace transform is applied to the solution of linear partial differential equations. Diffusion is treated in detail, and some transmission line problems are solved. Longitudinal and transverse vibrations of a bar are treated, as well as transverse vibrations in a string. The emphasis is upon the particular applications. Nothing like a unified theory of elliptic, hyperbolic, or parabolic equations is attempted.

In Chapter VIII, Gamma, Error, Fresnel, and Bessel functions are briefly introduced, as well as asymptotic series and some simple integral equations. The author is to be commended for employing the standard notation $Y_n(t)$ for the now universally employed second solution of the Bessel equation which is due to Weber. Jahnke and Emde erroneously attributed this function to Neumann and changed the notation to $N_n(t)$. Unfortunately many electrical engineers in this country have followed this schismatic practice.

Chapter IX treats some partial differential equations of a more complicated nature which involve the special functions introduced in Chapter VIII. A number of problems of practical interest are worked.

In the last chapter, Chapter X, the author introduces what he calls finite Fourier transforms. The resulting theory is equivalent to the theory of Fourier series. He also introduces what he calls the finite Hankel transform. The resulting theory is equivalent to the theory of Fourier-Bessel series.

This book will provide engineers and physical scientists with a good many techniques that they will find important. It will not be satisfactory as a last word on the subjects treated to the mathematics major (for instance, a mathematics major ought to be able to *prove* Fourier's theorem). On the other hand,

a good many mathematics students would be much better off for having had a course of this type. There might then be fewer members of that forlorn group of non-productive Ph.D.'s who know theorems precisely, but have no idea what in the world to *do* with them.

A valuable feature of the book is its many examples and illustrations. The typography is excellent.

E. PINNEY
University of California
Berkeley, California

Numerical Methods. By A. D. Booth. New York, Academic Press. 1955. 7+195 pages. $6.00.

This is a small book (187 pages) covering a wide range of topics in Numerical Analysis. Since it covers such a wide range of topics the treatment of each subject is necessarily brief. The author states that the book was prepared with automatic computing equipment in mind; therefore, there is emphasis on methods suitable for use with automatic computing equipment.

The book starts with interpolation formulas using finite differences. Numerical integration and differentiation is the subject of the next chapter. Summation of series and the solution of ordinary and partial differential equations, as well as the solution of linear and non-linear systems of equations, are treated. The book ends with approximating functions, Fourier synthesis and analysis, and Monte Carlo methods.

There are some misprints in the book. For example, on Page 2 read SWAC for SEAC. Clearly, formula 4.6.12 on Page 39 should read $s_m y^m/m$ instead of $s_m y^m$. In many places in Chapter 6 the error terms of formulas involve factors such as $f^{(6)}$ max. These should be replaced by terms of the form $f^{(6)}(s)$ where s is some point in the interval under consideration. With these indications of lack of care in the statement of formulas, one realizes that other formulas in the book must be used with caution.

However, the development of high speed computing equipment has increased the demand for material on numerical analysis so this book should be part of the library of all those interested in computation.

H. D. HUSKEY
University of California

Automatic Feedback Control System Synthesis. By J. G. Truxal. McGraw-Hill, New York. Electrical and Electronic Engineering Series, 1955, xiii+675 pages. $12.50.

This book is intended as an engineering text and reference. It achieves the second aim much more successfully than the first. The book is based on a second course in servomechanisms as taught by the author. The book contains many cases where initial references are made apparently to content material of the

first course. In some of these instances a text reference is given, but this procedure is confusing when reading the book *per se*.

The book is replete with completely solved examples. In fact there is a tendency to explain by using many specific illustrations. The book might have been considerably reduced in size with little loss if proofs were given or generalized illustrations used. The detail with which examples are worked out is also excessive. At this level for example, it should not be necessary to work out polynomial long divisions in detail.

The contents are divided as follows:

Chapter 1, Introduction; develops Laplace Transform Techniques.

Chapter 2, Signal-flow Diagrams and Feedback Theory; and Chapter 3, Synthesis of RC Networks; develop the basic theory of network synthesis and circuit stability.

Chapter 4, Root-locus Methods; Chapter 5, Synthesis through Pole-zero Configurations; and Chapter 6, Design in the s Plane; constitute the major contribution of the book. Here the details of techniques for linear network synthesis are described fully.

Chapter 7, Principles of Statistical Design; Chapter 8, Application of Statistical Design Principles; and Chapter 9, Sample-data Control Systems; develop the statistical treatment leading to system design techniques where noise is involved.

Chapter 10, Non-linear Systems and Describing-function Analysis; and Chapter 11, Phase-plane Analysis; discuss several methods of treating non-linear systems.

From a mathematical point of view it is distressing to find the author accentuating Laplace Transform theory as the only essential mathematical tool necessary for servo-engineers. A much healthier approach would be to accentuate the value of a strong mathematical background for all engineers.

Generally the book is well written and well printed. Few misprints were noted. The book is comprehensive and should prove useful as a reference for the field.

PAUL BROCK
Electro Data Corporation
Pasadena, California

NEW BOOKS RECEIVED

Modern Trigonometry. By W. A. Turledge and J. A. Pond. Englewood Cliffs, N. J., 1956. 11+243 pages.

An Introduction to Mathematics. By L. E. Boyer. New York, Henry Holt and Co., 1956. 16+528 pages. $5.25.

Nellie Landblom's Copybook for Beginners in Research Work. By Nellie Thompson Landblom. Multigraphed book available from author, P. O. Box 248, Fort Collins, Colorado, 1955. 118 pages. $2.95.

Elements of Business Mathematics for Colleges. Llewellyn R. Snyder. New York, McGraw-Hill Book Co., 1956. x+249 pages. $3.75.

Carl Friedrich Gauss: Titan of Science. A Study of His Life and Work. By G. W. Dunnington. New York, Exposition Press, 386 Fourth Ave., New York 16, N. Y., 1955. xi+479 pages. $6.00.

Surface Area. By Lamberto Cesari. Princeton, N. J., Princeton University Press, 1956. x+595 pages. $8.50.

Articles et Memoires—I, La variable complexe, II, Le champ reel—Notices— Arnaud Denjoy. Paris, Gauthier-Villars, 1955. Vol. 1, x+507 pages and Vol. 2, vi+600 pages. 2 volumes $15.03.

Calculus, Differential and Integral, with Problems and Solutions. By G. M. Peterson and R. F. Graesser. Ames, Iowa. Littlefield, Adams and Co., 1956. 10+321 pages. $1.75.

Through the Mathescope. By C. S. Ogilvy. New York, Oxford University Press. 1956. viii+162 pages. $4.00.

Elementary Topology. By D. W. Hall and G. L. Spencer. New York, John Wiley and Sons, 1955. xii+303 pages.

Tracts for Computers, No. XXVI, Correlated Random Normal Deviates. By E. C. Fieller, T. Lewis and E. S. Pearson. New York, Cambridge University Press, 1956. 60 pages. $2.00.

Spheroidal Wave Functions, Including Tables of Separation Constants and Coefficients. By J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little and F. J. Corbato. Published jointly by the Technology Press of M.I.T. and John Wiley and Sons, Inc., New York, 1956. xiii+613 pages, $12.50.

Plane Waves and Spherical Means Applied to Partial Differential Equations. By Fritz John. New York, Interscience Publishers, Inc., 1955. 180 pages. $4.50.

Experimental Design. By W. T. Federer. New York, The Macmillan Company, 1955. xix+544+47 pages of problems. $11.00.

Mathematics of Finance, Second Edition. By P. M. Hummel and C. L. Seebeck, Jr. New York, McGraw-Hill Book Company, 1956. xi: 372 pages. $4.75.

Analytic Geometry, Fifth Edition. By C. E. Love and E. D. Rainville. New York, The Macmillan Co., 1955. xiv+302 pages. $4.00.

Elementary Statistics for Students of Social Science and Business. By R. C. Sprowls. New York, McGraw-Hill Book Company, 1955. xiii+392 pages. $5.50.

Analytic Geometry and Calculus. By T. S. Peterson. New York, Harper and Brothers, 1956. ix+456 pages. $5.50.

Introduction to Numerical Analysis (International Series in Pure and Applied Mathematics). By F. B. Hildebrand. New York, McGraw-Hill Book Company, 1956. x+511 pages. $8.50.

Mathematical Theory of Elasticity, Second Edition. By I. S. Sokolnikoff. New York, McGraw-Hill Book Company, 1956. xi+476 pages. $9.50.

Plane Trigonometry, Third Edition. By A. L. Nelson and K. W. Folley. New York, Harper and Brothers, 1956. xi+134 pages. $3.50.

Numerical Analysis. By Zdenek Kopal. New York, John Wiley and Sons, Inc., 1955. xiv+556 pages. $12.00.

Methods in Numerical Analysis. By K. L. Nielsen. New York, The Macmillan Company, 1956. viii+382 pages. $6.90.

An Introduction to Linear Algebra. By L. Mirsky. New York, Oxford University Press, 1956. xii+433 pages. $5.60.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1956.

Michigan State University. June 26 to August 3: Professor Arnold, correlation analysis; Professor Hill, advanced calculus I, functions of a complex variable; Professor J. B. Kelly, differential equations, matrices and groups; Professor Meixner, potential theory, mathematical theory of continuous matter; Professor Stewart, projective geometry. June 26 to August 24: Dr. Berberian, mathematics of engineering; Professor Blair, functions of a real variable, algebraic systems of continuous functions; Dr. Faith, advanced calculus II; Professor Hannan, analysis of variance; Professor L. Kelly, differential equations, vector analysis, combinatorial topology; Mr. Laurent, statistical methods in engineering.

Stanford University. June 25 to August 17: Professor Weinberger (University of Maryland), approximation of quadratic functionals; Professor Lax (New York University), selected topics from analysis; Professor Spencer (Princeton University), seminar.

State University of Iowa. June 12 to August 8: Professor Hogg, introduction to mathematical statistics, statistical hypotheses; Professor Johnson, astronomy; Professor Knowler, operations research, actuarial aspects of casualty insurance; Professor Muhly, advanced calculus, group theory; Professor Price, the supervision of mathematics, laboratory in the construction of teaching aids in mathematics; Professor Smiley, matrices and determinants, introduction to advanced mathematics; Professor Woods, studies in secondary geometry, differential equations.

University of Michigan, School of Education. June 25 to July 13: The Workshop for College Professors will present a program designed to help college teachers develop new ideas and fresh materials for their academic courses. Such topics as course planning, teaching techniques, and evaluation will be discussed. For further information write to Director A. D. Henderson, 2442 U.E.S., University of Michigan, Ann Arbor, Michigan.

University of Minnesota, College of Science, Literature, and the Arts. June 11 to July 14: Professor Carlson, differential equations, non-euclidean geometry; Dr. Orey, Fourier series and orthogonal functions; Visiting Professor Roy, analysis of variance and multivariate analysis II, statistical models in the social sciences. July 16 to August 18: Professor Gelbaum, integral equations, intermediate differential equations; Professor Hatfield, Jr., Fourier series and orthogonal functions, II. June 18 to July 21: A Summer Institute for High School Teachers of Mathematics, carrying graduate credit, will be given with the support of the Hill Family Foundation.

University of Pittsburgh. June 11 to July 20, July 23 to August 31: Professors Blumberg and Christiano, differential equations, advanced calculus; Professor Bryson, vector analysis; Professor Taylor, functions of a complex variable; Professor Levine, modern algebraic theories; Professor Laush, infinite series, Lie theory of groups (second six weeks); Professor Bompiani, geometry of differential elements; Professor Elyash, advanced function theory (first six weeks). June 18 to August 10 (evenings): Professor Knipp, mathematics of modern engineering; Professor Bryson, Laplace transform theory and applications (second half of two semester course). July 2 to August 10: Professor Vogeley, differential equations for students of engineering; Professor Edwards, mathematical theory of statistics; Professor Teats, history of mathematics; Professor Myers, recreational mathematics for teachers, theory of equations; Mr. Kachun, navigation for teachers; Professor Knipp, teaching of secondary school mathematics.

University of Virginia. June 18 to August 14: Professor Ball, advanced calculus and applied mathematics; calculus of variations; Professor Floyd, topological groups. July 3 to August 14: Professor Ball, foundations of geometry.

CLIFFORD BREWSTER UPTON FELLOWSHIP

The American Book Company has established a fellowship in honor of Professor Emeritus Clifford Brewster Upton for graduate study in the Teaching of Mathematics at Teachers College, Columbia University. Three thousand dollars will be awarded annually to a student in the teaching of mathematics in elementary schools, in secondary schools, or in teacher training institutions. Inquiries should be made to the Chairman of the Committee on Fellowships and Scholarships, Teachers College, Columbia University, New York 27, New York, before May 1, 1956. Formal applications, together with all supporting materials, must be filed with the Committee by June 1, 1956.

CALIFORNIA CONFERENCE FOR TEACHERS OF MATHEMATICS

The 1956 California Conference for Teachers of Mathematics will be held at the University of California at Los Angeles from June 20 through July 3. The purpose of this conference is to bring together teachers interested in mathematics—arithmetic through calculus—to study common problems in the teaching of mathematics. For further information write to University of California Extension, Mathematics, Los Angeles 24, California.

PERSONAL ITEMS

Assistant Professor J. W. Lindsay of Texas Technological College has been awarded a Danforth Foundation Teachers Study Grant for the year 1956–1957. He will spend the year at the University of California at Los Angeles.

Illinois Institute of Technology announces the following: Mr. Michel McKiernan, previously a graduate assistant at the Institute, and Dr. Abe Sklar, formerly an assistant at California Institute of Technology, have been appointed to instructorships.

Kansas State College reports: Professor P. M. Young has been appointed Associate Dean of the School of Arts and Sciences; Professor R. G. Sanger, formerly the Acting Associate Dean of the School of Arts and Sciences, has resumed his duties as Head of the Department of Mathematics.

At Wartburg College: Professor J. O. Chellevold, head of the Department of Mathematics, has been appointed Acting Dean of the Faculty; Mr. Richard Wideranders of Estherville Junior College, Iowa, has been appointed to an instructorship.

Mr. R. G. Albert, formerly a graduate student at Harvard University, has a position as a programmer for the Datamatic Corporation, Newton Highlands, Massachusetts.

Mr. P. N. Armstrong, previously a technical engineer for the International Business Machines Corporation, Endicott, New York, is an engineer for the Magnavox Company, Santa Monica, California.

Mr. R. V. Benson of Los Angeles City College has been appointed to an instructorship at Long Beach City College.

Dr. W. J. Berger, formerly a mathematician at R.C.A. Service Company, Patrick Air Force Base, Cocoa, Florida, now has a position as a mathematician with the Raydist Navigation Corporation, Hampton, Virginia.

Assistant Professor D. R. Bey of Illinois State Normal University has been promoted to an associate professorship.

Dr. E. E. Blanche, previously an analyst with the Research and Development Division, Department of the Army, Washington, D. C., is President and Senior Research Scientist of Ernest E. Blanche and Associates, Rockville, Maryland.

Mr. H. L. Carlson, formerly an electronic applications specialist for the

National Cash Register Company, Dayton, Ohio, is employed as a computing engineer by Douglas Aircraft Company, Santa Monica, California.

Mr. D. A. Cope, recently an engineer for the Glenn L. Martin Company, Baltimore, Maryland, has accepted a position as a senior research engineer for the Consolidated-Vultee Aircraft Company, San Diego, California.

Mr. H. N. Garber, formerly National Science Foundation fellow at Massachusetts Institute of Technology, has a position as a management assistant with the Radio Corporation of America, Camden, New Jersey.

Miss Jane C. Ingersoll, previously an operations analyst at the Operations Research Office, Chevy Chase, Maryland, has been appointed to an instructorship at the University of Maryland.

Mr. Seymour Landau, formerly an instructor at the University of Connecticut, is now an actuary for the New York City Transit Authority, New York.

Mr. R. L. Mentzer of Teachers College of Connecticut has been promoted to an assistant professorship.

Associate Professor E. N. Nilson of Trinity College, Connecticut, has a position as a project engineer for Pratt and Whitney Aircraft Corporation, East Hartford, Connecticut.

Dean Emeritus and Professor J. R. Overman of Bowling Green State University is retiring in June 1956. He was the first member of the teaching staff, first Chairman of the Department of Mathematics, and first Dean of the College of Liberal Arts, Dean of Men, and Dean of Faculties. The University's Board of Trustees recently honored Dr. Overman by naming the new science building Overman Hall "in recognition of his unique service and contribution in the development and progress of the University."

Mr. J. L. Pallone, assistant actuary for Woodward and Fondiller, Inc., New York, New York, has been appointed to an instructorship at Polytechnic Institute of Brooklyn.

Assistant Professor H. C. Parrish of North Texas State College has been promoted to an associate professorship.

Mr. J. B. Roberts of Reed College has been promoted to an assistant professorship.

Dr. D. T. Sigley, formerly a physicist at Johns Hopkins University Physics Laboratory, is employed as Associate Director of the Engineering Laboratories, American Machine and Foundry Company, Greenwich, Connecticut.

Assistant Professor James C. Smith, Jr., of Loyola University of Los Angeles is a teacher in the Extension Division, University of California at Los Angeles.

Mr. G. E. Waldman, a teacher at Tottenville High School, Staten Island, New York, has been appointed Acting Chairman of the Department of Mathematics.

Dr. E. A. Walker, recently a graduate student at the University of Kansas, is employed as a mathematician at the National Security Agency, Washington, D. C.

Dr. G. P. Weeg, a mathematician at Engineering Research Associates, St. Paul, Minnesota, is now employed in the Systems Analysis Department, Remington Rand Univac Division of the Sperry Rand Corporation, St. Paul, Minnesota.

Mrs. Elizabeth S. Wolf, formerly an instructor at Indiana Technical College, has a position as a computer in the Advanced Development Division of the Avco Manufacturing Company, Stratford, Connecticut.

Dr. Albert Wolinsky, previously an associate physicist with the Farrand Optical Company, New York, New York, has a position as a staff member of the General Precision Laboratory, Pleasantville, New York.

Associate Professor Willie E. Clark, chairman of the Department of Mathematics of Agricultural, Mechanical, and Normal College, Pine Bluff, Arkansas, died on December 4, 1954.

Captain C. E. Dimick, United States Coast Guard (ret.), professor and head of the Department of Mathematics of the United States Coast Guard Academy, died on January 2, 1956. He was a charter member of the Association.

Professor Emeritus C. C. Grove of City College of the City of New York died on January 12, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania on November 26, 1955. Professor T. L. Koehler, Chairman of the Section, presided. There were 87 present, including 64 members of the Association. By-Laws for the Section were adopted by vote of those present.

Officers of the Section for the year 1955–1956 are: Chairman, Professor J. C. Oxtoby, Bryn Mawr College; Secretary, Professor G. C. Webber, University of Delaware; Members at Large—for three years, Professor N. J. Fine, University of Pennsylvania; two years, Professor T. L. Koehler, Muhlenberg College; one year, Dr. Bryant Tuckerman, Institute for Advanced Study.

The following papers were presented:

1. *A report on the Ford Foundation study on the integration of high school and college mathematics*, by Professor H. W. Brinkmann, Swarthmore College.

Professor Brinkmann reported on the present status of the School and College Study of Admission with Advanced Standing as it concerns mathematics. He gave a brief summary of the

results of the mathematics examinations given last spring and of the Conference of Mathematics Teachers held at Williams College in June. The work of the Committee on Mathematics, as well as that of the other committees, is being carried on by the College Entrance Examination Board. Examinations for advanced standing will therefore be available (beginning next May) for any candidates who wish to take advantage of them.

2. *Dedekind sums and classes of modular substitutions*, by Professor Hans Rademacher, University of Pennsylvania.

The modular matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (a, b, c, d \text{ integers}, ad - bc = 1).$$

form a group Γ. Two matrices M and M' are called similar if there exists an $L \in \Gamma$ such that

$$M' = L^{-1}ML.$$

The relation of similarity splits Γ into classes. An invariant for similarity is obviously the trace $s = a + d$. Another invariant can be defined as follows:

$$\Psi(M) = \begin{cases} \dfrac{b}{d} & \text{for } c = 0 \\ \dfrac{a+d}{c} - 12 \operatorname{sign} c \cdot s(a, |c|) - 3 \operatorname{sign}(c(a+d)) & \text{for } c \neq 0 \end{cases}$$

where $s(h, k)$ is the Dedekind sum

$$s(h, k) = \sum_{\mu=1}^{k-1} \frac{\mu}{k} \left\{ \frac{h\mu}{k} - \left[\frac{h\mu}{k} \right] - \frac{1}{2} \right\}, \qquad (h, k) = 1.$$

The invariant Ψ has the properties

$$\Psi(-M) = \Psi(M), \qquad \Psi(M^{-1}) = -\Psi(M).$$

The inhomogeneous modular group Γ^* (*i.e.* the factor group Γ/E, where E consists of two elements, the unit matrix I and $-I$) can be generated by two elements T, U with

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

such that

$$T^2 = U^3 = 1.$$

Any $M \in \Gamma^*$ has therefore a unique representation

$$M = U^{\epsilon_0} T U^{\epsilon_1} \cdots U^{\epsilon_\nu} T U^{\epsilon_{\nu+1}}$$

with $\epsilon_0, \epsilon_{\nu+1} = 0, \pm 1$, $\epsilon_j = \pm 1, j = 1, 2, \cdots \nu$. The paper gives then another determination of $\Psi(M)$ by means of the ϵ_i.

3. *Flexagons*, by Professor R. J. Wisner, Haverford College.

During the last few years, a number of people have shown interest in certain geometric objects usually called flexagons. Three notes have been published about two of these: the one with three and the one with six physical faces. A method of constructing flexagons with $n \geqq 3$ physical faces, some properties of "regular" flexagons, a possible algebraic definition, and the use of flexagons as realizations of certain groups are shown. The results are a part of joint work by Professor C. O. Oakley and the speaker.

4. *On differential operators*, by Professor William Feller, Princeton University.

The differential operator Ω defined by $\Omega f = af'' + bf' + cf$, where $a > 0$, has the characteristic property: if f has a local minimum at x_0 and $f(x_0) = 0$ then $\Omega f(x_0) \geqq 0$. The speaker described the form of the most general operator with this property, and the advantages arising from such generalized differential operators instead of Ω.

5. *Pea soup, tripe and mathematics*, by Professor Morris Kline, New York University.

The speaker contends that the usual material taught to freshmen, namely, college algebra and trigonometry, is meaningless, unmotivated, lacking in esthetic qualities and incoherent. These deficiencies account for the usual negative student reaction to freshman mathematics. This paper also presents objections to the replacement of this conventional material by rigorous abstract mathematics such as set theory, symbolic logic, Dedekind cuts, groups and fields and the like. The recommendation made in this talk is that mathematics be tied intimately to physical problems and to its cultural setting and significance. Physical problems should provide the motivation, meaning and significance at the same time that the role of mathematics in idealizing and reasoning about physical problems is given the chief attention so that the course is definitely one in mathematics and not a part of a physical science course. Other values of mathematics that can be presented in such an approach are included.

G. C. WEBBER, *Secretary*

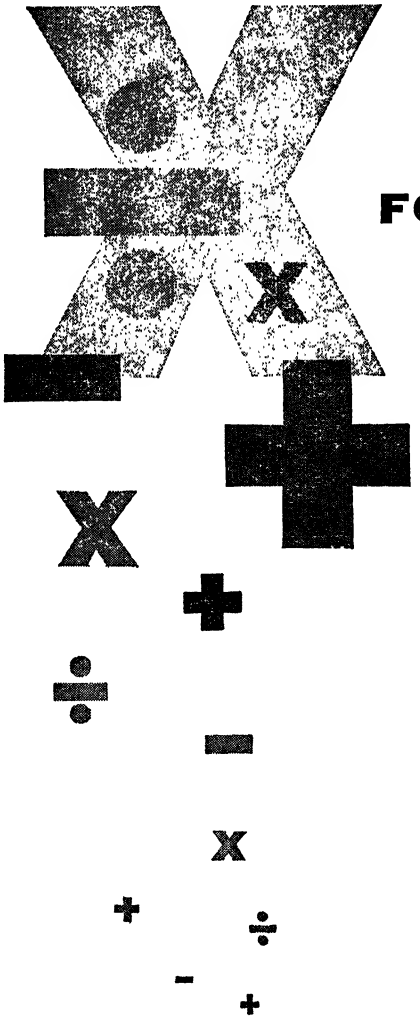
CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.
INDIANA, Wabash College, Crawfordsville, May 5, 1956.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, May 5, 1956.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Augsburg College, Minneapolis, May 5, 1956.
MISSOURI
NEBRASKA
NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 26, 1956.
PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.
PHILADELPHIA
ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Marquette University, Milwaukee, May 12, 1956.



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JUNE-JULY 1956

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THE BASIC CONCEPTS OF ALGEBRAIC LOGIC

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1. Introduction. It has often happened that a theory designed originally as a tool for the study of a physical problem came subsequently to have purely mathematical interest. When that happens, the theory is usually generalized way beyond the point needed for applications, the generalizations make contact with other theories (frequently in completely unexpected directions), and the subject becomes established as a new part of pure mathematics. The part of pure mathematics so created does not (and need not) pretend to solve the physical problem from which it arises; it must stand or fall on its own merits.

Physics is not the only external source of mathematical theories; other disciplines (such as economics and biology) can play a similar role. A recent (and possibly somewhat surprising) addition to the collection of mathematical catalysts is formal logic; the branch of pure mathematics that it has precipitated will here be called *algebraic logic.*

Algebraic logic starts from certain special logical considerations, abstracts from them, places them into a general algebraic context, and, *via* the generalization, makes contact with other branches of mathematics (such as topology and functional analysis). It cannot be overemphasized that algebraic logic is more algebra than logic. Algebraic logic does not claim to solve any of the vexing foundation problems that sometimes occupy logicians. All that is claimed for it is that it is a part of pure mathematics in which the concepts that constitute the skeleton of modern symbolic logic can be discussed in algebraic language. The discussion serves to illuminate and clarify those concepts and to indicate their connection with ordinary mathematics. Whether the subject as a whole will come to be considered sufficiently interesting and sufficiently deep to occupy a place among pure mathematical theories remains to be seen.

The literature of algebraic logic is not yet very extensive, and the few items that are available are highly technical. It is for that reason that this expository paper was written; its main purpose is to kindle interest in a young but promising subject. In such a context it does not seem to be appropriate to burden the reader with the usual scholarly references and assignments of credit. At the very least, however, the names of the principal contributors should be mentioned. Here they are: Curry, Henkin, Rasiowa, Sikorski, and Tarski. Many of the ideas of algebraic logic have been in the air for several years and were known, at least subconsciously, by most logicians. The greatest contributions are those of Tarski; especially relevant is his work (with Jónsson) on Boolean algebras with operators and his theory of cylindric algebras. (Most of the latter material is unfortunately unpublished.) The reader who wishes to study the details will find exact references in two papers on algebraic logic by the present author; the first is in *Compositio Mathematica* (1955), and the second is to appear in *Fundamenta Mathematicae* (1957).

2. Boolean algebras. The father of algebraic logic is George Boole; it is appropriate that a discussion of the subject begin with a quick review of the algebras that bear his name. The shortest definition of Boolean algebras is in terms of the theory of rings: a *Boolean algebra* is a ring with unit in which every element is idempotent (*i.e.*, if p is in the ring, then $p^2=p$). The simplest example, and one that plays a vitally important role throughout the theory, is the field of integers modulo 2; this Boolean algebra will be denoted by **O**.

Boolean algebras have an almost embarrassingly rich structure. It is, for instance, an easy consequence of the definition that a Boolean algebra always has characteristic 2 (*i.e.*, $p+p=0$) and that as a ring it is always commutative (*i.e.*, $pq=qp$). In every Boolean algebra there is, moreover, a natural order relation; it is defined by writing $p\leqq q$ if and only if $pq=p$. The algebraic structure and the order structure are as compatible as they can be. The algebraic zero is also the order zero (*i.e.*, the least element of the algebra), and the algebraic unit is also the order unit (*i.e.*, the greatest element of the algebra); in other words, $0\leqq p\leqq 1$ for every p. With respect to the order, the algebra turns out to be a complemented lattice. The lattice operations can be expressed in terms of the given algebraic operations, as follows: the complement of p, denoted by p', is $1+p$; the infimum of p and q, denoted by $p\wedge q$, is pq; and the supremum of p and q, denoted by $p\vee q$, is $p+q+pq$.

The process of defining many useful operations and relations in a Boolean algebra, in terms of addition and multiplication, is to a large extent reversible. This fact is responsible for the abundance of different axiomatic approaches to the subject. A Boolean algebra can be defined in terms of its partial order, or in terms of complements and suprema, or in terms of complements and infima, and so on and so forth *ad* almost *infinitum*. Thus, for instance, since sums and products can be expressed in terms of complements and infima ($pq=p\wedge q$ and $p+q=(p'\wedge q')'\wedge(p\wedge q)'$), it follows that if a set admits a unary operation and a binary operation satisfying the appropriate conditions, then that set is a Boolean algebra. (It isn't really, but it is close enough to make the distinction pedantic. What should be said is that if addition and multiplication are defined as indicated above, then the set, together with the defined operations, constitutes a Boolean algebra.) The appropriate conditions are simple to describe. They require that the underlying set contain at least two distinct elements, and that

(I 1) $$p\wedge(q\wedge r)=(p\wedge q)\wedge r,$$

(I 2) $$p\wedge q=q\wedge p,$$

(I 3) $$\text{if } p\wedge q'=r\wedge r', \text{ then } p\wedge q=p,$$

(I 4) $$\text{if } p\wedge q=p, \text{ then } p\wedge q'=r\wedge r',$$

for all elements p, q, and r. (Caution: (I 3) means that if $p\wedge q'=r\wedge r'$ for *some* r, then $p\wedge q=p$, and (I 4) means that if $p\wedge q=p$, then $p\wedge q'=r\wedge r'$ for *all* r.)

Since, inside a fixed non-empty set, set-theoretic complementation and the formation of set-theoretic intersections do satisfy these conditions, any class of sets that is closed under these two operations is a Boolean algebra. The class of all subsets of a non-empty set is the most easily described (but far from the only) example of a Boolean algebra obtained in this way.

[Terminological purists sometimes object to the Boolean use of the word "algebra." The objection is not really cogent. In the first place, the theory of Boolean algebras has not yet collided, and it is not likely to collide, with the theory of linear algebras. In the second place, a collision would not be catastrophic; a Boolean algebra is, after all, a linear algebra over the field of integers modulo 2. The last, but not the least, pertinent comment is a pragmatic one. While, to be sure, a shorter and more suggestive term than "Boolean algebra" might be desirable, the nomenclature is so thoroughly established that to change now would do more harm than good.]

It is amusing and instructive to compare the axiom system (I) with the following, somewhat unorthodox, system of axioms for groups. A group may be defined as a non-empty set with a unary operation $(p \to p^-)$ and a binary operation $((p, q) \to p \times q)$ such that

(II 1) $$p \times (q \times r) = (p \times q) \times r,$$

(II 2) $$(p \times q)^- = q^- \times p^-,$$

(II 3) $$\text{if } p \times q = r \times r^-, \quad \text{then} \quad p = q^-,$$

(II 4) $$\text{if } p = q^-, \quad \text{then} \quad p \times q = r \times r^-,$$

for all elements p, q, and r. (Caution: (II 3) means that if $p \times q = r \times r^-$ for *some* r, then $p = q^-$, and (II 4) means that if $p = q^-$, then $p \times q = r \times r^-$ for *all* r.) It is clear that if, in a group, p^- is defined to be the inverse of p, and $p \times q$ is defined to be the product of p and q (in that order), then the conditions (II) are satisfied. The fact that, conversely, the conditions (II) are characteristic of inversion and multiplication in groups is an easy exercise in elementary axiomatics.

[One comment should be made on the independence of the axiom set (II). The fact is that the set is *not* independent; the first three axioms are sufficient to characterize groups, and, in particular, they imply the fourth. The reason (II) is offered in the present form is to emphasize its similarity with the axiom set (I) for Boolean algebras. Each of the axioms (II 1), (II 2), and (II 3) is independent of the other three axioms of the set (II).]

3. Propositional calculi. To understand the connection between Boolean algebra and logic, a good way to begin is to examine how sentences are combined by means of sentential connectives. To ensure an unprejudiced approach to the subject, it is desirable to proceed as abstractly as possible. Suppose, therefore, that there is given an arbitrary non-empty set S; intuitively the elements of S are to be thought of as the basic sentences of some theory that is being studied. Suppose, moreover, that A and N are two distinct objects not contained in S; intuitively A and N are to be thought of as the connectives "and" and "not."

(Warning: in other expositions, A is frequently used for the dual connective "or".) Consider finite sequences whose terms are either elements of S or else A or N. (The empty sequence is not included.) If s is a sequence of length n, say, so that $s_0, \cdots, s_{n-1}$ are elements of $S \cup \{A, N\}$, let Ns be the sequence defined by

$$(Ns)_0 = N, \qquad (Ns)_i = s_{i-1} \qquad (i = 1, \cdots, n);$$

if s and t are sequences (of lengths n and m respectively), let Ast be the sequence defined by

$$(Ast)_0 = A, \quad (Ast)_i = s_{i-1} \ (i = 1, \cdots, n), \quad (Ast)_{n+j} = t_{j-1} \ (j = 1, \cdots, m).$$

Let S^* be the smallest set of sequences such that (1) if s is a sequence of length 1 whose unique term is in S, then $s \in S^*$, (2) if $s \in S^*$, then $Ns \in S^*$, and (3) if $s \in S^*$ and $t \in S^*$, then $Ast \in S^*$. In other words, S^* is the set of sequences generated from the one-term sequences of S by means of the operations of prefixing N to a sequence and prefixing A to the concatenation of two sequences. Intuitively S^* is to be thought of as the set of sentences generated from the basic sentences by the two basic connectives. The device of writing Ast instead of $s\ A\ t$ (an ingenious Polish invention) is designed to avoid the multiple layers of parentheses that the more intuitive infix notation necessitates.

The set S^* by itself is not quite a proper object of logical study. The trouble is that if, for instance, s and t are in S^*, then Ast and Ats are distinct elements of S^*, whereas common sense seems to demand that if s and t are sentences, then "s and t" and "t and s" should be sentences that "say the same thing" in some sense. Sentences admit, in other words, a natural equivalence relation; such a relation should therefore be introduced into S^*. A little thought about the intuitive interpretation of the elements of S^* will suggest many conditions that equivalence should satisfy. If the assertion that the elements s and t of S^* are equivalent is denoted by $s \equiv t$, then, for all elements s, t, and u of S^*, it should at the very least be required that

(III 1) $$AsAtu \equiv AAstu,$$

(III 2) $$Ast \equiv Ats,$$

(III 3) $$\text{if } AsNt \equiv AuNu, \text{ then } Ast \equiv s,$$

(III 4) $$\text{if } Ast \equiv s, \text{ then } AsNt \equiv AuNu,$$

(III 5) $$\text{if } s \equiv t, \text{ then } Ns \equiv Nt,$$

(III 6) $$\text{if } s \equiv t, \text{ then } Asu \equiv Atu.$$

In addition, of course, it is necessary that the concept of equivalence used here be an honest equivalence, *i.e.*, that it be reflexive, symmetric, and transitive.

There are likely to be many equivalence relations satisfying the conditions (III); one such is defined by writing $s \equiv t$ for all s and t. In order to avoid this triviality (and for other reasons), it is desirable to consider the smallest possible

equivalence (*i.e.*, the one in which the fewest possible pairs turn out to be equivalent) satisfying these conditions. This makes sense. Indeed, if an equivalence relation is thought of as a certain set of ordered pairs, then the intersection of all the equivalence relations satisfying (III) is an equivalence relation satisfying (III). If, from now on, the symbol $\equiv$ is used to denote this minimal equivalence, then the pair $(S^*, \equiv)$ is one of the simplest non-trivial logical structures; it is usually known as the *propositional* (or *sentential*) *calculus*. There are really many propositional calculi; there is one corresponding to each non-empty set S. It is clear, however, that the only thing that matters (all other differences between the various propositional calculi being essentially notational matters) is the cardinal number of S. It is customary (but not particularly profitable) to assume that S is countably infinite.

4. Axioms and rules. The time has come to make the notation more transparent. While the symbols A and N are technically convenient (no parentheses), it is usually a cumbersome job to decode the sentences involving them and to recapture their intended intuitive content. Accordingly, in what follows, $(s)'$ will be used as an alternative symbol for Ns, and, similarly, $(s) \wedge (t)$ will be used as an alternative symbol for Ast. Thus, for instance, $AsNt$ can be denoted by $(s) \wedge ((t)')$; with the usual mathematical conventions about omitting superfluous parentheses, this becomes $s \wedge t'$. It should now be clear that the conditions (III 1)–(III 4) are only notationally different from (I 1)–(I 4); the conditions (III 5) and (III 6) assert, in customary mathematical language, that $\equiv$ is a congruence relation with respect to the operations of attaching $'$ and infixing $\wedge$.

The equivalence relations that occur in algebra (*e.g.*, the congruence relations in a ring) are usually described by specifying a particular equivalence class (the kernel) and a particular operation (subtraction); it then turns out that two elements are congruent if and only if their difference belongs to the kernel. A similar procedure is available to define the equivalence $\equiv$ described above. In order to motivate the choice of the kernel and the choice of the pertinent subtraction operation, consider first of all an element of S^* that has the form $s \wedge s'$. If the elements of S^* are interpreted as sentences, then there is something obviously undesirable about such an element; in some intuitive sense it is "false." By the same token, the result of attaching $'$ to such an element converts it into something quite laudable; sentences such as that are "true." The kernel that is usually considered is the equivalence class of any particular "true sentence" so obtained. It is pleasant to be able to report that this kernel is independent of the arbitrary choice of its generator. The equivalence class of any element of the form $(s \wedge s')'$ contains all other elements of that form, and, in fact, it consists exactly of all the elements that common sense would declare to be "true." An element of this kernel is called a *tautology* of the propositional calculus. The subtraction operation is the one that associates with two elements s and t of S^* the element $(s \wedge t')' \wedge (s' \wedge t)'$. (In the original notation this reads $ANAsNtNANst$.) The perceptive reader will note that if s and t are interpreted as sentences, then the intuitive interpretation of the proposed "difference" between s and t is the

sentence "s if and only if t." This subtraction does what it was tacitly promised it would do; it is true that a necessary and sufficient condition that s be equivalent to t is that the difference of s and t belong to the kernel. (In customary logical language: a necessary and sufficient condition for $s \equiv t$ is that the biconditional of s and t be a tautology. It is a happy circumstance that biconditioning is a commutative operation; the order of s and t is immaterial.)

In principle it is clear how the procedure used above could be reversed. The tautologies could be defined by making a complete list of them, and equivalence could then be defined in terms of tautologies and the formation of biconditionals. The list of tautologies would be infinite, to be sure, but a clever classifier might nevertheless succeed in describing it in a finite number of words. Something like this is what is usually done. In most text-book presentations of the propositional calculus one finds a small list of special tautologies, together with a few easy instructions for manufacturing others; a general tautology is then defined to be a member of the subset of S^* generated from the special tautologies by repeated applications of the instructions. The given special tautologies (whose particular choice is largely a matter of individual taste) are called *axioms*, and the instructions for manufacturing others are called *rules of inference*. This "axiomatic" procedure (which has become traditional) has no particular advantages (or disadvantages) in comparison with the one followed above; its final result is merely an alternative description of the propositional calculus.

[Here, for the sake of completeness, are the details of one of the popular axiomatic approaches to the propositional calculus. Let $s \vee t$ and $s \rightarrow t$ be abbreviations for $(s' \wedge t')'$ and $(s \wedge t')'$ respectively (or, in the original notation, for $NANsNt$ and $NAsNt$ respectively); the axioms are most conveniently, and most understandably, described by means of these abbreviations. The axioms consist of all those elements of S^* that are of one of the following four forms:

$$(1) \qquad (s \vee s) \rightarrow s,$$

$$(2) \qquad s \rightarrow (s \vee t),$$

$$(3) \qquad (s \vee t) \rightarrow (t \vee s),$$

$$(4) \qquad (s \rightarrow t) \rightarrow ((u \vee s) \rightarrow (u \vee t)),$$

where s, t, and u are arbitrary elements of S^*. There is only one rule of inference (called *modus ponens*). According to that rule, if s and t are elements of S^* such that both s and $s \rightarrow t$ are tautologies, then t is a tautology.]

5. Free Boolean algebras. By whatever method the pair $(S^*, \equiv)$ is obtained, once it is at hand it is natural to form the set $\mathbf{A}^*$ of all equivalence classes. In analogy with calling an element of S^* a sentence, an element of $\mathbf{A}^*$ may be called a *proposition*. (The word "proposition" is not always defined this way. The intuitive reasons for using the definition are obvious, but, admittedly, they are not overwhelming.) The set $\mathbf{A}^*$ of propositions possesses, in a natural way, the structure of a Boolean algebra. If $p \in \mathbf{A}^*$, let s be any element of the equivalence class p and write p' for the equivalence class of Ns (*i.e.*, of s'). If both s_1 and s_2

belong to p, *i.e.*, if $s_1 \equiv s_2$, then, by (III 5), $Ns_1 \equiv Ns_2$, so that the definition of p' is unambiguous. If p and q are in $\mathbf{A}^*$, let s and t be corresponding representative elements (*i.e.*, $s \in p$ and $t \in q$) and write $p \wedge q$ for the equivalence class of Ast (*i.e.*, of $s \wedge t$). The necessary unambiguity argument is based on (III 6) this time. The fact that the conditions (I) are satisfied now follows immediately from a comparison of those conditions with the corresponding conditions (III).

The procedure for arriving at $\mathbf{A}^*$ is familiar to anyone who ever heard of the theory of free groups; the Boolean algebra $\mathbf{A}^*$ is, in fact, isomorphic to the free Boolean algebra generated by the originally given set S. In logical studies it is customary to describe the algebraic properties of $\mathbf{A}^*$ by some rather specialized terminology. Thus, for instance, the fact that $\mathbf{A}^*$ satisfies the first necessary condition for being a Boolean algebra (the possession of at least two distinct elements) is usually described by saying that the propositional calculus is consistent.

In view of what has already been said, it is not at all surprising that the construction of $\mathbf{A}^*$ has a very near parallel in group theory. Given an arbitrary non-empty set S, form the set S^* exactly as before and introduce into S^* the smallest equivalence relation $\equiv$ such that

(IV 1) $$AsAtu \equiv AAstu,$$

(IV 2) $$NAst \equiv ANtNs,$$

(IV 3) $$\text{if } Ast \equiv AuNu, \quad \text{then} \quad s \equiv Nt,$$

(IV 4) $$\text{if } s \equiv Nt, \quad \text{then} \quad Ast \equiv AuNu,$$

(IV 5) $$\text{if } s \equiv t, \quad \text{then} \quad Ns \equiv Nt,$$

(IV 6) $$\text{if } s \equiv t, \quad \text{then} \quad Asu \equiv Atu,$$

for all elements s, t, and u of S^*. The set of all equivalence classes possesses, in a natural way, the structure of a group. The obvious details may safely be omitted; suffice it to say that the proof depends on a comparison of (II) with (IV). The group so obtained is, in fact, isomorphic to the free group generated by the originally given set S.

[The method of sequences-*cum*-equivalence is essentially the only known way of constructing free groups. It is worth noting that for Boolean algebras the following much more elegant method is available. Given an arbitrary (possibly empty) set S, let X be the set of all subsets of S. For each s in S, let $p(s)$ be the set of all those elements of X that contain s. Let $\mathbf{A}^*$ be the Boolean algebra generated by all the sets of the form $p(s)$ with s in S, so that $\mathbf{A}^*$ is a Boolean subalgebra of the Boolean algebra of all subsets of X. Assertion: $\mathbf{A}^*$ is isomorphic to the free Boolean algebra generated by S. The proof of the assertion is an easy exercise in set theory. Unfortunately this method of constructing $\mathbf{A}^*$ is quite provincial; it works like a charm for Boolean algebras, but it does not work for very many algebraic systems, at least not without significant modifications.]

6. Quotients of free algebras. Since the algebraic end-product of the propositional calculus is a free Boolean algebra, that calculus, as it now stands, is too special for most logical purposes. (Freedom for an algebraic system means, roughly speaking, that the only relations among its elements are the trivial ones). The difficulty is visible in the starting point of the construction of the propositional calculus; its source is the fact that the generating set S is a set with no algebraic structure at all. In realistic situations the basic sentences of a theory are likely to be related to each other in various intricate ways, and, in fact, it is usually considered to be one of the chief functions of logic to study such relations and their consequences.

Suppose, for instance, that someone wants to undertake a logical examination of a fragment of the history of English literature. Among the basic sentences of the theory (*i.e.*, among the elements of S) there might occur "Shakespeare wrote Hamlet" and "Bacon wrote Macbeth"; call these sentences s_0 and t_0 respectively. ("Basic sentence" is not the same as "axiom." The sentences s_0 and t_0 need not, at first, be considered either "true" or "false"; they are just considered.) Under these circumstances the investigator would perhaps want to declare the sentence NAs_0t_0 to be an axiom. Since, however, NAs_0t_0 is not a tautology of the propositional calculus, the word "axiom" must be used here in a sense broader than the one discussed above. Such a broadening is perfectly feasible. One way to achieve it is to add another condition to the set (III); the new condition could, for instance, be

$$\text{(III 7)} \qquad As_0t_0 \equiv As_0Ns_0.$$

The effect of this adjunction is to enlarge the original equivalence relation; the new equivalence relation is not the smallest relation satisfying (III 1)–(III 6), but the smallest relation satisfying (III 1)–(III 7). The same purpose can be accomplished by adjoining to the axioms of the propositional calculus the *extra-logical axiom*

$$NAs_0t_0,$$

and then proceeding exactly as before. The result, with either approach, is that each new equivalence class is the union of several old ones. The tautologies, in particular, all belong to the same (new) equivalence class; an element of this equivalence class is usually called a *provable sentence* or a *theorem* of the modified propositional calculus. If there are at least two (new) equivalence classes (in logical language: if the adjoined axioms are consistent), the set **A** of all such classes possesses in a natural way the structure of a Boolean algebra, which, in general, is not free.

The "free" procedure for constructing non-free Boolean algebras has its group-theoretic parallel. One way to ensure that the generators of a group satisfy certain prescribed relations is to build those relations into the equivalence by means of which the quotient system (S^* modulo $\equiv$) is formed. The disadvantage

of such a procedure is that it is repetitious; it uses the method of an earlier construction instead of its result. It is more usual, and algebraically more satisfactory, to apply the combinatorial method only once; thereafter, all groups defined by generators and relations are constructed by forming a quotient group of one of the free groups already obtained. This works very smoothly; the main reason it works is that every group is a quotient group of a free group.

Since, similarly, every Boolean algebra is a quotient algebra of a free Boolean algebra, the group-theoretic shortcut is available for Boolean algebras too. Just as in the theory of groups, moreover, the core of the idea is applicable to algebras that are not necessarily free. It makes sense (and it is often useful) to force the elements of a Boolean algebra (or group) to satisfy some relations, even if the algebra (or group) is not free to start with. A well-known example is the process of reducing a group by its commutator subgroup and thereby forcing it to become abelian.

7. Filters and ideals. From the point of view of logic, the reduction of Boolean algebras is connected with the theory of provability. It turns out that from the point of view of algebraic logic the most useful approach to that theory is not to ask "What is a proof?" or "How does one prove something?" but to ask about the structure of the set of provable propositions. Suppose, therefore, that **A** is a Boolean algebra, whose elements are to be thought of, intuitively speaking, as the propositions of some theory, and suppose that a non-empty subset **P** of **A** has been singled out somehow; the elements of **P** are to be thought of as the provable propositions of the theory. Common sense suggests that if s and t are provable sentences, then "s and t" is a provable sentence, and if s is a provable sentence, then "s or t" is a provable sentence, no matter what the sentence t may be. In order to meet these demands of common sense, the set **P** cannot be arbitrary; it must be such that if both p and q belong to **P**, then $p \wedge q$ belongs to **P**, and if p belongs to **P**, then $p \vee q$ belongs to **P** for all q in **A**. If a non-empty subset of a Boolean algebra satisfies these conditions, it is called a *filter*. An illuminating comment is this: a necessary and sufficient condition that a subset **P** of a Boolean algebra be a filter is that $1 \in \mathbf{P}$ (*i.e.*, all tautologies are provable), and that if $p \in \mathbf{P}$ and $(p \rightarrow q) \in \mathbf{P}$ (recall that $(p \rightarrow q) = (p' \vee q)$), then $q \in \mathbf{P}$ (*i.e.*, *modus ponens* is a rule of inference).

A filter is not a commonly encountered mathematical object, but one of its first cousins (namely, an ideal) is known to every mathematician. A non-empty subset **M** of a Boolean algebra is an *ideal* (for occasional emphasis, a *Boolean ideal*) if it contains $p \vee q$ whenever it contains both p and q and if it contains $p \wedge q$ whenever it contains p. Although it looks slightly different, this definition is, in fact, equivalent to the usual one; a Boolean algebra is, after all, a ring, and a Boolean ideal in the present sense is the same as an ordinary algebraic ideal.

Each of the two concepts (filter and ideal) is, in a certain sense, the Boolean dual of the other. (Some authors indicate this relation by using the term *dual-ideal* instead of filter.) Specifically, if **P** is a filter in a Boolean algebra **A**, and if

$\mathbf{M}$ is the set of all those elements p of $\mathbf{A}$ for which $p' \in \mathbf{P}$, then $\mathbf{M}$ is an ideal in $\mathbf{A}$; the reverse procedure (making a filter out of an ideal) works similarly. This comment indicates the logical role of the algebraically more common concept; just as filters arise in the theory of provability, ideals arise in the theory of refutability. (A proposition p is called refutable if its negation p' is provable.) Duality is so ubiquitous in Boolean theory that every development of that theory (unless it is exactly twice as long as it should be) must make an arbitrary choice between two possible approaches. Logic is usually studied from the "1" approach, *i.e.*, the emphasis is on truth and provability, and, consequently, on filters. Since the dual "0" approach uses the algebraically more natural concept of ideal, the remainder of this exposition (addressed to mathematicians, rather than to professional logicians) will be couched in terms of the logically less pleasant concepts of falsehood and refutability.

8. Boolean logics. The preceding discussion was intended to motivate the following definition. A *Boolean logic* is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is a Boolean algebra and $\mathbf{M}$ is a Boolean ideal in $\mathbf{A}$. The elements of $\mathbf{A}$ will be called propositions; the elements of $\mathbf{M}$ will be called refutable propositions. The group-theoretic analogue of this concept (the structure consisting of a group and a specified normal subgroup) has not received very much attention, but it would strike most algebraists as a perfectly reasonable object of study.

The concept of a Boolean logic $(\mathbf{A}, \mathbf{M})$ has many similarities with the earlier concept of the propositional calculus $(S^*, \equiv)$. Both objects consist of (1) a set already endowed with some structure, and (2) a congruence relation in that set. (In Boolean theory, as in the rest of algebra, there is a natural one-to-one correspondence between ideals and congruence relations.) It should not be surprising therefore that the "axiomatic" method is the most common way of converting a Boolean algebra into a Boolean logic. In algebraic terms the axiomatic method amounts simply to this: given $\mathbf{A}$, select an arbitrary subset $\mathbf{M}_0$ of $\mathbf{A}$, and let $\mathbf{M}$ be the ideal generated by $\mathbf{M}_0$. (Because $\mathbf{M}$ consists of the refutable propositions, not the provable ones, the elements of $\mathbf{M}_0$ are "anti-axioms"; their negations are axioms in the usual sense.) Even this procedure has its group-theoretic analogue; for an example, recall the usual definition of the commutator subgroup of a group.

Most logical concepts have an algebraic alter-ego definable within the theory of Boolean logics. Two concepts of special importance (consistency and completeness) will be used to illustrate this point.

A Boolean logic $(\mathbf{A}, \mathbf{M})$ is called *consistent* if for no proposition p in $\mathbf{A}$ are both p and p' provable, or, equivalently, if for no p in $\mathbf{A}$ do both p and p' belong to $\mathbf{M}$. Since $\mathbf{M}$ is an ideal, it follows that $(\mathbf{A}, \mathbf{M})$ is consistent if and only if the ideal $\mathbf{M}$ is proper. To say that a Boolean logic is consistent is to say, roughly speaking, that the set of propositions that are refutable in it is not too large.

On intuitive (pragmatic) grounds it is desirable that the set of refutable propositions be not too small. (Recall that there is a natural one-to-one correspondence between refutable propositions and provable propositions.) The

simplest way to ensure that the set of refutable propositions is large enough is to insist that, for every proposition p, either p or p' be refutable. A Boolean logic satisfying this condition is called *complete*. In other words, $(\mathbf{A}, \mathbf{M})$ is complete if and only if, for every p in $\mathbf{A}$, either $p \in \mathbf{M}$ or $p' \in \mathbf{M}$.

Inconsistent Boolean logics, *i.e.*, logics of the form $(\mathbf{A}, \mathbf{A})$ are not very interesting from either the algebraic or the logical point of view. For a consistent logic $(\mathbf{A}, \mathbf{M})$, the concept of completeness has an elegant algebraic formulation: the logic is complete if and only if the ideal $\mathbf{M}$ is maximal in the algebra $\mathbf{A}$. ("Maximal ideal" in such contexts always means "maximal proper ideal.")

If $(\mathbf{A}, \mathbf{M})$ is a Boolean logic, it is natural to form the quotient system $\mathbf{A}/\mathbf{M}$. A necessary and sufficient condition that $\mathbf{A}/\mathbf{M}$ be a Boolean algebra is that it have at least two distinct elements, and a necessary and sufficient condition for that is exactly that $\mathbf{M}$ be a proper ideal in $\mathbf{A}$. On the other hand, a necessary and sufficient condition that $(\mathbf{A}, \mathbf{M})$ be complete is that $\mathbf{A}/\mathbf{M}$ have at most two distinct elements; if this is so, and if $\mathbf{M} \neq \mathbf{A}$, then $\mathbf{A}/\mathbf{M} = \mathbf{O}$. (Recall that $\mathbf{O} = \{0, 1\}$. On universal algebraic grounds, the condition that a proper ideal $\mathbf{M}$ be maximal in an algebra $\mathbf{A}$ is equivalent to the condition that the quotient algebra $\mathbf{A}/\mathbf{M}$ be *simple*, *i.e.*, that $\mathbf{A}/\mathbf{M}$ have no non-trivial proper ideals. The only simple Boolean algebra is $\mathbf{O}$.) Conclusion: a necessary and sufficient condition that a Boolean logic $(\mathbf{A}, \mathbf{M})$ be both consistent and complete is that $\mathbf{A}/\mathbf{M} = \mathbf{O}$.

For a deeper study of concepts such as consistency and completeness additional logical apparatus is needed; some of it is described below. First, a terminological warning. Consistency and completeness, as defined above, are usually called *simple consistency* and *simple completeness*; perhaps *syntactic* would be a more suggestive adjective here than *simple*. The point is that the propositional calculus, for instance, is usually considered to be a "language," and (simple) consistency and completeness are defined in terms of the intrinsic structure (syntax) of that language. Two related concepts, to be discussed below, are defined in terms of a possible external interpretation (meaning) of the language, and are therefore appropriately called *semantic*. The group-theoretic analog of a semantic concept is one that depends on constructions reaching outside the given group, *i.e.*, typically, on representations of the group. Thus, for instance, "characteristic subgroup" is a syntactic concept, while "character" is a semantic one.

In most of what follows it will be simpler to forget about Boolean logics and to consider Boolean algebras instead. The point is that if $(\mathbf{A}, \mathbf{M})$ is a consistent Boolean logic, then the quotient algebra $\mathbf{A}/\mathbf{M}$ can be formed and can be used to study virtually all the logically important properties of $(\mathbf{A}, \mathbf{M})$. Note that if a proposition in $\mathbf{A}$ is refutable, then its image in $\mathbf{A}/\mathbf{M}$ (its equivalence class modulo $\mathbf{M}$) is equal to 0, and, similarly, if a proposition in $\mathbf{A}$ is provable, then its image in $\mathbf{A}/\mathbf{M}$ is equal to 1. It is, accordingly, convenient to agree that if p is an element of a Boolean algebra, then "p is refutable" shall be a long way of saying "$p = 0$," and, similarly, "p is provable" shall be a long way of saying "$p = 1$."

9. Quantifiers. Within the framework of Boolean algebras (or logics) it is easy to give an algebraic formulation of the inference from the premises "Some Greeks are men" and "All men are mortal" to the conclusion "Some Greeks are men." This is not a misprint. Within the framework of Boolean algebras alone it is not possible to formulate the inference that allows, from the same premises, the conclusion "Some Greeks are mortal." The desired inference is justified not by manipulating with propositions as a whole, but by the intrinsic structure of its constituents, and, in particular, by a study of what "some" and "all" mean.

The clue is in the consideration of propositional functions. As their name indicates, propositional functions are functions whose values are propositions. If, for instance, for every natural number x, $p(x)$ is the sentence "x is even" and $q(x)$ is the sentence "$2x=1$," then p and q are propositional functions. (The present discussion is only heuristic, of course, but even so the contrasting use of "sentence" and "proposition" needs a little justification. The justification is a very common one in mathematics; it is convenient, though incorrect, to identify an equivalence class with a representative element. Recall that a proposition was defined above as an equivalence class of sentences. The solecism is the same as the one every analyst commits when he speaks of an element of L_2 as a function.) From the algebraic point of view, a propositional function is a function defined on an arbitrary non-empty set with values in a Boolean algebra.

To single out a particular theory for logical examination means, algebraically, to fix a Boolean algebra $\mathbf{B}$. If, in addition, a certain non-empty set X is selected, then the ground is prepared for a discussion of propositional functions. (Intuitively the elements of X may be thought of as the objects that the propositions of the theory talk about.) The set $\mathbf{A}$ of all functions from X to $\mathbf{B}$ is in a natural way a Boolean algebra (pointwise operations), and the same is true of many of its subsets. The constant functions (*i.e.*, the functions p obtained by selecting an element p_0 in $\mathbf{B}$ and writing $p(x)=p_0$ for every x in X) constitute a subalgebra of the algebra $\mathbf{A}$; that subalgebra is obviously isomorphic to $\mathbf{B}$. In other words, the propositions of a theory are (or, rather, may be identified with) particular propositional functions of that theory.

What is the effect of "some" in a sentence such as "for some x, x is even"? (It is understood here that the underlying set X is the set of natural numbers; the value-algebra $\mathbf{B}$ has not been, and need not be, specified). The most striking effect it has is to convert a propositional function into a constant, or, *via* the identification convention of the preceding paragraph, into a constant function. This effect is analogous to the effect of "sup" in "$\sup_x f(x)$" and to the effect of "$\int_0^1 \cdots dx$" in "$\int_0^1 f(x)dx$" (where the function f is, say, a real-valued continuous function defined on the closed unit interval). Accordingly, the algebraic analogue of "some" ought to be a mapping that sends a certain Boolean algebra (propositional functions) into a certain subalgebra (constant functions). Such a mapping (usually denoted by $\exists$) is indeed at the basis of the theory of propositional functions.

It is now wise to abstract from the motivation. Consider a perfectly arbitrary Boolean algebra **A**, and let $\exists$ be a mapping of **A** into itself. If **A** were an algebra of propositional functions, and if $\exists$ were the operator "some," then, presumably, $\exists$ would satisfy some rather special conditions; the problem now is to find a reasonable list of conditions that characterize such a special $\exists$. This is easy; here are some of them:

(Q 1) $$\exists 0 = 0,$$

(Q 2) $$p \leqq \exists p,$$

(Q 3) $$\exists(p \vee q) = \exists p \vee \exists q,$$

(Q 4) $$\exists\exists p = \exists p.$$

(In these conditions, and also in (Q 5) below, p and q are arbitrary elements of **A**). The intuitive grounds for the conditions are easy to see. Suppose, for instance, that $q(x)$ is "$2x = 1$"; here, once more, **A** is to be thought of as an algebra of propositional functions whose domain is the set of natural numbers. Since in all reasonable theories of the arithmetic of natural numbers the sentence "$2x = 1$" is refutable, each value of the propositional function q is refutable, and therefore (in accordance with the agreement concerning the reduction of Boolean logics to Boolean algebras) the propositional function q is equal to the constant function 0. The same considerations apply to the sentence "for some x, $2x = 1$" and thus serve to complete the illustration of (Q 1). The remaining conditions (Q 2), (Q 3), and (Q 4) are illustrated similarly. In intuitive terms, (Q 2) says that each value of p implies that "for some x, $p(x)$"; the construction of analogous readings of (Q 3) and (Q 4) is left as an exercise to the reader.

It is amusing to observe that (except for notation) the conditions (Q 1)–(Q 4) are well-known to most mathematicians (but in a very different context). They are, all but verbatim, the Kuratowski axioms for a closure operator on a topological space. They do not, however, serve to characterize the "some" operator; in technical language, the theory of closure algebras is not co-extensive with the monadic functional calculus. Another glance at the conditions as they now stand should arouse the suspicion that something is missing; the trouble is that they relate $\exists$ to $\vee$ only (0 and $\leqq$ can be defined in terms of $\vee$), and say nothing about the relation of $\exists$ to either $\wedge$ or $'$. The missing condition is this:

(Q 5) $$\exists(\exists p)' = (\exists p)'.$$

In intuitive terms, (Q 5) serves as at least a partial reminder of the fact that the constant functions form a Boolean algebra (so that, in particular, they are closed under complementation), and that "some" applied to a constant function has no effect.

[Two remarks are in order. (1) The conditions (Q 1)–(Q 5) are equivalent to a shorter set, namely to (Q 1), (Q 2), and

(Q 6) $$\exists(p \wedge \exists q) = \exists p \wedge \exists q.$$

The proof of this equivalence is just axiom-chopping. The longer set was selected for presentation here because of its greater intuitive content. (2) The motivation of (Q 1)–(Q 5), and also of (Q 6), could have been, and sometimes is, based on quasi-geometric considerations, involving the formation of (possibly infinite) Boolean suprema. The didactic danger of this motivation is its emphasis on the intuitive idea that "some" is just an infinite "or." The idea is not wholly unsound, but real progress in algebraic logic was achieved only after the realization that $\exists$ is really a unary operation, not an infinitary one.]

The way is now clear to a precise definition: an *existential quantifier* is a mapping $\exists$ of a Boolean algebra into itself satisfying the conditions (Q 1)–(Q 5). Dually, a *universal quantifier* is a mapping $\forall$ of a Boolean algebra into itself satisfying the conditions

$$\text{(Q' 1)} \qquad \forall 1 = 1,$$

$$\text{(Q' 2)} \qquad \forall p = p,$$

$$\text{(Q' 3)} \qquad \forall(p \wedge q) = \forall p \wedge \forall q,$$

$$\text{(Q' 4)} \qquad \forall\forall p = \forall p,$$

$$\text{(Q' 5)} \qquad \forall(\forall p)' = (\forall p)'.$$

It is easy to see that $\forall$ bears the same relation to the intuitive "all" as $\exists$ bears to "some."

There is a very close connection between existential quantifiers and universal ones: if $\exists$ is an existential quantifier on, say, a Boolean algebra $\mathbf{A}$, and if a mapping $\forall$ of $\mathbf{A}$ into itself is defined by

$$\forall p = (\exists p')',$$

then $\forall$ is a universal quantifier on $\mathbf{A}$; if, in reverse, a universal quantifier $\forall$ is given, and if $\exists$ is defined by

$$\exists p = (\forall p')',$$

then $\exists$ is an existential quantifier. ("Always p" is the same as "not sometimes not p," and "sometimes p" is the same as "not always not p"). The thoroughgoing symmetry of this situation justifies an asymmetric treatment; anything that can be said about an $\exists$ has an obvious dual about an $\forall$, and it is therefore sufficient to discuss in detail only one of these two objects. In what follows $\exists$ will be given preferred treatment, and, in fact, the word "quantifier" will be used from now on in the sense of "existential quantifier." (This is in line with the algebraists preference for ideals instead of filters; the dual of an algebraist would select $\forall$ for preferred treatment.) Universal quantifiers, whenever they must be considered, will always be given their full name.

10. Monadic algebras. Once the concept of a quantifier is at hand, it is immediately possible to generalize the concept of a Boolean algebra in a manner adapted to an algebraic answer to the question of why some Greeks are mortal. Technically the generalized algebras (called monadic algebras) play an inter-

mediate role; they are slightly more useful than Boolean algebras, but not nearly so useful as certain even more general algebras (the polyadic algebras that will be described a little later). Psychologically and historically, however, the intermediate generalization has some value. Its psychological value is that it exhibits in a simplified form some of the curious properties of its generalized version; its historical value is that it puts into a modern algebraic context the oldest known systematic treatment of logic, namely Aristotle's syllogistics.

A *monadic* (Boolean) *algebra* is a pair $(\mathbf{A}, \exists)$, where $\mathbf{A}$ is a Boolean algebra and $\exists$ is a quantifier on $\mathbf{A}$. (The word "monadic" serves as a reminder of the one additional operation that distinguishes monadic algebras from Boolean algebras.) The elementary theory of monadic algebras is a routine matter; subalgebras, homomorphisms, ideals, and similar universal algebraic concepts (with a qualifying "monadic" when clarity demands it) are defined in a completely unsurprising manner. Free monadic algebras can be defined and studied (if desired) by the techniques used in the study of free Boolean algebras; the analogue of the propositional calculus is called the *monadic functional calculus*.

The concept of a monadic logic arises naturally in connection with the theory of provability and refutability in the monadic functional calculus: a *monadic logic* is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is a monadic algebra (with quantifier $\exists$, say) and $\mathbf{M}$ is a monadic ideal in $\mathbf{A}$. (It is convenient here, as in other parts of algebra, to be mildly forgetful of the completely rigorous definitions, and, accordingly, to identify a monadic algebra with the underlying Boolean algebra. If this were not done, a monadic logic would have to be denoted by a symbol such as $((\mathbf{A}, \exists), \mathbf{M})$.) The fact that a monadic ideal is a Boolean ideal invariant under the application of $\exists$ corresponds to the logical fact that if p is a refutable propositional function, then $\exists p$ is also refutable.

To discuss the (simple, or syntactic) consistency and completeness of Boolean logics, it is desirable to introduce a new term: an element p of a monadic algebra will be called *closed* if $\exists p = p$. Intuitively, closed propositions correspond to what were called constant functions before, or, in other words, to propositions instead of propositional functions. There is a natural way of associating a Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ with every monadic logic $(\mathbf{A}, \mathbf{M})$; the algebra $\mathbf{A}_0$ is the set of all closed elements of $\mathbf{A}$ and the ideal $\mathbf{M}_0$ is the intersection of $\mathbf{M}$ with $\mathbf{A}_0$. A monadic logic $(\mathbf{A}, \mathbf{M})$ is called syntactically consistent (or complete) if the associated Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ is consistent (or complete).

Why is it necessary to modify the Boolean definitions of consistency and of completeness for the monadic situation? Consistency could conceivably be defined by the requirement that for no p in $\mathbf{A}$ should both p and p' belong to $\mathbf{M}$, and, similarly, completeness could be defined by the requirement that for every p in $\mathbf{A}$ either p or p' should belong to $\mathbf{M}$. What is wrong with these definitions? The answer is that there is nothing wrong as far as consistency is concerned, and very much is wrong as far as completeness is concerned. The alternative definition of consistency is equivalent to the one officially adopted above, but the alternative definition of completeness is out of harmony with both the official

definition and common sense. If, as in some previous examples, $p(x)$ is "x is even," then it is not at all reasonable to demand of a logic sufficient power to settle the provability or refutability of the propositional function p. The sentences "for some x, x is even" and "for all x, x is even" are closed; a reasonable logical theory of arithmetic should declare each of them to be either provable or refutable. The function p, however, is not a sentence; common sense demands that both it and its negation should fail to be either provable or refutable.

11. Syllogisms. The discussion of monadic algebras and logics up to this point was merely an adaptation of the corresponding discussion of Boolean facts. Some progress has been made, nevertheless; monadic logics (unlike Boolean logics) contain the general theory of syllogisms. For an example, consider again the premises "Some Greeks are men" and "All men are mortal," and consider the desired conclusion "Some Greeks are mortal." To make an algebraic model of the situation, let X be an appropriate set and let $\mathbf{B}$ be an appropriate Boolean algebra of propositions about the elements of X. The set X, for instance, could be the set of all animals, and $\mathbf{B}$ could be an algebra containing, for each x in X, the propositions "x is Greek," "x is a man," and "x is mortal." (This description is, of course, much too colloquial for complete precision, but there is no difficulty at all in converting it into honest mathematics.) If these propositions are denoted by $p(x)$, $q(x)$, and $r(x)$, respectively, and if $\mathbf{A}$ is the algebra of all propositional functions such as p, q, and r, then $\mathbf{A}$ (with the intuitively obvious "some" quantifier in the role of $\exists$) is a monadic algebra suitable for the study of the inference described above. The first premise is $\exists(p \wedge q)$, the second premise is $\forall(q \rightarrow r)$, and the conclusion is $\exists(p \wedge r)$. (The universal quantifier $\forall$ here is the natural dual of the given existential quantifier $\exists$.) The algebraic justification of the inference is that if $\mathbf{A}$ is the monadic algebra of a monadic logic such that both premises belong to the filter of provable propositions, then the conclusion also belongs to that filter.

There is a special aspect of the theory of syllogisms that still remains to be converted into algebra; it is exemplified by the classical premises "Socrates is a man" and "All men are mortal," together with the conclusion "Socrates is mortal." In highly informal language, the trouble with this syllogism is that the algebraic theory (so far) is equipped to deal with generalities only, and is unable to say anything concrete; Socrates, however, is a concrete individual entity. The preceding paragraph shows that a monadic algebra can be taught to say "All men are mortal." The reason is that "manhood" can easily be thought of as an element of a monadic algebra, since it is the obvious abstraction of the propositional function whose value at each point x of some set is "x is a man." Socrates, on the other hand, is a "constant," and there is no immediately apparent way of pointing to him. (The use of the word "constant" here and below is quite different from its earlier use in the phrase "constant function." The important concept now is what logicians call an individual constant.) A classical artifice, designed to deal with just this difficulty, is to promote Socrates to a propositional function, *i.e.*, to identify him with the function whose value at x is

"x is Socrates." This procedure is both intuitively and algebraically artificial; Socrates should not be a propositional function but a possible argument of such functions.

To find the proper algebraization of the concept of a constant it is necessary only to recall that the elements of a monadic algebra are abstractions of the concept of a propositional function, and to determine the algebraic effect of replacing the argument of such a function by some fixed element of its domain. If p is a propositional function with domain X, and if $x_0 \in X$, then $p(x_0)$ is a proposition, or, *via* the obvious identification convention, a propositional function with only one value. The mapping $p \to p(x_0)$ (the evaluation map induced by x_0) clearly preserves the Boolean operations (*i.e.*, suprema, infima, and complements). If the function p itself has only one value (equivalently, if $p = \exists q$ for some q), then that value is $p(x_0)$, *i.e.*, the mapping leaves the range of $\exists$ elementwise fixed. If, on the other hand, $\exists$ is applied to the (constant) function $p(x_0)$, the result is the same function, *i.e.*, $\exists$ leaves the range of the mapping elementwise fixed. These considerations motivate the following general definition: a *constant* of a monadic algebra $\mathbf{A}$ is a Boolean endomorphism c of $\mathbf{A}$ such that

$$c\exists = \exists \quad \text{and} \quad \exists c = c.$$

This definition is applicable to the mortality of Socrates. If, as before, q is manhood and r is mortality, and if Socrates is taken to be a constant, say c, of a monadic algebra containing q and r, then the algebraic justification of the classical syllogism is this: if $\mathbf{A}$ is the monadic algebra of a monadic logic such that both $\forall(q \to r)$ and cq belong to the filter of provable propositions, then cr also belongs to that filter.

Constants are much more important than their more or less casual introduction above might indicate; the concept of a constant (suitably generalized to the polyadic situation) is probably the most important single concept in algebraic logic. This should not be too surprising; in the intuitive interpretation, the constants of a theory constitute, after all, the subject matter that the propositions of the theory talk about. Algebraically constants play a crucial role in the representation theory of monadic (and polyadic) algebras.

12. Quantifier algebras. The theory of propositional functions of only one variable (and of their abstract algebraic counterparts) is as insufficient for the understanding of logic and its applications as the theory of ordinary numerical functions of one variable is insufficient for the understanding of calculus. Mathematics contains not only sentences of the form "x is positive," but also sentences such as "x is less than y" and "x is between y and z." Although a mathematician with no training in modern logic is likely to be suspicious when he is told that syllogisms are not enough for mathematics, the basis of that assertion is nothing more profound or esoteric than the need to consider propositional functions of several variables. This is all that DeMorgan meant when he said that the scholastics, after two millennia of Aristotelean tradition, were still unable to prove that if a horse is an animal, then a horse's tail is an animal's tail.

Suppose, to begin modestly, that one wants to study propositional functions of three variables. The most immediately apparent new phenomenon is the possibility of partial quantification: the theory must be able to treat sentences of the form "there is an x_1 and there is an x_3, such that $p(x_1, x_2, x_3)$." (Here, as before, it is enough to discuss existential quantifiers; the corresponding theory of universal quantifiers is obtained by a simple dualization.) Another way of expressing the phenomenon is to say that the appropriate algebraic system must possess not one quantifier but many. What appears to be going on is this: if I is the set of relevant indices, so that $I = \{1, 2, 3\}$ in the present example, then there is a quantifier (*i.e.*, an existential quantifier) corresponding to each subset (*e.g.*, to $\{1, 3\}$) of the set I. If $\exists(J)$ is the quantifier corresponding to the subset J of I, then, of course, the way that $\exists(J)$ depends on J should be specified. The example shows the way. If J is empty, then prefixing $\exists(J)$ to p should produce no change in p, and if both J and K are subsets of I, then prefixing $\exists(J)$ and $\exists(K)$ to p, in either order, should have the same effect as prefixing the quantifier corresponding to the union of J and K.

The preceding paragraph suggests the definition of an algebraic system, which, however, turns out to fall far short of what is needed. It is worth a brief look anyway. Call a *quantifier algebra* a triple $(\mathbf{A}, I, \exists)$, where $\mathbf{A}$ is a Boolean algebra, I is a set, and $\exists$ is a function from subsets of I to quantifiers on $\mathbf{A}$ such that

$$(\exists\ 1) \qquad \exists(\varnothing)p = p$$

whenever $p \in \mathbf{A}$, and

$$(\exists\ 2) \qquad \exists(J)\exists(K) = \exists(J \cup K)$$

whenever J and K are subsets of I. In analogy with logical usage, an element of the set I will be called a *variable* (or, in more detail, an *individual variable*) of the quantifier algebra $(\mathbf{A}, I, \exists)$. Caution: a variable, in this sense, does not vary at all; it merely serves as a reminder of the place into which a "variable," in the intuitive sense, could be substituted, if, that is, the elements of $\mathbf{A}$ were propositional functions and not elements of a quite abstract Boolean algebra.

The concept of a quantifier algebra is a proper generalization of the concept of a monadic algebra; a quantifier algebra $(\mathbf{A}, I, \exists)$ for which the set I of variables consists of exactly one element may be identified with the monadic algebra $(\mathbf{A}, \exists(I))$. The *degree* of a quantifier algebra is defined to be the cardinal number of the set of its variables; the last comment says that quantifier algebras of degree 1 are essentially the same as monadic algebras. It is crucial for applications to permit the consideration of quantifier algebras of infinite degree. While, to be sure, each particular propositional function that is likely to arise has only finitely many arguments, the number of arguments in a reasonably extensive theory (*e.g.*, in mathematics) is not likely to be bounded. To keep the generalization from running away with itself, it is often advisable to restrict the study of quantifier algebras to the locally finite case. The quantifier algebra $(\mathbf{A}, I, \exists)$ is

called *locally finite* if to every element p of $\mathbf{A}$ there corresponds a finite subset J of I such that $\exists(I-J)p=p$; this is the abstract formulation of the concrete promise that no propositional function under consideration will actually depend on infinitely many variables.

13. Polyadic algebras. The reason that quantifier algebras are an inefficient logical tool is that they do not allow the discussion of transformations of variables. Consider, for instance, a sentence of the form "if $p(x_1, x_2)$ and $p(x_2, x_1)$, then $x_1=x_2$." Such sentences usually occur in the definition of a partial order; they indicate the need for discussing the passage from $p(x_1, x_2)$ to $p(x_2, x_1)$. For a more complicated example of the same type consider the problem of converting $p(x_1, x_2, x_3, x_4)$ into $p(x_4, x_4, x_1, x_3)$. There is no way of describing such conversions in terms of Boolean operations and quantifications alone; the best way of incorporating them into the general theory is by postulating them outright. What appears to be going on is this: to every transformation τ on I (*i.e.*, to every mapping of the set I into itself) there corresponds a mapping, say $\mathbf{S}(\tau)$, of the set of propositional functions under consideration into itself. The mappings $\mathbf{S}(\tau)$ preserve the Boolean operations (*i.e.*, they are Boolean endomorphisms). If τ is the identity transformation on I (to be denoted by δ), then $\mathbf{S}(\tau)$ leaves every p invariant; if both σ and τ are transformations on I, then the effect of $\mathbf{S}(\sigma)$ on $\mathbf{S}(\tau)p$ is the same as the effect of $\mathbf{S}(\sigma\tau)$ on p.

In analogy with quantifier algebras, it is possible to define a *transformation algebra* as a triple $(\mathbf{A}, I, \mathbf{S})$, where $\mathbf{A}$ is a Boolean algebra, I is a set, and $\mathbf{S}$ is a function from transformations on I to Boolean endomorphisms on $\mathbf{A}$, such that

$$(\mathbf{S}\ 1) \qquad \mathbf{S}(\delta)p = p$$

whenever $p \in \mathbf{A}$, and

$$(\mathbf{S}\ 2) \qquad \mathbf{S}(\sigma)\mathbf{S}(\tau) = \mathbf{S}(\sigma\tau)$$

whenever σ and τ are transformations on I.

Neither quantifier algebras nor transformation algebras are of much logical significance; the important concept is one that possesses both structures at the same time. It is a matter of universal algebraic experience, however, that it is not enough to impose two different structures on the same set; in order to get a usable theory, it is necessary to describe the compatibility conditions that relate the two structures to each other. (A topological group is not merely a set that is simultaneously a group and a topological space.) The appropriate conditions are discovered by experimentation with the special structures that the theory is intended to generalize; their final justification is their success. To examine here the experimentation that led to the definition of polyadic algebras would be more boring than profitable; suffice it to report the results. A *polyadic* (Boolean) *algebra* is a quadruple $(\mathbf{A}, I, \mathbf{S}, \exists)$ subject to the following conditions: (i) ($\exists$ 1) and ($\exists$ 2) hold, or, more precisely, $(\mathbf{A}, I, \exists)$ is a quantifier algebra, (ii) ($\mathbf{S}$ 1) and ($\mathbf{S}$ 2) hold, or, more precisely, $(\mathbf{A}, I, \mathbf{S})$ is a transformation algebra, and (iii)

($\exists\mathbf{S}$ 1) $$\mathbf{S}(\sigma)\mathbf{\exists}(J) = \mathbf{S}(\tau)\mathbf{\exists}(J)$$

whenever J is a subset of I and σ and τ are transformations on I such that $\sigma i = \tau i$ for all i in $I - J$, and

($\exists\mathbf{S}$ 2) $$\mathbf{\exists}(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)\mathbf{\exists}(\tau^{-1}J)$$

whenever J is a subset of I and τ is a transformation on I such that two distinct elements of I are never mapped by τ onto the same element of J.

The conditions ($\exists\mathbf{S}$ 1) and ($\exists\mathbf{S}$ 2) were recorded here for the sake of completeness only; no technical use will be made of them in this merely descriptive report. It is perhaps worth remarking though that they (or, rather, the provisos that accompany them) are not so complicated as they seem on first sight. What they amount to is a condensation of the usual and intuitively obvious relations between quantifications and transformations. Suppose, for example, that i and j are distinct elements of I; let J be the singleton $\{i\}$ and let τ be the transformation that maps i onto j and everything else (including j itself) onto itself. Since τ agrees with δ outside J, it follows from ($\exists\mathbf{S}$ 1) that $\mathbf{S}(\tau)\mathbf{\exists}(J) = \mathbf{\exists}(J)$; this equation corresponds to the familiar fact that once a variable has been quantified, the replacement of that variable by another one has no further effect. To get another example, note, for the same τ and J, that $\tau^{-1}J = \varnothing$. It follows from ($\exists\mathbf{S}$ 2) that $\mathbf{\exists}(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)$; this equation corresponds to the familiar fact that once a variable has been replaced by another one, a quantification on the replaced variable has no further effect.

[Polyadic algebras stand in the same relation to the so-called *pure* first-order functional calculus as do Boolean algebras to the propositional calculus. The simplest example of an *applied* functional calculus is one that is equipped to discuss the concept of equality. By a suitable adaptation of standard logical methods, such applied calculi can be treated within the framework of polyadic algebras; the details are of no relevance here. What should be mentioned, however, is that there is another way of algebraizing the functional calculus with equality, namely, *via* Tarski's concept of a cylindric algebra. Roughly speaking, a cylindric algebra is a quantifier algebra together with certain distinguished elements; the distinguished elements play the role of sentences that assert equations among variables. The theory of cylindric algebras is an efficient algebraic tool for studying calculi with equality. The exact relations between polyadic algebras and cylindric algebras are of considerable technical interest; they are still in the process of being clarified. It is already known, however, that in most important cases the two concepts are equivalent. The way that transformations enter into the theory of cylindric algebras is an ingenious trick; the idea is that to say "$p(x_1, x_1)$" is the same as to say "there is an x_2 such that $p(x_1, x_2)$ and $x_2 = x_1$."]

14. Semantic concepts. The discussion of polyadic logics and their syntactic consistency and completeness proceeds in a straightforward manner. A *polyadic logic* is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is a polyadic algebra and $\mathbf{M}$ is a polyadic ideal in

A. An element p of a polyadic algebra **A** is called *closed* if $\exists(I)p=p$ (where I, of course, is the set of variables of **A**). Just as for monadic logics, there is a natural way of associating a Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ with every polyadic logic $(\mathbf{A}, \mathbf{M})$; the algebra $\mathbf{A}_0$ is the set of all closed elements of **A** and the ideal $\mathbf{M}_0$ is the intersection of **M** with $\mathbf{A}_0$. A polyadic logic $(\mathbf{A}, \mathbf{M})$ is called syntactically consistent (or complete) if the associated Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ is consistent (or complete).

All this is pretty routine stuff by now. The element of novelty is in the study of the "semantic" theory of polyadic logics, *i.e.*, in the study of "interpretations" of a logic in a "model" and in the study of "truth" and "validity." (It must be admitted that these concepts could have been introduced in connection with Boolean logics and monadic logics also. Since, however, in those simple situations, they exhibit the deceptive simplicity of a degenerate case, their premature study would have been more confusing than helpful.) The natural habitat of semantic theories is situated somewhere near a polyadic *logic*, but the machinery for examining such a theory is easier for a polyadic *algebra*. The reduction of (syntactically consistent) logics to algebras is an easy step: if $\mathbf{M}\neq\mathbf{A}$, then $\mathbf{A}/\mathbf{M}$ can be formed and can be used to study $(\mathbf{A}, \mathbf{M})$. (This kind of thing was done before; *cf.* the corresponding discussion in the Boolean case.) Some concepts disappear in the passage from $(\mathbf{A}, \mathbf{M})$ to $\mathbf{A}/\mathbf{M}$ and others take on an interesting new guise. Thus, for instance, to say that a polyadic algebra is (syntactically) consistent is to say merely that it is there; on the other hand, to say that a polyadic algebra is (syntactically) complete is to say that it is simple. (Just as for Boolean logics, it turns out that a polyadic logic $(\mathbf{A}, \mathbf{M})$, with $\mathbf{M}\neq\mathbf{A}$, is syntactically complete if and only if the polyadic ideal **M** is a maximal ideal in the polyadic algebra **A**.)

The usual way to describe a model is this: take a non-empty set X, interpret the variables (*i.e.*, the elements of I) as variables (in the intuitive sense) varying over X, and interpret the elements of **A** as propositional functions on an appropriate Cartesian power of X with values that are either true or false. This is somewhat vague; a slight change in language converts it, however, into an algebraically precise and clean definition. If X is a non-empty set and if I is an arbitrary set, the set of all functions from the Cartesian product X^I into the Boolean algebra **O** possesses in a natural way the structure of a polyadic Boolean algebra; a *model* is, by definition, a polyadic subalgebra of such a total functional algebra.

Once the concept of a model is known, the definitions of the remaining semantic concepts are automatic. The purpose of the following paragraphs is first to define the usual semantic concepts and then to examine their algebraic significance.

An *interpretation* of a polyadic algebra $(\mathbf{A}, I, \mathbf{S}, \exists)$ in a model is a polyadic homomorphism from **A** onto the model. An element p of **A** is *true* in an interpretation f if $fp=1$; if $fp=0$, then p is *false* in the interpretation. (Caution: fp is a function from X^I into **O** and is therefore not necessarily either 0 or 1. To say that p is true for f means that the image fp of p under f is the constant function

whose value is 1.) An element p of **A** is *valid* if it is true in every interpretation; if p is false in every interpretation, then p is called *contravalid*. An element p of **A** is *satisfiable* if there is an interpretation for which it is not false (*i.e.*, if p is not contravalid).

The algebra **A** is *semantically consistent* if it has an interpretation in a model. The definition of semantic completeness requires a little more motivation. It is a trivial consequence of the definition of Boolean homomorphism (sometimes even built in as part of the definition) that the unit element of **A** is always valid, and, dually, that the zero element is always contravalid; in logical terms this says that provable propositions are valid, and refutable ones are contravalid. (The assumption of syntactic consistency is usually, and quite properly, made explicit at this point. In the present discussion, dealing with algebras instead of logics, the role of that assumption is played by the fact that **A** is an algebra at all, or, in more detail, that **A** has a unit element distinct from zero.) Semantic completeness requires the converse assertions; the algebra **A** is *semantically complete* if the unit is the only valid element, or, equivalently, if every non-zero element is satisfiable. In logical terms these requirements say that everything valid is provable, or, equivalently, that everything that is not refutable is satisfiable.

All the preceding definitions are based on one of them, namely, on the definition of a model. The situation can be better understood if it is first generalized and then specialized in a new direction. A model was defined as an element of a certain particular class of polyadic algebras, but, in the subsequent definitions, no special properties of that class were used. Those definitions, in other words, are relative to the prescribed class $\mathfrak{M}$ of models, and to nothing else; a significant generalization of them is obtained if $\mathfrak{M}$ is replaced by an arbitrary class $\mathfrak{C}$ of algebras. The concepts so obtained may be referred to by using $\mathfrak{C}$ as a prefix, so that expressions such as "$\mathfrak{C}$-interpretation" and "$\mathfrak{C}$-valid" now make sense. (Only in case $\mathfrak{C}=\mathfrak{M}$ may the prefix be omitted.) An interesting specialization is obtained by letting the class $\mathfrak{S}$ of all simple algebras play the role of $\mathfrak{C}$. The discussion that follows will be in terms of $\mathfrak{S}$ rather than $\mathfrak{M}$; the vitally important relation between $\mathfrak{S}$ and $\mathfrak{M}$ will be treated afterward.

To say that an algebra **A** is semantically $\mathfrak{S}$-consistent means, by definition, that there exists a homomorphism from **A** onto a simple algebra. This, in turn, is equivalent, on universal algebraic grounds, to the existence of a maximal ideal in **A**. The problem of the existence of maximal ideals is well known in algebra; the cases in which the solution is affirmative can usually be settled by a straightforward application of Zorn's lemma. The theory of polyadic algebras is one of these pleasant cases; Zorn's lemma applies and proves that maximal ideals always do exist. In other words, every polyadic algebra is (semantically) $\mathfrak{S}$-consistent. The situation is reminiscent of syntactic consistency, and, in fact, it turns out that the two concepts are the same. It is profitable to return (but just for a moment) to logics instead of algebras. Recall that a logic (**A**, **M**) is syntactically consistent if and only if **M** is a proper ideal in **A**. The general definition

of semantic $\mathfrak{C}$-consistency for a logic $(\mathbf{A}, \mathbf{M})$ requires the existence of an algebra $\mathbf{C}$ in the class $\mathfrak{C}$ and the existence of a homomorphism f from $\mathbf{A}$ onto $\mathbf{C}$ such that $fp = 0$ whenever $p \in \mathbf{M}$. If $\mathfrak{C}$ is $\mathfrak{S}$, then this is equivalent to the requirement that $\mathbf{M}$ be included in some maximal ideal. Since Zorn's lemma can be used to show not only that maximal ideals exist, but that, in fact, every proper ideal is included in some maximal ideal, syntactic consistency and semantic $\mathfrak{S}$-consistency are the same whether they are approached *via* logics ($\mathbf{M}$ is a proper ideal) or *via* algebras ($\mathbf{A}$ is a non-degenerate algebra).

To say that an algebra $\mathbf{A}$ is semantically $\mathfrak{C}$-complete means, by definition, that for every non-zero element p of $\mathbf{A}$ there exists an algebra $\mathbf{C}$ in the class $\mathfrak{C}$ and there exists a homomorphism f from $\mathbf{A}$ onto $\mathbf{C}$ such that $fp \neq 0$. In this formulation $\mathfrak{C}$-completeness may not look familiar even to a professional algebraist; an easy argument serves, however, to prove that the definition is equivalent to one that is very well known indeed. The fact is that $\mathbf{A}$ is $\mathfrak{C}$-complete if and only if $\mathbf{A}$ is a *subdirect sum* of algebras belonging to the class $\mathfrak{C}$. The argument (and even the definition of subdirect sum) will be omitted here; they are mentioned only to show that the concept ($\mathfrak{C}$-completeness for an arbitrary $\mathfrak{C}$) does fit into the framework of algebra.

In view of the relation between maximal ideals on the one hand and homomorphisms onto simple algebras on the other hand, the definition of semantic $\mathfrak{S}$-completeness for an algebra $\mathbf{A}$ reduces to this: to every non-zero element of $\mathbf{A}$ there corresponds a maximal ideal not containing it. Equivalently: the intersection of all the maximal ideals of $\mathbf{A}$ is the singleton $\{0\}$. This too is a well-known algebraic phenomenon. In analogy with some other parts of algebra, a polyadic algebra will be called *semisimple* if the intersection of all its maximal ideals is trivial; the result is that a polyadic algebra is $\mathfrak{S}$-complete if and only if it is semisimple, or, equivalently, if and only if it is a subdirect sum of simple algebras.

15. The Gödel theorems. The title of this paper promised a discussion of the basic concepts of algebraic logic, and indeed the definitions of the basic *concepts* were emphasized much more than the basic *results* concerning them. As a partial rectification of this unbalanced state of affairs the present (concluding) section is devoted to the statement of two of the deepest theorems of the field. The original formulations and proofs of these justly celebrated theorems are due to Gödel; the results are known as the Gödel completeness theorem and the Gödel incompleteness theorem.

The word "completeness" in "completeness theorem" refers to semantic completeness. The algebraic statement of the theorem is short and elegant: *every locally finite polyadic algebra of infinite degree is semantically complete* (or, equivalently, $\mathfrak{M}$-complete). The crux of the proof is a characterization of simple polyadic algebras. It is very easy to see that every model is a simple algebra; the hard thing to prove is that, conversely, every simple, locally finite polyadic algebra of infinite degree is (isomorphic to) a model. The proof makes use of the polyadic generalization of what was called a "constant" of a monadic algebra.

In terms of the symbols used above, the easy fact is that $\mathfrak{M} \subset \mathfrak{S}$ and the hard one is that (in the presence of local finiteness and infinite degree) $\mathfrak{S} \subset \mathfrak{M}$. In view of the equation $\mathfrak{S} = \mathfrak{M}$, semantic completeness (*i.e.*, $\mathfrak{M}$-completeness) is the same as $\mathfrak{S}$-completeness, and, consequently, the Gödel completeness theorem implies that every locally finite polyadic algebra of infinite degree is semisimple. In fact, *every* polyadic algebra is semisimple, and this assertion is sometimes taken to be the algebraic version of the Gödel completeness theorem. The proof of semisimplicity is quite easy; the crucial fact is not that something is semisimple but that in the most important cases semisimplicity is the same as semantic completeness.

The word "incompleteness" in "incompleteness theorem" refers to syntactic incompleteness. The result here is not so general as the completeness theorem; the subject matter is not all polyadic algebras, but only relatively few of them. The algebras covered by the theorem are usually described by saying that they are adequate for elementary arithmetic. Because a precise explanation of what this means would involve a rather long and technical detour, no such explanation will be presented here, and, consequently, the incompleteness theorem will be described rather than stated. The striking qualities of the theorem are sufficiently great to remain visible even under such cavalier treatment.

Even if the definition of the class of algebras under consideration is not made explicit, it is convenient to have a short phrase in which to refer to them; in what follows "Peano algebra" will be used instead of "polyadic algebra that is adequate for elementary arithmetic." The Gödel incompleteness theorem asserts the existence of "undecidable propositions." Since in the passage from logics to algebras the statement that an element p is refutable (or provable) was identified with the statement that $p = 0$ (or $p = 1$), and since "undecidable" means "neither refutable nor provable," the assertion reduces to the existence of a (closed) element different from both 0 and 1. The ideal generated by such an element is a non-trivial proper ideal. Conversely, every non-trivial proper ideal contains an undecidable (closed) element. These facts indicate that the Gödel incompleteness theorem asserts the existence, in Peano algebras, of non-trivial proper ideals; the definition of syntactic completeness shows now that the theorem does indeed assert that something is (syntactically) incomplete. The usual, logical, formulation explicitly makes the assumption that the underlying logic is consistent; the treatment of algebras instead of logics makes it unnecessary to mention such an assumption here.

The Gödel theorem does not assert that every Peano algebra is syntactically incomplete. It asserts, instead, that the definition of Peano algebras is not a faithful algebraic transcription of all intuitive facts about elementary arithmetic. In algebraic terms this means that while some Peano algebras may be syntactically complete, there definitely exist others that are not. The situation is analogous to the one in the theory of groups. A class of polyadic algebras could be defined that is adequate for a discussion of elementary group theory. For this class the assertion that every two elements of a group commute would be un-

decidable, in the obvious sense that the assertion is true for some groups (and, therefore, corresponds to the unit element of some of the polyadic algebras under consideration) and false for others.

What has been said so far makes the Gödel incompleteness theorem take the following form: not every Peano algebra is syntactically complete. In view of the algebraic characterization of syntactic completeness this can be rephrased thus: not every Peano algebra is a simple polyadic algebra. This is the description that was promised above. What follows is another rephrasing of this description; the rephrasing, possibly of some mnemonic value, makes its point by making a pun. Consider one of the systems of axiomatic set theory that is commonly accepted as a foundation for all extant mathematics. There is no difficulty in constructing polyadic algebras with sufficiently rich structure to mirror that axiomatic system in all detail. Since set theory is, in particular, an adequate foundation for elementary arithmetic, each such algebra is a Peano algebra. The elements of such a Peano algebra correspond in a natural way to the propositions considered in mathematics; it is stretching a point, but not very far, to identify such an algebra with mathematics itself. Some of these "mathematics" may turn out to possess no non-trivial proper ideals, *i.e.*, to be syntactically complete; the Gödel theorem implies that some of them will certainly be syntactically incomplete. The conclusion is that the crowning glory of modern logic (algebraic or not) is the assertion: *mathematics is not necessarily simple.*

ZERO INTEGRAL HOMOGENEOUS SYMMETRIC FUNCTIONS IN CERTAIN RINGS*

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In this paper†, we shall discuss a type of function which we shall define at the start. If $f(c_0, c_1, \cdots, c_n)$ is a function with the c's in a ring, then f is called an *integral homogeneous symmetric function of degree n*, written IHSF, of degree n provided it satisfies:

(i) f is a polynomial in the c's which is unchanged by any permutation of the c's.

(ii) $f(tc_0, tc_1, \cdots, tc_n) = t^n f(c_0, c_1, \cdots, c_n)$.

Each elementary symmetric function is an IHSF. It is well known that in the field of real numbers each elementary symmetric function of the nth roots of unity is zero except the one of highest degree. More generally if

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0$$

has a decomposition in a field, then in this field, $a_i = S_i(-1)^i$, where S_i is the

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sum of the products of the roots of the above equation, taken i at a time, $i=1, 2, \cdots, n$.

Many theorems are known which give us congruences involving symmetric functions on the residue classes prime to a modulus. In some of these theorems the modulus is prime; in others, it is a power of a prime; and in still others, the modulus may have several distinct prime divisors. Some theorems have been given which relate symmetric functions with Bernoulli numbers [3, 7, 9]. Several theorems [1, 3, 9, 10] are also known which involve congruence relations on symmetric functions of residue classes not necessarily prime to a composite modulus. Symmetric functions have often been studied as products of the elementary symmetric functions; however, P. Hensel [6] studied the class of integral homogeneous symmetric functions and using the property of homogeneity, as we do in the proof of our main theorem, equation (2), obtained easily:

An IHSF of the numbers of the set A: $1, 2, \cdots, p-1$, *for* p *a prime, such that the degree of the function is not a multiple of* $p-1$, *is divisible by* p.

LaGrange's identity yields as a special case of Hensel's theorem, the well-known theorem:

If S_i *is the sum of the products of the elements of the set* A *above, taken* i *at a time,* $0<i<p-1$, *then* $S_i \equiv 0 \pmod{p}$.

Authors of recent textbooks have neglected both the method of Hensel and his theorem, but mention this special case. Wolstenholme's theorem is also often mentioned:

If t *varies over the set* A *and* $1/t$ *is defined by* $(1/t)t \equiv 1 \pmod{p^2}$, *then* $\sum 1/t \equiv 0 \pmod{p^2}$, *for* $p>3$.

It is the object of this paper to prove a theorem which contains as a special case a corollary which contains Hensel's theorem and one similar to that of Wolstenholme. Both the methods used and the results obtained are similar to methods and results of Vandiver's paper [9]. This together with the fact that our theorem also applies to symmetric functions involving non-units as well as units in certain rings suggests its considerable generality. The statement that u is a *unit* of the ring R means that R contains a multiplicative identity e, and that u divides this identity; e is called the *unity* element.

In a recent paper [11] the writer discussed cosets and divisor groups in semi-groups. If a commutative semi-group S contains a subsemi-group S' and a is an element of S, the *coset* aS' was defined to be the set of elements obtained by multiplying a by each element of S'; a was called the *coset multiplier* of aS'. A subgroup G of S was called a *divisor group* if: (i) Each element of S is in some coset aG; (ii) The cosets of S with respect to G are disjoint; and (iii) This division into cosets with respect to G is unique in the sense that each two divisions of S into cosets with respect to G contain the same cosets. For instance the group of units modulo 6 is a divisor group of the multiplicative semi-group modulo

6. A minimal set M of coset multipliers such that each element of S can be written in the form $a_i g$ where a_i is an element of M and g is an element of G will be called, in the present paper, a *complete set of coset multipliers of S with respect to G*. We shall need Theorem 7 and a special case of Theorem 1 of [11]. We state them respectively below as theorems B and A.

THEOREM A. *A finite semi-group S contains the subgroup G as a divisor group if the identity of G is an identity of S.*

THEOREM B. *If a semi-group is divided into left cosets with respect to a left divisor group G, a necessary and sufficient condition that two left cosets aG and bG be identical is that there exist an element g of G such that $a = bg$.*

We now proceed with our principal theorem.

THEOREM. *Let a finite commutative ring R with unity e contain among its units a multiplicative divisor group G of order a and an element i such that $i^{ad} - e$ is also a unit for d a positive integer; then each IHSF of degree d of the a-th powers of a complete set of coset multipliers of R with respect to G is zero in R.*

Proof. By using Theorem A we can write the distinct elements of R as

$$
\begin{aligned}
&w_1,\ w_1g_{s_1},\ w_1g_{s_2},\ \cdots,\ w_1g_{s_x},\\
&w_2,\ w_2g_{t_1},\ w_2g_{t_2},\ \cdots,\ w_2g_{t_y},\\
&\cdots\cdots\cdots\cdots\\
&w_b,\ w_bg_{u_1},\ w_bg_{u_2},\ \cdots,\ w_bg_{u_z},
\end{aligned}
\tag{1}
$$

where the g's are in G, and the w's form a complete set of coset multipliers of R with respect to G. Let S be an IHSF of degree d of the a-th powers of the w's. Then

$$
S((iw_1)^a, (iw_2)^a, \cdots, (iw_b)^a) = i^{ad} S(w_1^a, w_2^a, \cdots, w_b^a). \tag{2}
$$

If $1 \leqq k \leqq b$, we have from (1) that there exists an integer m, $1 \leqq m \leqq b$, and an element g_r of G such that

$$
iw_k = w_m g_r. \tag{3}
$$

Hence, since R is commutative

$$
(iw_k)^a = (w_m g_r)^a = w_m^a. \tag{4}
$$

Now as w_k varies over the complete set of coset multipliers in (3), so also does w_m, for if

$$
iw_k = w_m g_{r_i}
$$

and

$$
iw_{k_i} = w_m g_{r_i},
$$

where the w's are coset multipliers in (1) and the g's are in G, then

$$w_k = i^{-1}w_m g_{r_i},$$

and

$$w_{k_1} = i^{-1}w_m g_{r_j} = g_{r_n} i^{-1} w_m g_{r_i} = g_{r_n} w_k,$$

where g_{r_n} is an element of G such that $g_{r_n} = g_{r_j}$. This contradicts theorem B unless w_k is w_{k_i}. Therefore, by (2) and (4)

$$S((iw_1)^a, (iw_2)^a, \cdots, (iw_b)^a) = S = i^{ad}S,$$

and

$$(i^{ad} - e)S = 0.$$

Since $i^{ad} - e$ is a unit, it is not a divisor of 0. Hence, $S = 0$.

We can easily illustrate our theorem. Consider the ring R of residue classes modulo 21. It has as a divisor group G of order three of units, the group whose least positive residues are 4, 16, and 1, and a cyclic subgroup of order two of units, generated by the residue class whose least positive residue is 20. And $20^3 - 1 \equiv 19 \pmod{21}$, where 19 is the least positive residue of a unit. A complete set M of coset multipliers of R with respect to G has as least positive residues the integers 1, 2, 3, 5, 7, 9, 10, 14, and 21. In the terminology of our theorem, $a = 3$, $n = 2$. Let $d = 1$ and $S \equiv \sum w_i^3 \pmod{21}$, where w varies over M. Then it is easy to verify that $S \equiv 0 \pmod{21}$.

Again let R be a Galois field with unity 1 and order p^n, and let a divide $p^n - 1$ and d be an integer such that $p^n - 1$ does not divide ad. Then R contains a multiplicative divisor group G of order a and a multiplicative cyclic subgroup H of order $p^n - 1$. Let the generator of H be i; then $i^{ad} - 1$ is a unit since $p^n - 1$ does not divide ad. Hence, an IHSF of degree d of the a-th powers of a complete set of coset multipliers of R with respect to G is zero in R unless $p^n - 1$ divides ad. This is theorem IV of Vandiver's paper [9].

COROLLARY. *Let r vary over the set A: 1, 2, $\cdots$, $p-1$ where p is an odd prime, and let d not be a multiple of $p-1$, k be prime to p, and t be an integer; then each IHSF of degree d of the p^{n-1}-st powers of the integers of the set B: $kr+tp$ is congruent to zero modulo p^n.*

Proof. We shall show that this corollary is a special case of the principal theorem. Let G be the cyclic [2] multiplicative subgroup of units modulo p^n of order $\phi(p^n) = p^{n-1}(p-1)$; and let g be a primitive root modulo p^n. Then there exists a subgroup H of order p^{n-1}; also there exists an element $g^{p^{n-1}}$ of order $p-1$; furthermore $g^{dp^{n-1}} - 1$ is a unit modulo p^n, otherwise $g^{dp^{n-1}} \equiv 1 \pmod{p}$ which contradicts the fact that d is not a multiple of $p-1$, for g is also a primitive root modulo p. To show that the elements of the set B are a complete set of coset multipliers of G with respect to H, we first notice that the elements of H are:

$$g^{vp},\ v = 1, 2, \cdots, p^{n-1}.$$

Certainly each element of the set B is an element of G. If

$$(kr_j + tp)g^{v(p-1)} \equiv kr_m + tp \pmod{p^n}$$

for r_j and r_m elements of A, and $1 \leqq v \leqq p^{n-1}$, then since $g^{v(p-1)} \equiv 1 \pmod{p}$ and $(k, p) = 1$, it follows that $r_j \equiv r_m \pmod{p}$, and $r_j = r_m$; hence by theorem B and Lagrange's theorem, the set B is a complete set of the coset multipliers of G with respect to H. Let R be the ring modulo p^n. One complete set C of the coset multipliers of R with respect of H is therefore the set B together with a set D of non-units. Then by our main theorem each IHSF of degree d of the p^{n-1}-st powers of the elements of C is 0 in the ring. Denote by $s_i(T)$ the sum of the products taken i at a time of the elements of a given set T. Let S be an IHSF of degree d of the p^{n-1}-st powers of the elements of the set B. Then S is expressible [8] as $S \equiv f(s_1(B'), s_2(B'), \cdots, s_{p-1}(B')) \pmod{p^n}$ where B' is the set of p^{n-1}-st powers of the elements of the set B. But $f \equiv g(s_1(C'), s_2(C'), \cdots, s_{p-1}(C')) \pmod{p^n}$ where C' is the set of p^{n-1}-st powers of the elements of the set C, since the p^{n-1}-st power of a non-unit of C is zero in R. Since S may be expressed as the sum of the non-zero terms of the IHSF g which is zero in R, it follows that $S \equiv 0 \pmod{p^n}$, and the corollary is proved.

If in our corollary $k = n = 1$ and $t = 0$, we obtain the theorem of Hensel, mentioned earlier.

If in our corollary $k = 1$, $n = 2$, $t = 0$, and $d = p^2 - p - 1$, we consider the function $S \equiv \sum r^{p(p^2-p-1)} \pmod{p^2}$ and find that $S \equiv 0 \pmod{p^2}$. This is the theorem already mentioned which resembles that of Wolstenholme; for the latter theorem may be expressed as $\sum r^{p^2-p-1} \equiv 0 \pmod{p^2}$ for $p > 3$.

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STACKING COLORED CUBES

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1. Introduction. The six faces of a cube can be painted with six different colors in thirty different ways. Thirty cubes so painted not only are the basis for a number of interesting games and puzzles, but also are excellent models for illustrating group theory, combinatorial counting, and cataloguing. This paper generalizes a puzzle of W. W. R. Ball [1] and extends the knowledge of these thirty cubes. It is hoped these results will help mathematicians make more use of this simple, attractive, yet complicated model. Possible applications to the theory of cubic crystal alloys also exist.

We set aside one cube, called the "key" cube. The three main theorems discussed are the following:

THEOREM I. *There are* 144,500 *different ways in which eight of the twenty-nine non-key cubes can be selected and stacked to form a* $2\times2\times2$ *cube such that* (1) *each face of the* $2\times2\times2$ *cube is a solid color, and* (2) *the colors have the same arrangement as those on the key cube.*

THEOREM II. *If a set of eight non-key cubes will stack properly, it will do so in* 2, 4, 8 *or* 16 *ways.*

THEOREM III. *There are* 67,260 *different sets of eight non-key cubes that will stack properly.*

These numerical results are valuable for some of the games and puzzles. However, actually proving these results, recognizing differences and avoiding duplications in the counting process, is even more interesting.

To fix our ideas, we call the colors A, B, C, D, E, F. Let us suppose the key cube to be the cube with A on top, F on the bottom, and $BCDE$ on the sides in counterclockwise order looking down. We call it $ABCDEF$.

The secret of the proof is to concentrate on the key corners rather than on the cubes. A key corner is one like one of the eight corners of the key cube. ABC is a key corner, with the colors listed in counterclockwise order as seen by an external observer looking directly at the corner.

Corner ABC lies on five different cubes. For, after three faces are painted A, B, and C, the remaining three can be painted D, E, and F in six different ways, forming six different cubes. One of these is the key cube, leaving five among the 29.

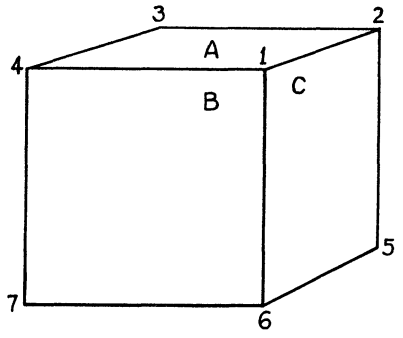
No cube can have three key corners except the key cube. For, if a cube has three key corners, two of them must be diagonal corners of one of the faces of the key cube. This would require that five of the faces be colored like the key cube, and hence all six.

A careful look at the five non-key cubes with corner ABC shows that each has exactly one other key corner as well. $ABCFDE$ and $ABCEFD$ have the key corner DFE. We note that ABC and DFE are end points of a solid diagonal of

the key cube. The other three cubes, $ABCDFE$, $ABCFED$, $ABCEDF$ have the three key corners ACD, AEB, BFC respectively. These three corners, with ABC, are the end points of the three edges of the key cube which meet at ABC.

Symmetry shows similar results for the other key corners. Hence the 29 cubes divide into three groups: (1) eight cubes with two key corners which are ends of the solid diagonals of the key cube, two for each of the four solid diagonals, (2) twelve cubes with two key corners which are ends of the twelve edges of the key cube, one for each edge, and (3) nine cubes with no key corners at all. The cubes of groups (1) and (2) are called "diagonal" and "edge" cubes respectively. It is easily seen that the nine cubes of group (3) are the mirror images of the key cube and the eight diagonal cubes of group (1).

2. Counting the number of ways eight cubes can be stacked. Clearly it is possible to find eight cubes which will stack. All one needs to do is to pick out eight key corners on eight different cubes. In particular, the eight diagonal cubes of group (1) will stack. However, not every set of eight cubes will stack. For example, if a cube of group (3) is included, the set will not stack, since this cube will fit nowhere.



To assist in the counting, the key corners are numbered according to the diagram in the figure. With this system every edge is bounded by one odd and one even numbered corner, and each solid diagonal is bounded by corners whose sum is nine. A cube with key corners is now named by the numbers of its two corners. The twelve edge cubes of group (2) are named 12, 14, 16, 32, 34, 38, 52, 56, 58, 74, 76, 78. The eight diagonal cubes of group (1) are named 18, 18′, 36, 36′, 54, 54′, 72, 72′, where prime merely distinguishes one of the two cubes which have the given diagonal key corners.

Since each corner appears on five different cubes, every freshman knows there are 5^8 or 390,625 different sets of eight different key corners. If each of these sets appeared on eight different cubes, this would be the number of ways of stacking eight cubes. Unfortunately, among these sets of corners will be those where both key corners of one, two, three, or even four cubes have been used. These sets of corners lie on less than eight cubes, which, therefore, do not stack. We subtract the number of such sets.

Let S, P, T, Q be the number of times both corners of a single, a pair, a triple, a quadruple of cubes are used. For example, consider the set of eight corners where corners one and two are taken from cube 12, corners three and four are taken from cube 34, and the remaining four corners from four different cubes. This set contributes two to the value of S and one to the value of P.

The number of different sets of eight corners on less than eight different cubes is $S-P+T-Q$. The total number of ways eight cubes can be stacked is $390{,}625-S+P-T+Q$.

Clearly both corners of the cube 12 are used in 5^6 sets, 5^6 being the number of ways the remaining six corners could be chosen after corners one and two are assigned to cube 12. Symmetry shows that the same is true for each cube. Hence $S=20\times5^6=312{,}500$.

Suppose both corners of a pair of cubes are used which include corners one and three. There are 19 such pairs, as is easiest seen by cataloguing. Each such pair is used in 5^4 sets of 8 corners. Since there are 6 pairs of odd corners like one and three, $P=19\times5^4\times6=71{,}250$.

In counting T and Q some caution is required because edge cubes and diagonal cubes have different effects in counting patterns. However, there are 49 triples of cubes including corners one, three and five. Hence $T=49\times5^2\times4=4900$.

Every quadruple of cubes with both corners used will contain a triple with both corners used including corners one, three and five. Thirty-three of the 49 such triples are made a quadruple by only one edge cube. Sixteen are made a quadruple by two diagonal cubes. Hence $Q=33\times1+16\times2=65$.

Hence eight cubes can be stacked in the following number of ways:

$$390{,}625 - 312{,}500 + 71{,}250 - 4900 + 65 = 144{,}500.$$

3. Multiple stacking.

THEOREM II. *If a set of eight non-key cubes will stack properly, it will do so in* 2, 4, 8 *or* 16 *ways.*

Proof. Every cube which is used has two key corners. Draw the eight line segments connecting these key corners. These line segments will form a space polygon which will contain one, two, three, or four closed polygons. For example, the cubes 12, 23, 34, 41 would yield a closed polygon. So would the cubes 18, 18′, although this polygon is somewhat degenerate, having but two sides, one on top of the other.

The set of cubes forming a closed polygon can be placed in two positions in stacking corresponding to the two senses of the polygon. This is the only way cubes may be stacked in more than one manner. Hence the cubes can be stacked in 2, 4, 8, 16 different ways as their space polygon has one, two, three or four closed polygons.

The eight diagonal cubes are the only set which will stack in 16 different ways. This set is also the only set which can be stacked with all abutting faces the same color. They are thus the solution to Ball's puzzle mentioned above.

4. Sets of cubes which will stack. The variety of ways sets of eight cubes will stack suggests the question "How many different sets of 8 cubes from the 29 will stack?" The answer is in the following theorem.

THEOREM III. *There are* 67,260 *different sets of eight non-key cubes that will stack properly.*

The proof of theorem III is somewhat abbreviated. Eight cubes from the set of twenty will stack unless (I) one or more key corners is missing, or (II) some, say x, of the key corners appear on fewer than x cubes, or (III) both. If I, II, and III also represent the numbers of sets of eight cubes respectively described, then the number of sets which stack is $C(20, 8) - \mathrm{I} - \mathrm{II} + \mathrm{III}$ where $C(20, 8)$ is the standard symbol for the combinations of 20 objects 8 at a time.

Sets of eight cubes exist with one, two, or even three key corners missing. In counting these it is necessary to be aware of the different arrangements of the missing corners. For example, two missing key corners might be ends of an edge, a face diagonal or a solid diagonal of the key cube. It turns out that single corners are missing 51,480 times, pairs of corners are missing 4500 times, and triples of corners are missing 24 times. Hence there are $51{,}480 - 4500 + 24 = 47{,}004$ sets of eight cubes with one or more key corners missing.

There are sets of eight cubes where two key corners appear on only one cube, three corners on only two cubes, four corners on only three cubes, and four corners on only two cubes. Again it is necessary to keep account of whether the doubled cubes are edge or diagonal cubes, and how the corresponding edges and diagonals are related. The total number of such sets is 12,378.

There are also 672 sets with one corner missing and two corners on one cube, or three corners on two cubes. Hence the total number of cubes which will stack is

$$C(20,8) - 47{,}004 - 12{,}378 + 672 = 67{,}260.$$

Two interesting corollaries are the following. The probability that eight cubes will stack when chosen at random from the twenty edge and diagonal cubes is $67{,}260/C(20, 8) = .54$. The probability that eight cubes will stack when chosen from 29 (all except the key cube) is $67{,}260/C(29, 8) = .016$.

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A NATIONAL WEAKNESS

Editorial note. This article is reprinted from the *Quarterly Report of the Carnegie Corporation of New York*, vol. 4, 1956.

Concern over the mathematical incompetence of the average—and even above-average—American has become almost a national preoccupation. Science and industry cry in vain for more and better mathematicians. Ordinary businesses ask only that their employees be able to do simple arithmetic. Neither the extravagant nor the modest demands of society for mathematicians—or arithmeticians even—are being met. And public concern grows.

A recent study shows that there are more than adequate explanations for this national weakness, and that many of them lie in the caliber of mathematical instruction. Under a grant from Carnegie Corporation, the Educational Testing Service (ETS) recently conducted a survey to identify some of the reasons why so many elementary and secondary school children either fail or drop mathematics or, at best, just manage to limp through. Here are some of the basic facts which inspired the ETS to undertake its study:

A recent national survey showed that since 1910 there has been a consistent decline in the proportion of high school students who take college preparatory mathematics. It is true that the character of the high school population has changed with the vast increase in numbers. But this does not explain away the fact that many students starting mathematics courses in high school do not see them through. Some drop elementary algebra before they finish the first year; more quit in the middle of geometry; still more leave intermediate algebra unfinished. These facts would suggest a certain indifference toward the subject on the part of many youngsters; actually in many cases it is positive detestation rather than mere indifference. Almost all available data show that students have a poorer attitude toward math than toward any other school subject; in one survey 40 per cent of the pupils honored math by electing it the subject they dislike most.

Another basic fact which impressed the ETS is that mathematical incompetence is widespread even among students of superior intellectual ability. In one high school, of 526 students with I.Q.'s above 114, 135 either dropped math, averaged C or lower, or were retarded one grade or more. The mean I.Q. of these 135 students was 123.

Faced with these bewildering and discouraging figures, the ETS committee assembled data and expert opinions on the two important factors which many people believe lie at the heart of the difficulty: teaching and curriculum. They surveyed the relevant literature, sent questionnaires to experts in a variety of fields, observed classroom procedures, and interviewed students. From all these sources, they learned some astonishing facts about math teaching in the United States today.

For instance, although all states require *education* courses for secondary

mathematics teachers, a third of the states require no *mathematics* for certification of math teachers. At the elementary level the situation is even worse. In the majority of instances a prospective elementary school teacher can enter a teachers college without any credits in secondary school math. In most states, a teacher can be certified to teach elementary school math without any work in math at the college level.

Under such circumstances, it is no surprise that one professor states: "Elementary teachers, for the most part, are ignorant of the mathematical basis of arithmetic. . . . " As for the secondary school teachers, one math professor interested in teacher education says "They are not as good as our run-of-mine juniors."

An observer who visited 60 representative math classrooms in different regions of the country came to the conclusion that genuine and efficient mathematical learning was going on in only eight of them. He got the inescapable impression from the 36 elementary classrooms he observed that in most cases the instruction is carried on in such a routine, haphazard way that learning of any kind is probably largely accidental. And his observation of 24 high school classrooms led him to think that teaching on that level is no better, and perhaps even worse, than it is on the elementary level.

As for the curriculum, little change has occurred in it despite extensive changes in the nature and knowledge of mathematics and despite the fact that since 1894 many authorities have been recommending changes. The result is that the curriculum is out of touch with the real needs for math in the world today, and also out of touch with the interests of the pupils. The only major development has been the growth in a ninth-grade "general mathematics" course, a subject that the students tend to regard with disdain but often take, as the lesser of two evils, rather than algebra.

Although the total picture of mathematical education that emerges from the ETS survey is hardly encouraging, there are some moves toward reform, particularly in the curriculum. Further reforms may be instituted as the educators and public who must meet the problems are provided with information, such as that contained in the ETS survey, about the actual situation today. To give wide distribution to these facts, the ETS will soon publish a pamphlet summarizing the findings of the survey. The hope is that circulation of these findings will stimulate schools to re-examine their programs in mathematics to see what measures might be taken to improve them.

COMMENTARY: MATHEMATICS IN CRISIS

There is grave cause for alarm in the failure of our schools to provide more youngsters with a reasonable command of mathematics. It is a problem which merits the attention of all men and women accustomed to concerning themselves about the future of America.

In recent years, the astonishing advances in science and technology and the prompt application of these advances to our industrial and national defense

needs have created an unprecedented demand for intelligent and highly trained young men and women. It is now clear that unless heroic measures are taken the demand simply cannot be met.

What we are facing is not a shortage of talent, but of *trained talent*. Intelligent youngsters are not in short supply, and if we had made good use of the resources available to us we would be in no difficulty. This is not an easy fact for us to grasp. Throughout history, human societies have managed to be extremely wasteful of individual talent. The discovery and nurture of talent has been largely accidental. The needs of society have not in the past put a premium either upon gifts or upon training. Even a generation ago the world did not reckon highly intelligent, highly trained men among the most marketable commodities. Now for the first time in history a radically different condition exists. The nation needs its talented and highly trained men and women, and it needs them badly.

The national need for men with scientific and mathematical competence exists at all levels. It is not just that we need more creative scientists at the Nobel Prize level. Behind the great creative minds in science moves an army of able and superbly trained men who test and confirm (or reject) new discoveries, conduct the more mundane testing and experimenting of the scientific world, and make the numberless lesser contributions on which the structure of science rests. And beside them works an immense corps of well-trained and skillful laboratory technicians and assistants.

In the industrial world a similar situation exists. In the spotlight are the great designers, inventors, and innovators, but back of them stands an immensely able corps of technically brilliant men to put their discoveries into operation. And back of these men stretch rank after rank of highly trained technicians and mechanics. Only in recent years have we learned that much of the strength and vitality of American technology is in this "training in depth." Several of the leading countries of the world can match us for brilliant men at the top. But we outstrip most of them in competent, excellently trained second, third, fourth, and fifth level workers.

When it became apparent two or three years ago that we were going to experience a severe shortage of scientists and engineers, and that this shortage constituted cause for the gravest national concern, some first-rate minds were put to work to diagnose the problem. Why the shortage? How can it be corrected?

There are many reasons for the shortage, and these need not concern us here. When the experts turned to the question of how to correct it, they very soon were brought up short before the great stumbling block of mathematics. You cannot make scientists out of youngsters who have not been adequately trained in the early years of mathematics. And the youngsters who had received such adequate training proved to be all too rare.

That was the situation which led to the study described in the accompanying article, "A National Weakness." It would be foolish in the extreme to waste our

time and energy in fixing the blame for this disturbing state of affairs. Whatever critics may say, our schools as a whole are *not* deteriorating. They are doing a good job under extraordinarily difficult circumstances. Any attempt to find a scapegoat for the present situation will simply waste everyone's emotional energies, and divert us from constructive action.

But action is necessary. We must act to draw a vastly greater number of first-rate young men and women into the teaching of mathematics. We must act to strengthen and upgrade many of those who are already in it. And we must improve the training of those who teach.

A few months ago Carnegie Corporation made a grant of $300,000 to the American Association for the Advancement of Science to support a program for the improvement of teaching in science and mathematics. The ETS study reported here is another effort to draw national attention to a truly critical problem.

JOHN W. GARDNER

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE ASYMPTOTIC DENSITY OF SOME k-DIMENSIONAL SETS

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Let A be a k-dimensional set whose elements are k-tuples of positive integers, and let $A(n)$ be the number of k-tuples, $(a_1, \cdots, a_k)$, in A such that $a_i \leqq n$ for $i=1, 2, \cdots, k$. Then $\delta(A)=\lim_{n\to\infty}\{A(n)/n^k\}$ is defined to be the *asymptotic density of the set* A. It can also be said that $\delta(A)$ represents the *probability* that any k-tuple of positive integers, chosen at random, will be in the set A.

Suppose k positive integers ($k\geqq 2$) are chosen at random. What is the probability that they have greatest common divisor t, where $t\geqq 1$? Or the probability that t be square free? These questions will be answered in this note. For $k=2$ and $t=1$, Mertens [3] showed that the probability is $1/\zeta(2)$ where $\zeta(s)$ is the well known Riemann-Zeta function. For $k=1$, Gebenbauer [1] has shown that the probability that an integer be square free is also $1/\zeta(2)$.

Definition. Let $\phi_k(n)$ *be the number of integers,* a, *such that* $0<a\leqq n$ *and* $(a, n)=k$, *where* (a, n) *denotes the greatest common divisor of* a *and* n. It is easy to show that $\phi_k(kn)=\phi(n)$, where $\phi(n)$ is the well known Euler ϕ-function. We are now ready to state

THEOREM 1. *Let* t *be a positive integer. Then the probability that two positive*

integers, selected at random, have greatest common divisor t is $1/t^2\zeta(2)$.

Proof. There exist t^2n^2 pairs of numbers (a, b) such that $0<a\leqq tn$ and $0<b\leqq tn$. Let N_n be the number of these pairs for which $(a, b)=t$. Then, by counting, it is seen that

$$N_n = \phi_t(t) + 2\phi_t(2t) + \cdots + 2\phi_t(nt).$$

And so, by the remark following the above definition,

$$N_n = 2[\phi(1) + \phi(2) + \cdots + \phi(n)] - 1.$$

But it is well known (see *e.g.*, Hardy and Wright [2]) that

$$\phi(1) + \phi(2) + \cdots + \phi(n) = \frac{3}{\pi^2} n^2 + O(n \log n).$$

Thus

$$\lim_{n\to\infty} \frac{N_n}{t^2n^2} = \frac{6}{t^2\pi^2} = \frac{1}{t^2\zeta(2)}$$

which is the desired probability.

THEOREM 2. *Let a_1 and a_2 be two positive integers selected at random. The probability that their greatest common divisor, (a_1, a_2), be square free is $1/\zeta(4)$.*

Proof. In the ordinary Cartesian plane consider all pairs of integers, (x, y), in the square $0<x\leqq n$ and $0<y\leqq n$. Call this square Q_n, and note that Q_n contains n^2 pairs of positive integers. By Theorem 1 and its proof, it is seen that Q_n contains

$$\frac{1}{t^2\zeta(2)} n^2 + O\left(\frac{n}{t} \log \frac{n}{t}\right)$$

pairs which have greatest common divisor t. Let N be the number of pairs in Q_n for which t is square free. Then

$$N = \sum_{t=1}^{n} \left\{ \frac{|\mu(t)|}{t^2\zeta(2)} n^2 + O\left(\frac{n}{t} \log \frac{n}{t}\right) \right\}$$

where $\mu(t)$ is the well known Möbius function. Now

$$\lim_{n\to\infty} \frac{N}{n^2} = \frac{1}{\zeta(2)} \sum_{t=1}^{\infty} \frac{|\mu(t)|}{t^2} + \lim_{n\to\infty} \frac{\sum_{t=1}^{n} O\left(\frac{n}{t} \log \frac{n}{t}\right)}{n^2}$$

$$= \frac{1}{\zeta(2)} \cdot \frac{\zeta(2)}{\zeta(4)} = \frac{1}{\zeta(4)}.$$

Three theorems are now stated which are the generalizations of Mertens' result, Theorem 1, and Theorem 2 to k-dimensions.

THEOREM 3. *If $k \geqq 2$, then the probability that k integers, selected at random, are relatively prime is $1/\zeta(k)$.*

THEOREM 4. *Let t be a positive integer. If $k \geqq 2$, then the probability that k integers, selected at random, have greatest common divisor t is $1/t^k\zeta(k)$.*

THEOREM 5. *Let $a_1, \cdots, a_k$ be $k(k \geqq 1)$ positive integers selected at random. The probability that the greatest common divisor, $(a_1, \cdots, a_k)$, of these integers be square free is $1/\zeta(2k)$.*

Note that when $k=1$, Theorem 5 becomes Gebenbauer's result, and that all three theorems have been established for the case $k=2$. By a straightforward induction on k they may be established for all integers k.

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A BASIC SET OF POLYNOMIAL SOLUTIONS FOR THE EULER-POISSON-DARBOUX AND BELTRAMI EQUATIONS

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The equations

$$E_{ij}\colon \sum_{n=1}^{m} u_{x_n x_n} + (-1)^i[u_{tt} + jkt^{-1}u_t] = 0 \qquad i = 0, 1; j = 0, 1$$

are called: E_{10}, the wave equation, E_{00}, the Laplace equation, E_{11}, the Euler-Poisson-Darboux equation and E_{01}, the Beltrami equation. In a recent article [1] the authors presented a basic set of polynomial solutions for $E_{10}[E_{00}]$; in this note we modify certain of these polynomials to obtain the basic set of polynomial solutions for $E_{11}[E_{01}]$ associated with the Cauchy problem for $E_{11}[E_{01}]$ where $k>0$.

The following lemma is used to carry out the modification.

LEMMA. *Let $H_{10}[H_{00}]$ be a polynomial solution of $E_{10}[E_{00}]$ which is even in t; then the polynomial $\{H_{10}\}[\{H_{00}\}]$ obtained from $H_{10}[H_{00}]$ by replacing t^{2n} by*

$$t^{\{2n\}} = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(1+k)(3+k)\cdots(2n-1+k)}\,t^{2n}$$

is a solution of $E_{11}[E_{01}]$.

Proof. In operator notation we may write $E_{ij} \sim [\nabla^2 + D_{ij}]u = 0$. We note that

$$D_{10}[t^{2n}] = -2n(2n-1)t^{2n-2}, \quad \text{and} \tag{1}$$

$$\begin{aligned} D_{11}[t\{2n\}] &= -\left[\frac{2n(2n-1)\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{(1+k)(3+k)\cdots(2n-1+k)}\right. \\ &\quad \left. + \frac{k\cdot 2n\ 1\cdot 3\cdot 5\cdots(2n-1)}{(1+k)(3+k)\cdots(2n-1+k)}\right] t^{2n-2} \qquad (2) \\ &= -2n(2n-1)\frac{1\cdot 3\cdot 5\cdots(2n-3)}{(1+k)(3+k)\cdots(2n-3+k)}\, t^{2n-2} \\ &= -2n(2n-1)t\{2n-2\}. \end{aligned}$$

Any H_{10} may be written as $\sum_{s=0}^{p} P_s[x]t^{2s}$ where each P_s is a polynomial in $x_1, \cdots, x_m$. Since H_{10} is a solution of E_{10} we have

$$[\nabla^2 + D_{10}]H_{10} = \sum_{s=0}^{p} Q_s[x]t^{2s} \quad \text{and} \tag{3}$$

$$Q_s[x] = 0, \qquad s = 0, 1, \cdots, p. \tag{4}$$

Writing

$$\{H_{10}\} = \sum_{s=0}^{p} P_s[x]t\{2s\}$$

and making use of (1), (2), (3) and (4), we see immediately

$$[\nabla^2 + D_{11}]\{H_{10}\} = \sum_{s=0}^{p} Q_s[x]t\{2s\} = 0, \tag{5}$$

which completes the proof of the unbracketed form of the lemma. The bracketed form follows in similar fashion.

The authors' basic set [1] for E_{10} may be given as follows. For any set of non-negative integers (b_j) such that $b_0 \leqq 1$ and $\sum_{j=0}^{m} b_j = n$, let $H^n{}_{b_0,b_1,\cdots,b_m}(x_1, x_2, \cdots, x_m, t)$ denote

$$\sum \frac{\left[\frac{a_0}{2}\right]!}{\prod_{j=0}^{m} a_j! \prod_{j=1}^{m}\left[\frac{b_j - a_j}{2}\right]!}\, t^{a_0} \prod_{j=1}^{m} x_j^{a_j} \tag{6}$$

where the summation is extended over all non-negative (a_j) such that

$$\begin{aligned} &(1)\ a_j \equiv b_j \bmod 2, && j = 0, 1, 2, \cdots, m \\ &(2)\ \sum_{j=0}^{m} a_j = n \qquad (7) \\ &(3)\ a_j \leqq b_j, && j = 1, 2, \cdots, m. \end{aligned}$$

The polynomials (6) are even in t if $b_0=0$ and odd in t if $b_0=1$. We can readily see that no polynomial solution of $E_{11}[E_{01}]$ contains an odd power of t if $k>0$. For suppose the coefficient $P_n[X]$ of t^{2n+1} is the first nonvanishing coefficient of an odd power of t occurring in $P(x, t)$, a polynomial solution of E_{11}. The coefficient of t^{2n-1} in $(\nabla^2+D_{11})P(x, t)$ would then be $-P_n[x]\cdot(2n+1)(2n+k)$, a nonvanishing function, since k being positive cannot equal $-2n$, $n\geqq 0$.

We form our basic set for E_{11} from (6) by restricting b_0 to the value 0 and replacing t^{a_0} by $t^{\{a_0\}}$. That the resulting polynomials

$$(8)\quad H^n_{0b_1,\cdots,b_m}(x_1, x_2, \cdots, x_m, \{t\}) = \sum \frac{\left(\frac{a_0}{2}\right)!}{\prod_{j=0}^{m} a_j! \prod_{j=1}^{m}\left(\frac{b_j-a_j}{2}\right)!}\, t^{\{a_0\}} \prod_{j=1}^{m} x_j^{a_j}$$

are solutions of E_{11} follows from our lemma.

Since a particular monomial $\prod_{j=1}^{m} x_j^{b_j}$ occurs only in the polynomial (8) determined by its exponent set $0, b_1, \cdots, b_m$ we readily conclude that the polynomials (8) are independent. We call this monomial the generator for the associated polynomial. To show that this set of independent homogeneous polynomials is a basic set of solutions we need only to show that it contains the correct number of members. The polynomials (8) of degree n are

$$\binom{n+m-1}{m-1}$$

in number since they are generated in turn by the individual monomials appearing in $[\sum_{j=1}^{m} x_j]^n$.

Let us write a general homogeneous polynomial of degree n in the $m+1$ variables $t, x_1, \cdots, x_m$ which contains no first powers of t, as $P^n(t, x_1, \cdots, x_m) = P_1^n(x_1, \cdots, x_m)+t^2P_2^{n-2}(t, x_1, \cdots, x_m)$; thus P^n has

$$\binom{n+m-1}{m-1}+\binom{n+m-2}{m}$$

coefficients. Equating $(\nabla^2+D_{11})P^n$ to zero gives exactly

$$\binom{n+m-2}{m}$$

independent conditions on these coefficients, leaving

$$\binom{n+m-1}{m-1}$$

of them arbitrary. Hence the number of independent homogeneous polynomial solutions of degree n for E_{11} is

$$\binom{n+m-1}{m-1}$$

and the polynomials (8) are a basic set for E_{11}, $k>0$.

Since the polynomials (6′) obtained from (6) by replacing t^{a_0} by $(-1)^{[a_0/2]}t^{a_0}$ form a basic set [1] for E_{00}, we readily observe that the polynomials (8′) obtained from (8) by replacing $t^{\{a_0\}}$ by $(-1)^{(a_0/2)}t^{\{a_0\}}$ form a basic set for E_{01}.

We point out the association of the basic set (8) with the solution of the analytic Cauchy problem for E_{11}:

$$u_t(x_1, \cdots, x_m, 0) = 0$$

$$u(x_1, \cdots, x_m, 0) = f(x_1, \cdots, x_m) = \sum_{n=0}^{\infty}\Big[\sum A_{b_1,b_2,\cdots,b_m}\prod_{j=1}^{m} x_j^{b_j}\Big]$$

where the bracketed summation is over all non-negative b_j, $\sum_{j=1}^{m} b_j = n$. A formal solution is seen to be

$$u(x_1, \cdots, x_m, t) = \sum_{n=0}^{\infty}\Big[\sum A_{b_1,b_2,\cdots,b_m}\prod_{j=1}^{m}(b_j)!H^{n}_{0,b_1,\cdots,b_m}\Big]. \tag{9}$$

As an illustration we write, using (9), the solution, known to be unique [2], for the Cauchy problem for the E. P. D. equation

$$u_{xx} + u_{yy} - (u_{tt} + kt^{-1}u_t) = 0, \qquad k > 0;$$

$$u(x, y, t) = x^3y^2;\ u_t(x, y, 0) = 0.$$

The solution is

$$u(x, y, t,) = x^3y^2 + \frac{(3xy^2 + x^3)t^2}{1+k} + \frac{3xt^4}{(1+k)(3+k)}. \tag{10}$$

The solution (10) is valid for $k<0$, $k\neq -1, -3$ but is not unique.

Note added in proof. A theory which gives these and certain more general results in a simpler manner has been developed in the authors' paper, "The Cauchy problem for linear partial differential equations with restricted boundary conditions", to appear shortly in the *Canadian Journal of Mathematics*.

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ON MATRIC SOLUTIONS OF A CYCLIC EQUATION

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A. R. Richardson* has exhibited a set of matric conjugates M_i of the cyclic equation $x^3+x^2-2x-1=0$, such that M_{i+1} is obtained from M_i by cyclically permuting rows and columns. These conjugates arose incidentally and no comment was made on this particular property of the matrices. It is possible to show that the occurrence of such a set was not a coincidence, but that every cyclic equation possesses a set of conjugate matric solutions with this property.

THEOREM. *If $f(x)=0$ is a cyclic equation over the rational field, there exists a conjugate set of rational matric solutions B_i such that B_{i+1} is obtained from B_i by cyclicly permuting rows and columns.*

Let the complex roots of $f(x)=0$ be $x_1, x_2, \cdots, x_n$, where $x_{i+1}=\theta(x_i)$. Then if A_1 is the companion matrix of $f(x)$, the matrices $A_1, A_2=\theta(A_1), \cdots, A_n=\theta(A_{n-1})$ form a conjugate set of rational matric solutions. If $H=VUV^{-1}$, where V is the Vandermonde matrix (x_j^{i-1}) and U is obtained from the identity matrix of order n by cyclically permuting columns, then H is rational and of determinant $(-1)^{n-1}$. Albert† has shown that $H^{-1}A_1H=A_2$ whence, in general, we have

$$H^{-1}A_iH = A_{i+1}, \qquad i = 1, 2, \cdots, n-1. \tag{1}$$

While the relation $H=VUV^{-1}$ does not establish the similarity of H and U in the rational field, there does exist a rational matrix T such that‡

$$T^{-1}HT = U. \tag{2}$$

Now let $B_i=T^{-1}A_iT$. It follows immediately that the B_i constitute a conjugate set of rational matrices. Moreover, from (1) and (2), we have

$$\begin{aligned} B_{i+1} &= T^{-1}A_{i+1}T \\ &= T^{-1}H^{-1}A_iHT \\ &= T^{-1}H^{-1}TB_iT^{-1}HT \\ &= U^{-1}B_iU. \end{aligned}$$

But the transformation $U^{-1}B_iU$ has the effect of cyclically permuting the rows and columns of B_i, and the theorem is proved.

The set B_i having this property is by no means unique since it is not necessary that the transforming matrix H be as defined above. For example, when the equation $f(x)=0$ is cubic, the general matrix H for which (1) holds has its elements expressible in terms of three parameters and, in general, it is possible to choose these in a variety of ways so that the determinant of H is unity. For any

* Quarterly Journal of Mathematics, Oxford Series, vol. 7, 1936, pp. 256–270.

† Modern Higher Algebra, University of Chicago Press, 1937, p. 241.

‡ *Ibid.*, p. 77, corollary.

such H, the general matrix T satisfying (2) is also a three parameter matrix so that T may be constructed in an infinite number of ways. Thus, for the cyclic cubic equation $x^3+x^2-2x-1=0$, where

$$A_1=\begin{bmatrix}-1 & 2 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\quad A_2=A_1^2-2=\begin{bmatrix}1 & -1 & -1\\ -1 & 0 & 1\\ 1 & 0 & -2\end{bmatrix},$$

$$A_3=A_2^2-2=\begin{bmatrix}-1 & -1 & 0\\ 0 & -1 & -1\\ -1 & -1 & 1\end{bmatrix},$$

on setting

$$H=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$$

and solving the system of equations obtained by equating corresponding elements of $A_1H=HA_2$, for a, b, c, d, e, and g in terms of h, f, and i, we obtain

$$H=\begin{bmatrix}h-f-3i & f+i & -h+3i\\ -f-i & -h-f+2i & f\\ h-f-2i & h & i\end{bmatrix}.$$

The determinant of H is $7(i^3+i^2f-2i^2h+h^2i-hif)-f^3-h^3+2h^2f+hf^2$, and it is clear that this has the value unity for various choices of the parameters f, h, and i.

With either

$$H=\begin{bmatrix}1 & -1 & 0\\ 1 & -1 & -1\\ 1 & 0 & 0\end{bmatrix}\quad\text{and}\quad T=\begin{bmatrix}2 & 4 & 1\\ 1 & 2 & -3\\ 4 & 1 & 2\end{bmatrix}$$

or

$$H=\begin{bmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\quad\text{and}\quad T=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix},$$

we obtain a set of symmetric matrices

$$B_1=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & -1\\ 0 & -1 & -1\end{bmatrix},\quad B_2=\begin{bmatrix}-1 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\end{bmatrix},\quad B_3=\begin{bmatrix}0 & -1 & 1\\ -1 & -1 & 0\\ 1 & 0 & 0\end{bmatrix},$$

while if

$$H = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix},$$

we are led to a set of non-symmetric conjugates

$$B_1 = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}.$$

ON A FUNCTION OF RAMANUJAN

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Let

$$\phi(x) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} x^{n-1}e^{-nx}.$$

It has been conjectured by S. Ramanujan that

$$(-1)^k \frac{d^k\phi}{dx^k} \geqq 0, \qquad x \geqq 1, \qquad k = 1, 2, 3, \cdots,$$

and this conjecture has been proved for $k=1, 2, 3, 4$ [2]. In this note we prove

THEOREM 1. *The function $x\phi(x)$ is not completely monotonic* for $x>1$.*

Proof. Let

$$\beta(t) = 0, \qquad \text{for } 0 \leqq t < 1$$

$$= \sum_{1\leqq n\leqq t} \frac{n^{n-2}}{(n-1)!} e^{-n}, \qquad \text{for } t > 1.$$

Since the series $\sum_1^\infty n^{n-2}e^{-n}/(n-1)!$ is convergent, $\beta(t)$ is bounded and non-decreasing for $t\geqq 0$. Further,

$$F(x) = \int_0^\infty e^{-xt}\alpha\beta(t) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} e^{-n(x+1)}$$

is convergent for $0\leqq x<\infty$ and so by Bernstein's theorem [3; p. 160] $F(x)$ is completely monotonic for $x\geqq 0$. Let $\psi(t)=t-\log(1+t)$; then $\psi(t)$ is real and continuous for $t\geqq 0$, $\psi(0)=0$, and $\psi(t)>0$ for $t>0$, and $\psi(t)=\int_0^t h(t)dt$ where $h(t)=t/(1+t)$ is not completely monotonic in any interval $0\leqq a\leqq t\leqq b$. Hence by a theorem due to Bochner and Schoenberg [1; pp. 43–44] it follows that

* For definition see [3; pp. 144–5].

$F(\psi(x))$ is not completely monotonic for $x>0$. But

$$F(\psi(x)) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} \exp\{-n(x - \log(1+x) + 1)\}$$
$$= (1+x)\phi(1+x).$$

Hence $(1+x)\phi(1+x)$ is not completely monotonic for $x>0$ and so $x\phi(x)$ is not completely monotonic for $x>1$.

Remark. If we take $\psi(t) = t + \log(1+t)$ we find that

$$\sum_1^\infty \frac{n^{n-2}}{(n-1)!} e^{-nx}x^{-n}$$

is completely monotonic for $x>1$.

THEOREM 2. *The function $e^x\phi(x)$ is not completely monotonic for $x>1$.*

Let $\beta(t)$ be a step function having jumps $(n+1)^{n-1}e^{-n}/n!$ at $t=n(n=1, 2, 3, \cdots)$ and consider

$$\int_0^\infty e^{-t(x-\log(1+x))}d\beta(t) = e^{(x-1)}\phi(x+1) - 1.$$

Hence by the previous argument, $e^{(x+1)}\phi(x+1)-1$ is not completely monotonic for $x>0$, and so $e^x\phi(x)-1$ is not completely monotonic for $x>1$. Now for $x \geqq 0$,

$$e^x\phi(x) \geqq 1 + xe^{-x}$$

and hence $e^x\phi(x)$ is not completely monotonic for $x>1$.

THEOREM 3. *The function $e^x\phi(x)$ is an absolutely monotonic function of $y=xe^{-x}$ for all x such that $0 \leqq xe^{-x} < e^{-1}$. Moreover,*

$$e^x\phi(x) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} y^{n-1},$$

and the last series is convergent for $0 \leqq y < e^{-1}$.

Since all the coefficients of the last series are positive, $e^x\phi(x)$ is an absolutely monotonic function of $y=xe^{-x}$ for $0 \leqq y < e^{-1}$, that is, for all x such that $0 \leqq x < 1$ and $1 < x \leqq \Delta$.

Remark. The function $x\phi(x)$ is an absolutely monotonic function of xe^{-x} for all x such that $0 \leqq x < 1$ and $1 < x \leqq \Delta$.

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A NOTE ON THE DIVIDING STREAM LINE IN HYDRODYNAMICS

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1. In the two-dimensional irrotational motion of an incompressible fluid past a fixed body, it is well known that the fluid meets the body at right angles to its boundary curve at a stagnation point [1]. By using the elements of vectorial analysis, the object of this note is to show that this result may be extended to three dimensional flows which are unsteady, rotational and compressible. We have the following result:

THEOREM. *In any fluid motion past a fixed solid body, the necessary and sufficient condition that the dividing stream line should meet the boundary surface of the solid at right angles is that any existing vorticity of the fluid at the point of impact should lie along the normal to the boundary surface.*

In this theorem no assumption is made as to the nature of the fluid, thus it is a general theorem on vector fields.

2. For convenience, denote the boundary surface by S and the point of impact by P. Let $\mathbf{h}$ be the unit outward-drawn vector normal to S at P. Let $\mathbf{q}$ be the velocity vector of the fluid, and form the dyadic $\nabla\mathbf{q}$ at P. We shall suppose that this dyadic does not vanish. (This will most certainly be the case if the fluid motion is rotational at P).

Since $\mathbf{q}=0$ at P, $\delta\mathbf{q}=\delta\mathbf{r}\cdot\nabla\mathbf{q}$ gives the velocity distribution in the neighbourhood of P, where $\delta\mathbf{r}$ is the infinitesimal distance vector. If $\delta\mathbf{r}$ lies in S, we must have $\delta\mathbf{q}\cdot\mathbf{h}=0$, *i.e.*, $\delta\mathbf{r}\cdot\nabla\mathbf{q}\cdot\mathbf{h}=0$. Hence $\nabla\mathbf{q}\cdot\mathbf{h}$ is parallel to $\mathbf{h}$ and we may write

$$\nabla\mathbf{q}\cdot\mathbf{h} = \lambda\mathbf{h}, \tag{2.1}$$

where λ is a scalar.

Now

$$\nabla\mathbf{q} = \mathbf{q}\nabla - \rho \times I, \tag{2.2}$$

where $\rho(=\text{curl }\mathbf{q})$ is the vorticity vector of the fluid and I is the unit dyadic (see (32) on p. 149 of [2]). Since $(\mathbf{q}\nabla)\cdot\mathbf{h}=\mathbf{h}\cdot\nabla\mathbf{q}$, (2.1) and (2.2) give

$$\mathbf{h}\cdot\nabla\mathbf{q} = \rho\times\mathbf{h} + \lambda\mathbf{h}. \tag{2.3}$$

The differential form of the equation of a stream line which meets S at P is

$$\delta\mathbf{r}\cdot\nabla\mathbf{q} = \mu\delta\mathbf{r}, \tag{2.4}$$

where μ is a scalar. The theorem now follows by comparing the last two equations. If $\delta\mathbf{r}$ is parallel to $\mathbf{h}$, then (2.4) holds with $\delta\mathbf{r}$ replaced by $\mathbf{h}$; hence $\lambda=\mu$ and $\rho\times\mathbf{h}=0$. Conversely, if $\rho\times\mathbf{h}=0$, then (2.3) shows that there is a stream line through P in the direction of $\mathbf{h}$.

3. The statement of the theorem has supposed the dividing stream line to be unique. Suppose that in addition to the stream line in the direction of $\mathbf{h}$ we

† This work was carried out at the Bristol Aeroplane Company.

have a stream line in the direction of $\delta\mathbf{r}^*$ which meets S obliquely at P. Let $\mathbf{t}$ be the unit vector along the projection of $\delta\mathbf{r}^*$ in the tangent plane to S at P, and let $\mathbf{b}=\mathbf{t}\times\mathbf{h}$.

From (2.2), (2.3) and (2.4), with $\rho\times\mathbf{h}=0$, we have

$$\lambda\mathbf{h}\cdot\delta\mathbf{r}^* = \mathbf{h}\cdot\nabla\mathbf{q}\cdot\delta\mathbf{r}^* = \delta\mathbf{r}^*\cdot\nabla\mathbf{q}\cdot\mathbf{h} = \mu\delta\mathbf{r}^*\cdot\mathbf{h} = \mu\mathbf{h}\cdot\delta\mathbf{r}^*.$$

Since $\delta\mathbf{r}^*$ does not lie in S, $\mathbf{h}\cdot\delta\mathbf{r}^*\neq 0$. Hence $\lambda=\mu$. That is, $\delta\mathbf{r}^*$ and $\mathbf{h}$ are two characteristic vectors of the dyadic $\nabla\mathbf{q}$ corresponding to the same characteristic number $\lambda=\mu$. It follows that $\mathbf{t}$ is also a characteristic vector of $\nabla\mathbf{q}$ with the same characteristic number, for $\mathbf{t}$ is parallel to $\delta\mathbf{r}^*-(\delta\mathbf{r}^*\cdot\mathbf{h})\mathbf{h}$. We thus have

$$\mathbf{t}\cdot\nabla\mathbf{q} = \lambda\mathbf{t}, \quad \text{and} \quad \mathbf{h}\cdot\nabla\mathbf{q} = \lambda\mathbf{h}. \tag{3.1}$$

Since $\rho\times\mathbf{h}=0$, we may write $\rho=\rho\mathbf{h}$. From (2.2), $\mathbf{t}\cdot\nabla\mathbf{q}=\mathbf{t}\cdot(\mathbf{q}\nabla)+\mathbf{t}\times\rho=\nabla\mathbf{q}\cdot\mathbf{t}+\rho\mathbf{b}$. Also, $\mathbf{h}\cdot\nabla\mathbf{q}=\nabla\mathbf{q}\cdot\mathbf{h}$. Thus (3.1) is equivalent to

$$\nabla\mathbf{q}\cdot\mathbf{t} = \lambda\mathbf{t} - \rho\mathbf{b}, \quad \text{and} \quad \nabla\mathbf{q}\cdot\mathbf{h} = \lambda\mathbf{h}.$$

Hence $(\mathbf{b}\cdot\nabla\mathbf{q}+\rho\mathbf{t})\cdot\mathbf{t}=(\mathbf{b}\cdot\nabla\mathbf{q}+\rho\mathbf{t})\cdot\mathbf{h}=0$; *i.e.*, $\mathbf{b}\cdot\nabla\mathbf{q}+\rho\mathbf{t}$ is parallel to $\mathbf{b}$ and we write

$$\mathbf{b}\cdot\nabla\mathbf{q} = \Lambda\mathbf{b} - \rho\mathbf{t}, \tag{3.2}$$

where Λ is a scalar.

The equations given by (3.1) and (3.2) determine the dyadic $\nabla\mathbf{q}$ since $\mathbf{t}$, $\mathbf{h}$ and $\mathbf{b}$ form an orthonormal set of vectors. We have

$$\begin{aligned}\nabla\mathbf{q} &= \lambda\mathbf{tt} + \lambda\mathbf{hh} + \Lambda\mathbf{bb} - \rho\mathbf{bt} \\ &= \lambda I + (\Lambda - \lambda)\mathbf{bb} - \rho\mathbf{bt}.\end{aligned}$$

This equation shows that if there is more than one (non-tangential) stream line through P, then all rays through P in the normal plane through any other determine directions of stream lines. Furthermore, if there are three non-coplanar (non-tangential) stream lines through P, then $\rho=0$ and $\Lambda=\lambda$, and any ray through P determines the direction of a stream line.

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A CONDITION FOR DIAGONABILITY OF MATRICES

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A square matrix is said to be *diagonable* if it is similar to a diagonal matrix. Various necessary and sufficient conditions for the diagonability of matrices are to be found in the literature. (See, for example, **1.**) In this note we establish yet another such condition, which does not appear to have been stated previously.

By a *group matrix* we shall understand any matrix which is an element in

some multiplicative group of matrices. Every non-singular matrix is, of course, a group matrix; but groups of singular matrices also exist and were first investigated by Ranum (2). We shall prove the following

THEOREM. *A matrix A is diagonable if and only if $A-\omega I$ is a group matrix for every (complex) value of ω.*

All matrices considered below are assumed to be of type $n\times n$. We denote by I the unit matrix, by $R(A)$ the rank of A, and by diag $(\omega_1, \cdots, \omega_n)$ the diagonal matrix with diagonal elements $\omega_1, \cdots, \omega_n$. If $\mathfrak{V}$ is a vector space, then $d(\mathfrak{V})$ denotes its dimension.

A few preliminary results will be required.

LEMMA 1. *All matrices in a multiplicative matrix group have the same rank.*

Let A be an arbitrary element and U the unit element of a matrix group Γ. Denoting by B the inverse element of A, we have $U=AB$, $A=AU$. Therefore

$$R(U) = R(AB) \leqq R(A) = R(AU) \leqq R(U),$$

so that $R(A)=R(U)$ for all $A\in\Gamma$.

LEMMA 2. *If $R(AB)=R(B)$, then $R(ABC)=R(BC)$.*

Denote by $\mathfrak{N}_T$ the space of all vectors x such that $Tx=0$. Then

$$d(\mathfrak{N}_T) = n - R(T). \tag{1}$$

Hence, by hypothesis, $d(\mathfrak{N}_{AB})=d(\mathfrak{N}_B)$. But, trivially, $\mathfrak{N}_{AB}\supset\mathfrak{N}_B$, and so

$$\mathfrak{N}_{AB} = \mathfrak{N}_B. \tag{2}$$

Next, let $x\in\mathfrak{N}_{ABC}$, so that $ABCx=0$. Then $Cx\in\mathfrak{N}_{AB}$ and therefore, by (2), $Cx\in\mathfrak{N}_B$, *i.e.*, $BCx=0$. Thus $x\in\mathfrak{N}_{BC}$, and we have shown that $\mathfrak{N}_{ABC}\subset\mathfrak{N}_{BC}$. But, trivially, $\mathfrak{N}_{ABC}\supset\mathfrak{N}_{BC}$; and therefore $\mathfrak{N}_{ABC}=\mathfrak{N}_{BC}$. The assertion now follows by (1).

LEMMA 3. *If $\mu(x)$ is the minimum polynomial of A and $h(x)$ is a polynomial such that $h(A)=O$, then $\mu(x)$ divides $h(x)$.*

LEMMA 4. *If all zeros of the minimum polynomial of A are simple, then A is diagonable.**

Both these results are well-known.

Proof of the theorem. (i) Suppose that A is diagonable, and write $S^{-1}AS$ $=$diag $(\omega_1, \cdots, \omega_n)$. Then

$$S^{-1}(A - \omega I)S = \text{diag } (\omega_1 - \omega, \cdots, \omega_n - \omega). \tag{3}$$

Now the matrix on the right-hand side of (3) is an element in the matrix group Γ consisting of the matrices

* The converse statement is also true, but not necessary for our purpose.

$$\text{diag } (\epsilon_1 x_1, \cdots, \epsilon_n x_n),$$

where $x_1, \cdots, x_n$ take all non-zero (complex) values and ϵ_r is defined as 0 or 1 according as $\omega = \omega_r$ or $\omega \neq \omega_r$. Hence $A - \omega I$ is an element of the group $S\Gamma S^{-1}$, and so is a group matrix.

(ii) Suppose, next, that $A - \omega I$ is a group matrix for every value of ω. Then, by Lemma 1,

$$R(A - \omega I) = R\{(A - \omega I)^2\} \qquad \text{(all } \omega\text{)}. \tag{4}$$

Now let ω be any zero of the minimum polynomial $\mu(x)$ of A, so that $\mu(x) = (x-\omega)^k \phi(x)$, where $k \geqq 1$ and $\phi(\omega) \neq 0$. There exist polynomials $f(x)$ and $g(x)$ such that

$$(x - \omega)^k f(x) + \phi(x) g(x) = 1. \tag{5}$$

This implies that

$$(A - \omega I)^k f(A) + \phi(A) g(A) = I. \tag{6}$$

Writing $E = \phi(A) g(A)$, multiplying both sides of (6) by E, and using the fact that $\mu(A) = O$, we obtain $E^2 = E$. Putting $Z = (A - \omega I)E$, we therefore have

$$Z^m = (A - \omega I)^m E, \tag{7}$$

so that

$$Z^k = O. \tag{8}$$

In view of (4), (7), and Lemma 2, we at once obtain $R(Z) = R(Z^2)$. Hence, by Lemma 2 and (8),

$$R(Z) = R(Z^2) = \cdots = R(Z^k) = 0,$$

so that $Z = O$, *i.e.*,

$$(A - \omega I)\phi(A) g(A) = O.$$

It follows by Lemma 3 that $(x-\omega)^k \phi(x)$ divides $(x-\omega)\phi(x)g(x)$. But, by (5), $g(\omega) \neq 0$ and therefore $k = 1$. All zeros of $\mu(x)$ are therefore simple; and by Lemma 4, A is diagonable.

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CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON THE REPRESENTATION OF POWERS OF PRIMES BY POLYNOMIALS WITH INTEGRAL ZEROS

J. G. CAMPBELL, University of Kentucky

Let $f(x)$ represent a polynomial with integral coefficients. Let $(r_1, r_2, \cdots, r_j)$ represent the total set of integral zeros of $f(x)$ and note that $f(x)$ may have additional non-integral zeros. If $f(v)=p^a$, where p is a prime and a is an integer, then the integral value v is definitely restricted to a finite set of integers, dependent on the r's. Let r_i, $1 \leqq i \leqq j$, be any integral zero of $f(x)$. Then $f(x)=(r_i-x)q(x)$ and $f(m)=(r_i-m)q(m)$, where m is any integer not a zero of $f(x)$. Designate the total set of positive and negative divisors of $f(m)$ by the set $(d_1, d_2, \cdots, d_k)$. But (r_i-m) is one of the divisors of $f(m)$ and $(r_i-m)+m=r_i$. Hence, r_i is in the set $(d_1+m, d_2+m, \cdots, d_k+m)$. We use this fact to prove the following theorem.

THEOREM 1. *Let $(g_1, g_2, \cdots, g_n)$ represent the total set of zeros of $f(x)$. Let the subset $(r_1, r_2, \cdots, r_j)$, $j \geqq 1$, represent all integral zeros of $f(x)$. If $f(v)=p^a$, where p is a prime and a is a positive integer, then $v=r_s \pm p^t$, $s=1, 2, \cdots, j$, $t=0, 1, 2, \cdots, a$, for at least one s and one t.*

Proof. Finding a value v for which $f(v)=p^a$ is equivalent to finding an integral zero v of $F(x)=f(x)-p^a$. It is clear that $m=r_s$, $s=1, 2, \cdots, j$, is not a zero of $F(x)$ for any s, since $F(m)=-p^a$. The total set of divisors of $F(m)$ is $(1, p, p^2, \cdots, p^a, -1, -p^2, \cdots, -p^a)$. Hence, the zeros of $F(x)$ are included in the set $(r_s \pm p^t)$, $s=1, 2, \cdots, j$, $t=0, 1, 2, \cdots, a$. This completes the proof of the theorem.

It follows immediately that $f(v)$ represents a composite number for all integral $v \neq r_s \pm p^t$, $s=1, 2, \cdots, j$, $t=0, 1$. It is possible in certain cases to restrict the value v to a single integer if the polynomial $f(x)$ has only one integral zero. $F(x)=x^n-k^n-p^b$ is an example. Here n is an odd prime, the integer b is less than or equal to n, and k is a known integer. The only integral zero of x^n-k^n is k. The integral zeros of $F(x)=x^n-k^n-p^b$ are in the set $(k \pm p^t)$, $t=0, 1, 2, \cdots, b$. If $v^n=k^n+p^b$, then the inequalities $k<v<k+p$ hold. We must take the positive sign for p^t in the above set, since v is greater than k. The only positive value in the set for which $v<k+p$ is $k+1$. Hence, $v=k+1$. The following theorem is immediate.

THEOREM 2. *Let $x^n-y^n=p^b$, where n is an odd prime, p is a prime, and $b \leqq n$. Then x and y are consecutive integers.*

Equations of this type actually occur, *e.g.*, $8^3-7^3=13^2$.

ON "SOLVING" DIFFERENTIAL EQUATIONS

R. C. Buck, University of Wisconsin

Most texts in differential equations regard a problem as solved when the solution of a differential equation is reduced to the evaluation of one or more integrals. In many cases, the form in which such an answer appears gives little indication of the qualitative nature of the solutions. The purpose of this note is to point out that the mean value theorem can sometimes be applied to obtain quite precise information; this serves as an additonal "practical" motivation for the introduction of these tools into the engineering-type of calculus course.

Let us consider the first order linear equation

$$\frac{dy}{dx} - 2xy = x^2, \qquad x \geqq 0. \tag{1}$$

By the usual procedure, we immediately obtain the general solution

$$\begin{aligned} y &= e^{x^2}\int_0^x t^2e^{-t^2}dt + Ce^{x^2} \\ &= e^{x^2}\left\{\int_0^x t^2e^{-t^2}dt + C\right\}. \end{aligned} \tag{2}$$

However, the student would have considerable difficulty in visualizing the behavior of the solution curves from this. The following is certainly more useful.

THEOREM. *The general solution of* (1) *is*

$$y = Y_0(x) + Ae^{x^2} \tag{3}$$

where the particular solution Y_0 *obeys the inequalities*

$$-\frac{x}{2} \geqq Y_0(x) \geqq -\frac{x}{2} - \frac{1}{4x}, \qquad x \geqq 2. \tag{4}$$

COROLLARY. *The equation* (1) *has precisely one solution (namely,* $y = Y_0(x)$*) for which* dy/dx *is bounded, as* $x \to \infty$.

Proof. Examining the solution (2) we notice that the improper integral $\int_0^\infty t^2e^{-t^2}dt$ converges. Let its value be K. [The fact that K may be evaluated precisely, with $K = \sqrt{\pi}/4$, is not relevant to our argument.] Then

$$K = \int_0^x t^2e^{-t^2}dt + \int_x^\infty t^2e^{-t^2}dt,$$

so that (2) may be written:

$$y = e^{x^2}\left\{K + C - \int_x^\infty t^2e^{-t^2}dt\right\}.$$

Choosing $C=-K$, we obtain as a particular solution of (1),

$$y = Y_0(x) = -e^{x^2}\int_x^\infty t^2e^{-t^2}dt.$$

To estimate the rate of growth of $Y_0(x)$ as x increases, we write this as

$$Y_0(x) = \frac{-\int_x^\infty t^2e^{-t^2}dt}{e^{-x^2}} = \frac{F(x)}{G(x)}. \tag{5}$$

Observing that $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} G(x) = 0$, we apply the mean value theorem, and conclude that there is a point $\bar{x}>x$ such that $Y_0(x) = F'(\bar{x})/G'(\bar{x})$. However, $F'(x) = x^2e^{-x^2}$ and $G'(x) = -2xe^{-x^2}$, so that $Y_0(x) = -\frac{1}{2}\bar{x}$. Since $\bar{x}>x$, we have shown that $Y_0(x) < -\frac{1}{2}x$ for any $x>0$.

Consider now

$$Y_0(x) + \frac{1}{2}x = -e^{x^2}\int_x^\infty t^2e^{-t^2}dt + \frac{1}{2}x$$

$$= \frac{-\int_x^\infty t^2e^{-t^2}dt + \frac{1}{2}xe^{-x^2}}{e^{-x^2}} = \frac{f(x)}{g(x)}.$$

Applying the same method, we have $Y_0(x)+\frac{1}{2}x = f'(\bar{x})/g'(\bar{x})$ for some $\bar{x}>x$. This time, $f'(x) = \frac{1}{2}e^{-x^2}$ and $g'(x) = -2xe^{-x^2}$ so that $Y_0(x)+\frac{1}{2}x = -1/(4\bar{x})$. Since $\bar{x}>x$, this gives $Y_0(x)+\frac{1}{2}x > -1/(4x)$. Together, these inequalities give formula (4) and the theorem is proved.

We observe that what has been shown is that $Y_0(x) = -\frac{1}{2}x+O(x^{-1})$, with an estimate on the error. To prove the corollary, we return to the original differential equation (1) and, putting $y = Y_0(x)$, we have

$$\frac{dy}{dx} = 2xy + x^2 = 2x\left\{-\frac{1}{2}x + O(x^{-1})\right\} + x^2$$

$$= -x^2 + O(1) + x^2 = O(1).$$

Thus, the solution Y_0 has a bounded derivative. To see that this is the only one, we observe that

$$\frac{d}{dx}\left\{Y_0(x) + Ae^{x^2}\right\} = Y_0'(x) + 2Axe^{x^2},$$

which can never be bounded as x increases, unless $A=0$.

It is worth pointing out that this process can be carried out further to yield additional terms, arriving at (for example)

$$Y_0(x) = -\frac{1}{2}x - \frac{1}{4}x^{-1} + \frac{1}{8}x^{-3} - \frac{3}{16}x^{-5} + O(x^{-7}).$$

This suggests that one should seek a series representation for Y_0 of this form; one easily obtains the formal expansion

$$Y_0(x) = -\frac{1}{2}x + \sum_{k=0}^{\infty}(-1)^{k-1}\frac{(2k)!}{k!}\frac{x^{-2k-1}}{4^{k+1}}.$$

However, this series is divergent for every x; the student is now ready for the general notion of asymptotic series and their application.

TWO METHODS FOR FINDING THE ANGLE OF ROTATION

O. J. KARST, Stevens Institute of Technology

In eliminating the xy term from the general second degree equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ by rotation of axes, the rotation angle θ is given by the formula, $\tan 2\theta = B/(A-C)$. In order to obtain the $\sin\theta$ and $\cos\theta$ to use in the rotation transformation equations

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta,$$

the half angle formulas are generally used.

There are two other methods which are equally effective and are based on more familiar elementary methods than the half angle formulas. The first method is geometrical and is most easily shown by an example.

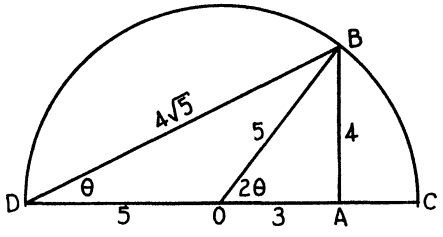


FIG. 1

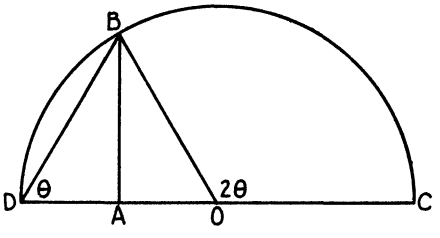


FIG. 2

Let $\tan 2\theta = 4/3$ (see Figure 1). Then construct 2θ as the central angle in a semi-circle, whose radius will then be 5. Since 2θ is measured by arc BC, angle BDC will be measured by $\frac{1}{2}$ arc BC and hence angle $BDC=\theta$. Hence from triangle DAB we obtain

$$\sin\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\theta = \frac{2}{\sqrt{5}}.$$

If $\tan 2\theta$ is negative, the figure is essentially the same as shown in Figure 2.

The second method is analytic. Let

$$\tan 2\theta = \frac{B}{A - C} = K$$

then

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = K$$

or

$$K \tan^2 \theta + 2 \tan \theta - K = 0.$$

Hence $\tan \theta = (-1 \pm \sqrt{1+K^2})/K$ from the quadratic formula. The positive root gives the usual acute positive angle of rotation. The negative root leads to a negative angle of rotation which is the numerical complement of the first angle. Either one will eliminate the xy term from the original equation.

GRAPHICAL CONSTRUCTION OF LOGARITHMIC SPIRALS

DIRAN SARAFYAN, Lamar State College of Technology

Let $A_1A_2A_3$ and $A_2A_3A_4$ be any two right triangles having in common the side A_2A_3 (Fig. 1). By drawing straight lines, departing from A_4, parallel to $\overline{A_1A_2}$ and $\overline{A_2A_3}$ consecutively, we obtain a graph which we shall call a rectangular spiral.

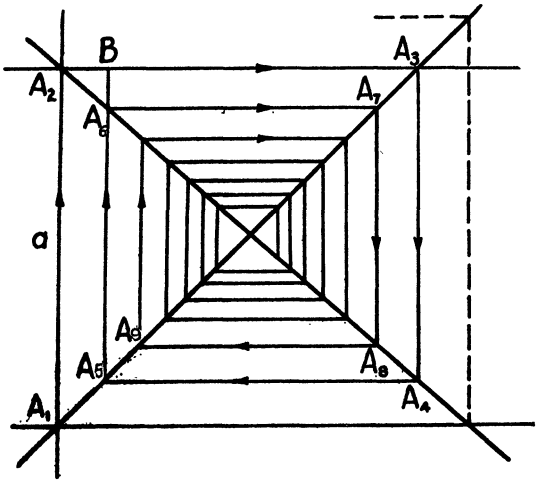


FIG. 1

Let $\overline{A_1A_2} = a$ and the ratio $A_3A_4/A_1A_2 = r$. Then $\overline{A_3A_4} = ar$.

Considering that the triangles $A_1A_2A_3$, $A_3A_4A_5$, $A_5A_6A_7$ *etc.* are similar, it can be easily seen that the magnitudes of the opposite sides of a rectangular spiral form the geometric progression

$$\overline{A_1A_2} = a \qquad \overline{A_3A_4} = ar \qquad \overline{A_5A_6} = ar^2 \qquad \overline{A_7A_8} = ar^3 \cdots .$$

Thus, the following theorem can be announced:

THEOREM: *The magnitudes of the opposite sides of a rectangular spiral form a geometric progression.*

Through the use of this theorem, the logarithmic spiral, the exponential and the logarithmic curves and the damped sinusoidal waves can be traced. We shall construct only the logarithmic spiral since the construction of the others follows more or less the same pattern.

Let the logarithmic spiral be represented by

$$\rho = a_0 a^{-\theta} \qquad 0 < a \neq 1.$$

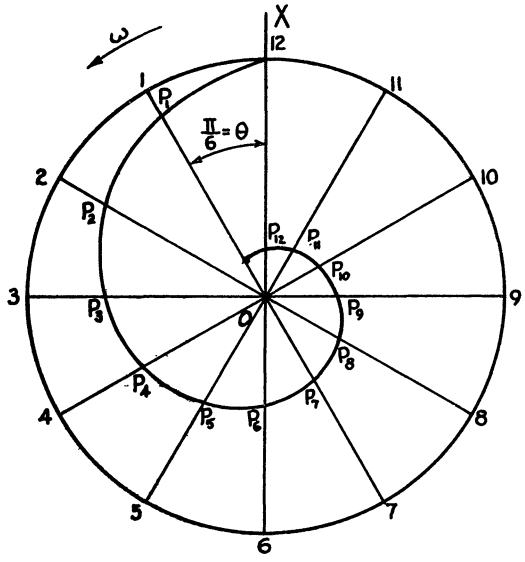


FIG. 2

Let OX be the polar axis (Fig. 2). For $\theta = 0$, $\rho = a_0$. Consider a circle with radius a_0 and with its center at O. This circle will be divided into n equal sectors, where n is an integer divisible by 4. In Figure 2, $n = 12$ has been chosen.

From the equation of the spiral, it is seen that the magnitude of the polar vectors OP_i with $i = 0, 1, 2, \cdots$, form a geometric progression of common ratio $a^{-\pi/6}$.

If on Figure 1, we take $\overline{A_1A_2} = a_0$ and $\overline{A_3A_4} = a_0 a^{-\pi/6}$, then in accordance with the previous theorem, the vertical sides of the rectangular spiral will be equal in length to the magnitude of polar vectors OP_i which will enable us to plot the points $P_1, P_2, P_3, \cdots$, of the logarithmic spiral.

If we consider the sequence of points P_i by successive groups of three points through which we draw an arc of a circle we obtain a graph which will almost coincide with a logarithmic spiral. However when great accuracy is not essential, one can as well join the sequence of points P_i by a smooth curve.

References

1. Irmgard Flugge-Lotz, Discontinuous Automatic Control, Princeton University Press, Princeton, New Jersey, 1953, pp. 163–65.

2. Timoshenko and Young, Vibration Problems in Engineering, Van Nostrand Co., New York, Third Edition, 1955, pp. 72–73.

THE NUMBER OF COMPONENTS OF CERTAIN GRAPHS

W. R. Utz, University of Missouri

Let $s, \alpha_1, \alpha_2, \cdots, \alpha_n$ be positive integers and let $a_1 > a_2 > \cdots > a_n$ be reals. The graph of

$$y^s = (x - a_1)^{\alpha_1}(x - a_2)^{\alpha_2} \cdots (x - a_n)^{\alpha_n} \tag{1}$$

is obviously connected if s is odd or if $n = 1$. Define, for each positive integer p,

$$g(p) = \begin{cases} 0, & \text{if } p \text{ is even;} \\ 1, & \text{if } p \text{ is odd.} \end{cases}$$

We shall prove the following theorem.

THEOREM. *If s is even and $n > 1$, the graph of* (1) *has*

$$1 + \sum_{i=1}^{n-1} g\left(\sum_{j=1}^{i} \alpha_j\right) \tag{2}$$

components (each isolated point of the graph being counted as one component).

The theorem is easily generalized to give the number of components of

$$y^s = G(x)(x - a_1)^{\alpha_1}(x - a_2)^{\alpha_2} \cdots (x - a_n)^{\alpha_n},$$

wherein s is a positive integer, $\alpha_1, \alpha_2, \cdots, \alpha_n$, are integers (not necessarily positive) and $G(x)$ is defined and of one sign for all real x. Besides its interest in an algebra or analytic geometry class the theorem answers a natural topological question. Similar, but generally less elementary, theorems are found, for example, in [1]

Proof of the theorem. The proof is by induction. Let $n = 2$. If α_1 is odd, it is easily verified that the graph of (1) has two components (one of which is a point in case α_2 is even). If α_1 is even, the curve is easily seen to be connected.

Before proceeding to the second step in the induction we observe that since

$$(x - a_{r+1})^{\alpha_{r+1}} > 0 \quad \text{for} \quad x > a_{r+1},$$

it is clear that the components of

$$y^s = (x - a_1)^{\alpha_1} \cdots (x - a_r)^{\alpha_r} \tag{3}$$

and

$$y^s = (x - a_1)^{\alpha_1} \cdots (x - a_r)^{\alpha_r}(x - a_{r+1})^{\alpha_{r+1}} \tag{4}$$

will be the same in number for $x > a_{r+1}$.

Suppose that (2) holds for $n = 2, 3, \cdots, r$.

Case 1. $\alpha_1 + \alpha_2 + \cdots + \alpha_{r-1}$ is odd. In this case $(x-a_1)^{\alpha_1} \cdots (x-a_{r-1})^{\alpha_{r-1}} < 0$ for $x < a_{r-1}$. Thus, if α_r is odd the components of (3) and (4) are the same regardless of whether α_{r+1} is even or odd and, since α_r is odd, the added term in (2) is 0. Similarly, if α_r is even, (4) has one more component (being a point if α_{r+1} is even) than (3) regardless of whether α_{r+1} is even or odd and, since α_r is even, the added term, $g(\alpha_1 + \alpha_2 + \cdots + \alpha_r)$, in (2) is 1.

Case 2. $\alpha_1 + \alpha_2 + \cdots + \alpha_{r-1}$ is even. In this case

$$(x - a_1)^{\alpha_1} \cdots (x - a_{r-1})^{\alpha_{r-1}} > 0 \qquad \text{for } x < a_{r-1}.$$

If α_r is odd, (4) has one more component than (3) has regardless of whether α_{r+1} is even or odd and (2) is increased by 1 since $\alpha_1 + \cdots + \alpha_r$ is odd. If α_r is even, (4) and (3) must have the same number of components and, since $\alpha_1 + \cdots + \alpha_r$ is even, the added term in (2) is 0. This completes the proof of the theorem.

Reference

1. Wilfred Kaplan, The topological structure of a variety defined by an equation, Compositio Mathematica, vol. 5, 1938, pp. 327–346.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1221. *Proposed by S. W. Golomb, University of Oslo, Norway*

If n is an integer greater than 1, then $2^n - 1$ is never a perfect square, cube, or higher power.

E 1222. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If we designate by C_1, C_2, C_3 the sides of the regular convex heptagon and of the two regular star heptagons inscribed in a circle of radius R, then

$$C_1^2 + C_2^2 + C_3^2 = 7R^2.$$

E 1223. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Starting with two arbitrary non-negative numbers a_1 and a_2, build a sequence a_k, $k=1, 2, 3, \cdots$, by taking alternately the arithmetic and the geometric means of the two preceding numbers, for example,

$$0, 1, 1/2, \sqrt{2}/2, (1+\sqrt{2})/4, \cdots.$$

Evaluate $\lim a_k$ as $k \to \infty$.

E 1224. *Proposed by G. K. Wenceslas, Providence, R. I.*

Let G be a finite group of rigid motions (or more generally affine transformations) of a Euclidean space. Then there is a point of the space which is left fixed by all the transformations of G.

E 1225. *Proposed by L. R. Ford, Illinois Institute of Technology*

Find necessary and sufficient conditions that it be possible to lay an $a \times b$ rug on an $A \times B$ floor. (Take $a \geqq b$, $A \geqq B$.)

SOLUTIONS

Bingo Cards

E 1188 [1955, 655]. *Proposed by Alice Anderson, Student, East Tennessee State College*

Each card used in the game of Bingo is made up of a square array of 25 numbers. The five numbers in the first column are selected from 1 to 15, those in the second column from 16 to 30, those in the third column from 31 to 45, those in the fourth column from 46 to 60, and those in the fifth column from 61 to 75. The space common to the third column and third row is left blank. If no card can have a row, column, or diagonal identical to a row, column, or diagonal of another card, how many different cards can be made?

Solution by A. C. Goddard and F. I. John, Dugway, Utah. The number of different cards certainly cannot exceed the number of cards with different middle columns. This is, in fact, the total number of possible different cards since the number of cards with different rows and diagonals is greater than this number $C(15, 4) = 1365$.

Also solved by G. B. Charlesworth and A. R. Hyde.

Much Ado About Nothing

E 1192 [1955, 728]. *Proposed by Edgar Karst, Independence, Missouri*

Let AB, BC be two adjacent sides of a regular nonagon inscribed in a circle of center O. Let M be the midpoint of AB and N the midpoint of the radius perpendicular to BC. Show that angle $OMN = 30°$. (Attention Professor Umbugio.)

Editorial Note. Professor Umbugio need not involve himself in a lengthy geometric or trigonometric argument, for, since O, A, M, N are concyclic, angle OMN = angle $OAN = 30°$.

Also solved by W. A. Al-Salam, J. L. Baker, Leon Bankoff, A. P. Boblétt, G. B. Charlesworth, A. E. Danese, J. E. D'Atri, Hüseyin Demir, J. M. Elkin, Edward Fleisher, H. E. Gould, Bernard Greenspan, W. T. Gregorzak, Emil Grosswald, Vern Hoggatt, J. P. Hoyt, Raymond Huck, A. R. Hyde, P. W. M. John, P. B. Kantor, M. S. Klamkin, Sam Kravitz, M. A. Laframboise, Elmer Latshaw, B. R. Leeds, D. C. B. Marsh, A. R. Nolstad, C. S. Ogilvy, Margaret Olmsted, Walter Penney, P. W. A. Raine, A. P. Rhodes, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, J. W. Ross, C. M. Sandwick, Sr., J. P. Scholz, E. P. Starke, Sister M. Stephanie, J. A. Tierney, Chih-yi Wang, Alan Wayne, Hazel S. Wilson, Samuel Wolf, David Zeitlin, and the proposer. Late solutions by Louisa R. Alger, G. W. Day, Roger Hou, F. I. John, Josef Langr, C. F. Pinzka, and A. I. Thaler.

Two Problems Connected with Rocket Research

E 1193 [1955, 728]. *Proposed by L. C. Barrett, University of Utah, and Herbert Knothe, Holloman Air Development Center*

(1) An ellipse has one focus at the center of two concentric circles, is tangent internally to the larger circle, and has at least one point in common with the smaller circle. Find the length of the maximum major axis of all such ellipses.

(2) Let the inner of two concentric circles represent a homogeneous spherical mass and the outer a prescribed orbit. Determine the range of specific energy values a particle may be given if it is to traverse a plane elliptical path from the surface of the sphere and just reach, but not cross, the circular orbit.

Solution by the Proposers. (1) Denote the radii of the circles by R and $R+h$, and let a typical ellipse have major axis $2a$, minor axis $2b$, and focal distance c. We may then write the polar equation of the ellipse as

$$r = b^2/(a - c\cos\phi).$$

For the ellipse to intersect the inner circle, it must be possible to determine ϕ's for which the equation

$$R = (R + h)(2a - R - h)/[a - (R + h - a)\cos\phi]$$

is satisfied. Solving this for $\cos \phi$ we obtain

$$\cos \phi = [(R + h)^2 - (R + 2h)a]/R(R + h - a).$$

As a increases, $\cos \phi$ decreases so that the maximum a occurs when $\cos \phi = -1$. Using the last equation, we find that the maximum value of $2a$ is $2R+h$, or the sum of the radii of the two circles.

(2) Since the particle moves in an inverse square force field, one focus of its elliptical path is at the center of the circles. Let R and $R+h$ be their radii and k the gravitational constant. Then the specific energy E of a particle traversing an orbit of major axis $2a$ is

$$E = - kM/2a.$$

The required range of specific energies is thus determined by the maximum and minimum values $2a$ can assume. These, from part (1), are $2R+h$ and $R+h$, respectively. If $2a = R+h$, the ellipse degenerates into a straight line segment. Hence

$$-kM/(R + h) < E \leqq kM/(2R + h).$$

Also solved by D. C. B. Marsh (partially), J. W. Ross, and R. H. Wilson, Jr.

The Proposers stated that these problems arose in connection with one phase of rocket research at the Holloman Air Development Center.

Construction of a Triangle

E 1194 [1955, 728]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a triangle ABC given A, $ma+b$, $na+c$, where m and n are given positive integers. (Dedicated to Professor W. B. Carver, the geometer of precision.)

Solution by R. L. Caskey, Oklahoma A. and M. College. If we construct triangle ADE with $AD = na+c$, $AE = ma+b$, angle $DAE = A$, we note that the problem becomes that of drawing a secant meeting the sides AD, AE in B, C, respectively, such that $DB:BC:CE = n:1:m$.

To solve this problem we construct, as follows, a quadrilateral $DAHI$ homothetic to the required quadrilateral $DBCE$ and having D as center of homothety. On EA lay off $EF = (m/n)AD$. Draw FG parallel to DE. Draw circle $(A, AD/n)$, cutting line FG in H. Draw HI parallel to FE, cutting DE in I.

We now obtain quadrilateral $DBCE$ by drawing DH, letting it intersect AE in C, and then drawing CB parallel to HA. If, in this construction, B falls between A and D and C between A and E, then triangle ABC is a solution of the problem. For, by construction, $DA:AH:HI = 1:1/n:m/n = n:1:m$. Therefore $DB:BC:CE = n:1:m$, and thus $DA = DB+BA = na+c$ and $EA = EC+CA = ma+b$.

Examples can be produced showing two, one, or no solutions. The number of solutions is controlled by the fact that circle $(A, AD/n)$ may cut line FG in two, one, or no points, and also by the fact that for the construction to be acceptable, B must fall between A and D and C between A and E.

We may remove the restriction that m and n be positive integers by letting m and n be represented by line segments and by letting the unit segment be given.

Also solved by W. B. Carver, Roger Hou, A. R. Hyde, Edgar Karst, M. S. Klamkin, Beckham Martin (three ways), D. C. B. Marsh, H. D. Ruderman, C. M. Sandwick, Sr., and the proposer. Late solutions by Vern Hoggatt and Josef Langr.

For the Proposer's treatment of this problem see *Mathesis*, vol. LXIV (1955), nos. 6–7–8, p. 286, art. 42.

Carver communicated a lengthy and remarkably complete algebraic treatment, which more than upholds the description of him in the dedication. The interested reader might like to carry out Caskey's construction on the following numerical examples given by Carver. The examples exhibit the possibility of two solutions and show how the construction may break down in various ways to give one or no solution.

$5a + b = 3$	$3a + b = 16$	$3a + b = 4$	$5a + b = 3$
$2a + c = 2$	$2a + c = 7$	$2a + c = 7$	$2a + c = 2$
$\cos A = -1/2$	$\cos A = 1/2$	$\sin A = 1/5$	$\cos A = 19/20$

$5a + b = 10$	$5a + b = 3$	$5a + b = 3$	$9a + b = 2$
$4a + c = 9$	$2a + c = 2$	$4a + c = 5$	$2a + c = 1$
$\cos A = 1/2$	$\cos A = 43/45$	$\cos A = 99/100$	$\cos A = -1/2$

$2a + b = 5$	$3a + b = 5$	$9a + b = 2$
$2a + c = 4$	$3a + c = 4$	$2a + c = 1$
$\cos A = 197/208$	$\cos A = 17/18$	$\cos A = (2 + \sqrt{504})/25$

A Conditional Inequality

E 1195 [1955, 728]. *Proposed by G. E. Bardwell, University of Denver*

If $n=2, 3, 4, \cdots$, and m is fixed and positive, for what positive values of p less than 1 is $\ln n < mn^p$?

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If $mn^p > \ln n$, then

$$p > \ln [(\ln n)/m]/\ln n = (\ln x - \ln m)/x,$$

where $x = \ln n$. Now the maximum value of $(\ln x - \ln m)/x$ occurs when $x = em$ and is equal to $1/em$. Thus

$1 > p > 1/em$ provided $m > 1/e$.

Also solved by G. B. Charlesworth, A. E. Danese, Nathaniel Grossman, A. R. Hyde, Lawrence Isenecker, D. C. B. Marsh, Walter Penney, E. P. Starke, Chih-yi Wang, David Zeitlin, and the proposer. Late solutions by A. J. Goldman and Roger Hou.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4693. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!n!}{(m+n+2)!}=\frac{\pi^2}{6}.$$

4694. *Proposed by R. W. Bagley, University of Kentucky*

There are simple examples which show that an uncountable topological space in which limits are unique (hence T_1) is not necessarily Hausdorff. Are there such examples for countable spaces? Here "limit" is used in the usual sense of limit of a sequence where the directed set is the positive integers rather than limit of a generalized sequence as defined by Kelley and others. With this general definition (where the directed set is allowed to vary), Kelley proved that a space is Hausdorff if, and only if, limits are unique. (See, Convergence in Topology, *Duke Mathematical Journal*, (1950), pp. 277–283.)

4695. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

4696. *Proposed by T. M. Apostol, California Institute of Technology*

Let $\Lambda(n)$ denote Mangoldt's function, defined as follows:

$$\Lambda(n) = \begin{cases} \log p \text{ if } n \text{ is a power of the prime } p \\ 0 \text{ otherwise.} \end{cases}$$

For $n>1$, let $f_m(n)$ denote the number of representations of n as a product of m factors, where each factor is greater than 1 and the order of the factors is taken into consideration. (*E.g.*, $f_2(24)=6$ and $f_3(16)=3$.) Prove the identity

$$\Lambda(n) = (\log n)\sum_{m=1}^{\nu(n)} \frac{(-1)^{m+1}}{m} f_m(n), \qquad (n > 1),$$

where $\nu(n)$ is the total number of prime factors of n, *i.e.*, $\nu(n)=a_1+a_2+\cdots$ if $n=p_1^{a_1}p_2^{a_2}\cdots$.

4697. *Proposed by Leonard Carlitz, Duke University*

Find the roots of the equation

$$y^n - ny^{n-2} + \frac{n(n-3)}{2!}y^{n-4} + \cdots + (-1)^r \frac{n(n-r-1)\cdots(n-2r+1)}{r!}y^{n-2r} + \cdots = 2a,$$

in particular when $a=1$.

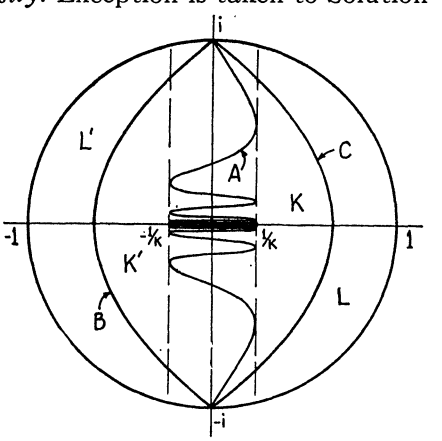
SOLUTIONS

Zero of a Complex Function

4594 [1954, 427]. *Proposed by Edgar Reich, Rand Corporation, Santa Monica, California*

If $f(z)$ is a complex-valued continuous function of the complex variable z, such that $f(z)=z$ whenever $|z|=1$, show that $f(z)$ has at least one zero in $|z|<1$.

IV. *Note by S. L. Gulden, Lehigh University.* Exception is taken to Solution I [1955, 591]. In this solution if one sets $f(z)=u(z)+iv(z)$ it is required that the set $\{z|u(z)=0\}$ contain a path joining i to $-i$. The following shows that this need not be true. (See figure).



Let

$$D = \{z| \, |z| \leqq 1\},$$

$$E = \left\{ z \,|\, z = \frac{1}{k} \sin \frac{\pi}{|y|} + iy;\ 0 < y \leqq 1 \right\},$$

$$F = \left\{ z \,|\, z = x;\ -\frac{1}{k} \leqq x \leqq \frac{1}{k},\ k > 100 \right\}.$$

Let $A = E \cup F$. Let B and C be circular arcs which do not meet A save at i and $-i$ (by choosing k sufficiently large this can be arranged) and assume them parametrized by y, *i.e.*, B is given by $x = x_1(y)$, C by $x = x_2(y)$, $-1 \leqq y \leqq 1$.

Let

$$G = \left\{ z \,|\, z = x + iy,\ \frac{1}{k} \sin \frac{\pi}{|y|} \leqq x \leqq x_2(y),\ y \neq 0 \right\},$$

$$H = \left\{ z \,|\, z = x,\ -\frac{1}{k} \leqq x \leqq x_2(0) \right\},$$

$$K = G \cup H,$$

$$G' = \left\{ z \,|\, z = x + iy,\ x_1(y) \leqq x \leqq \frac{1}{k} \sin \frac{\pi}{|y|},\ y \neq 0 \right\},$$

$$H' = \left\{ z \,|\, z = x,\ x_2(0) \leqq x \leqq \frac{1}{k} \right\},$$

$$K' = G' \cup H',$$

$$L = \left\{ z \,|\, z = x + iy,\ x_2(y) \leqq x \leqq (1 - y^2)^{1/2} \right\},$$

$$L' = \left\{ z \,|\, z = x + iy,\ -(1 - y^2)^{1/2} \leqq x \leqq x_1(y) \right\}.$$

The sets K, K', L, L' are all closed.

Define $u(z)$ on D as follows: Let d be the usual metric on D,

$$(d(z_1, z_2) = |z_1 - z_2|);$$

on K define

$$u(z) = \frac{d(z, A)}{5},$$

on K' define

$$u(z) = -\frac{d(z, A)}{5};$$

on L let

$$t = \frac{x - x_2(y)}{(1 - y^2)^{1/2} - x_2(y)}$$

and define

$$u(i) = u(-i) = 0,$$

$$u(z) = (1 - t)\frac{d(x_2(y) + iy, A)}{5} + t(1 - y^2)^{1/2}, \qquad z \neq i, -i;$$

on L' let

$$t = \frac{x_1(y) - x}{x_1(y) + (1 - y^2)^{1/2}}$$

and define

$$u(i) = u(-i) = 0,$$

$$u(z) = (1 - t)\left[-\frac{d(x_1(y) + iy, A)}{5}\right] - t(1 - y^2)^{1/2}, \qquad z \neq i, -i.$$

The following facts are not difficult to establish:

1. $u(z)$, thus defined on D, is continuous.
2. $\{z \mid u(z) = 0\} = A$.
3. There is no path in A joining i to $-i$.
4. For $|z| = 1$, $u(z) = \mathrm{Re}(z)$.

If one now takes for v the function defined by $v(z) = \mathrm{Im}(z)$ and for f the function $f(z) = u(z) + iv(z)$, where u is as defined above, then the argument of Solution I fails to show that $f(z)$ has a zero in D.

It should be noted, however, that although the supporting argument does not hold, the theorem: *Under the given hypotheses, for any* z_1 *with* $|z_1| < 1$ *there is a* $z_2 \in D$ *such that* $f(z_2) = z_1$, is nevertheless true and can be proved by a simple generalization of solution II of the problem.

Confocal Conics

4630 [1955, 127]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given two confocal conics C_1 and C_2, let M_1 on C_1 and M_2 on C_2 be so chosen that the tangents at these points are perpendicular. Show that the envelope of M_1M_2 is another conic having the same foci and having asymptotes passing through the intersections of C_1 and C_2.

Solution by E. J. F. Primrose, University College, Leicester, England. We assume that the geometry is complex. If M_1 is given on C_1 there are two diametrically opposite points M_2 on C_2 which satisfy the condition. Similarly, if M_2 is given on C_2 there are two points M_1 on C_1 which satisfy the condition. Hence there is a (2, 2) correspondence between M_1 and M_2.

Now let O be a general point of the plane. Since an arbitrary line through O

meets C_1 in two points M_1, and C_2 in two points M_2, there is a (4, 4) correspondence between OM_1 and OM_2. Such a correspondence has eight double lines, so there are eight lines M_1M_2 which pass through a general point O. The class of the envelope M_1M_2 is therefore eight.

Now let the common points of C_1 and C_2 be Q, R, S, T, where Q and S are a diametrically opposite pair, and so are R and T. Then Q as a point of C_1 corresponds to Q as a point of C_2, since confocal conics cut orthogonally. Hence OQ is one of the double lines, and so are OR, OS, OT. It follows that part of the envelope consists of the four points Q, R, S, T. The rest of the envelope is of class four.

Of the four common tangents of C_1 and C_2, two pass through I and two through J, where I and J are the circular points at infinity. Let one common tangent touch C_1 at N_1 and C_2 at N_2. Then N_1 corresponds to N_2, since a line through a circular point is self-perpendicular. Also, if O is a point on the tangent, N_1 counts twice as an intersection of ON_1N_2 with C_1. Hence N_1N_2 counts twice as a line of the envelope, and so do the other three common tangents. Also, Q on C_1 corresponds to S on C_2 and S on C_1 corresponds to Q on C_2, so QS, quite apart from belonging to the pencil with vertex Q and to the pencil with vertex S, counts twice as a line of the envelope: so does RT.

Now if six lines are double lines of an envelope of class four, either

(i) the six lines are the sides of a quadrangle, so the envelope consists of four points, or

(ii) the envelope is a repeated conic.

Clearly (i) is not the case, so the envelope is a repeated conic. Since it touches the four common tangents of C_1 and C_2, it is confocal with C_1 and C_2. Since it touches QS and RT, and these are two lines through the center, they must be the asymptotes.

It is worth noting that, since every tangent to the envelope counts doubly, if M_1M_2 meets C_1 again at P_1 and C_2 again at P_2, the tangents at P_1 and P_2 must also be perpendicular. Incidentally the dual argument would show that the locus of the point of intersection of the perpendicular tangents consists of the four common tangents and the circle $QRST$ counted twice: hence the other tangents from the point of intersection are also perpendicular.

The problem is solved in Baker, *Principles of Geometry*, vol. II, pp. 127–133, but the proof is long and difficult, since the problem is treated as a particular case of a remarkable general theorem. A proof may also be found in C. Smith, *Conic Sections*, p. 311. Smith does not mention the asymptote property but this is easily proved on the basis of the given analysis.

Also solved by Josef Langr, E. J. F. Primrose (second solution), Sister M. Stephanie, Chih-yi Wang, and the Proposer. The Proposer points out that the proposition of the problem is the reciprocal with respect to L of the following considerably easier proposition: Being given two circles C_1 and C_2 and points M_1 and M_2 on C_1 and C_2 respectively, so chosen that angle $M_1LM_2=\pi/2$, where

L is one of the limit points of the pencil of circles determined by C_1 and C_2, then the tangents to C_1 and C_2 at M_1 and M_2 intersect on a circle belonging to the pencil of circles.

A Permutation Problem

4633 [1955, 258]. *Proposed by W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

An even number, n, of chairs are arranged in a circle and numbered 1 through n. An equal number of people are numbered 1 through n and line up in an arbitrary manner. Prove that they can be seated in such a way as to preserve their order and such that no person's number is the same as that of his chair.

Solution by John Isbell, Aberdeen Proving Ground, Maryland. The permutation f of the persons makes seating without coincidence impossible if and only if for each j there is an i such that $f(i)+j\equiv i \pmod{n}$. For each i there is exactly one j, and hence $i\to j$ is a permutation g.

Thus we have a problem in the ring of integers modulo n. Now if n is odd there is a permutation $f(i)=g(i)=i/2$ such that $f(i)+g(i)=i$. But if n is even, the sum of all elements of the ring is $1+2+\cdots+n\equiv n/2$. Thus the sum of all $f(i)$ is n, and all $g(i)$ is n, and $i\to f(i)+g(i)$ is not a permutation.

Also solved by W. J. Blundon, Marshall Freimer and S. W. Golomb, A. J. Goldman, R. E. Greenwood, J. E. Householder, Free Jamison, D. C. B. Marsh, Katsuyoshi Memezawa, Leo Moser, D. J. Newman, J. V. Pennington, James Singer, and Michael Skalskyj.

Elliptic Integrals

4634 [1955, 258]. *Proposed by Chih-yi Wang, University of Minnesota*

Let a real number k, $0<k<1$, be given. Define the elliptic integrals

$$A_s=\int_0^{\pi/2}\sin^{2s}t\sqrt{1-k^2\sin^2t}\,dt, \qquad B_s=\int_0^{\pi/2}\frac{\sin^{2s}t\,dt}{\sqrt{1-k^2\sin^2t}}$$

for $s=0, 1, 2, \cdots$. Prove that

$$\text{(a) } B_r=\frac{B_{r-1}-A_{r-1}}{k^2}, \qquad \text{(b) } A_r=\frac{(2rk^2-1)A_{r-1}+(1-k^2)B_{r-1}}{(2r+1)k^2},$$

for $r=1, 2, 3, \cdots$.

Solution by T. F. Mulcrone, S. J., St. Charles College, Grand Coteau, La. Part (a) follows immediately upon substitution of the values for A_{r-1} and B_{r-1} in terms of the defining integrals.

For part (b), we have

$$A_{r-1} - A_r = \frac{1}{2r-1}\int_0^{\pi/2} \cos t\sqrt{1 - k^2\sin^2 t}\cdot d(\sin^{2r-1}t),$$

and an integration by parts gives

$$A_{r-1} - A_r = \frac{1}{2r-1}\left[A_r + k^2(B_r - B_{r+1})\right],$$

from which part (b) follows upon two applications of (a).

Also solved by W. J. Blundon, L. Carlitz, P. L. Chessin, G. B. Findley, V. D. Gokhale, Peter Henrici, H. F. Hunter, A. R. Hyde, M. S. Klamkin, G. N. Lance, W. R. McEwen, D. C. B. Marsh, R. E. Messick, Kovina Milosevich, C. D. Olds, B. E. Rhoades, P. G. Rooney, M. R. Spiegel, O. E. Stanaitis, J. A. Tierney, and the Proposer.

The Inversion in an Additive Group

4635 [1955, 259]. *Proposed by A. E. Babbitt, Jr., Rutgers University*

Let G be an additive Abelian group which has no element of order 2 and let f be a one to one mapping of G onto G. Then f is the inversion (*i.e.*, $f(x) = -x$) if and only if the following two conditions are satisfied:

I. The neutral element 0 is the only element of G invariant under f.

II. For every element $a \in G$, there exists an element $b = b(a)$ in G such that, for all $x \in G$,

$$f(x - a) = f^{-1}(x + b) - a.$$

Solution by W. R. Scott, University of Kansas. The hypothesis that G be commutative can be dropped provided that the equation in II is replaced by

II′. $$f(x - a) = -a + f^{-1}(x + b).$$

If $f(x) = -x$, then I is obvious; and II′ is satisfied with $b = -2a$. Conversely, assume I and II′ are true. Then $f(0) = 0$, and letting $x = a$ in II′ we get $a = f^{-1}(a + b)$ or

$$b = -a + f(a).$$

Again, with $x = 2a$, II′ yields $a + f(a) = f^{-1}(2a - a + f(a))$, or

$$f(a + f(a)) = a + f(a).$$

Hence, by I, $f(a) = -a$.

Also solved by J. D. Baum and Angelo Margaris, B. J. Boyer, R. H. Breusch, T. A. Brown, R. G. Crook and Velma R. Vogler, David Ellis, M. P. Epstein, V. D. Gokhale, A. J. Goldman, C. D. Gorman, D. S. Greenstein, Virginia S. Hanly, E. H. Hietbrink, Douglas Holdridge, S. P. Hughart, H. H. Johnson, W. G. Leavitt, H. D. Lipsich, D. C. B. Marsh, Katuyosi Memezawa, R. H. Oehmke, F. P. Palermo, B. E. Rhoades, P. G. Rooney, H. D. Ruderman,

P. T. Schaefer, Richard Scoville, James Singer, R. E. Swett, G. H. M. Thomas, Olga Taussky-Todd and John Todd, C. J. Vanderlin, Chihi-yi Wang, J. V. Whittaker, H. H. Wicke, F. J. Witt, Donald Ylvisaker, and the Proposer.

Editorial Note. This result is equivalent to a result obtained by de Kerekjarto, *C. R. Acad. Sci.*, Paris, 210, 1940, pp. 288–9. It was generalized to arbitrary groups in a paper by Taussky and Todd, Inversion in groups, *Quarterly Journal of Mathematics*, 12 (1941), pp. 65–7. Ellis adds the following references: *Anzeiger der Akad. der Wissenschaften* (Wien), 1930, 140–2; *Math. Zeitschr.*, 33, 1931, 412; *Publicationes Math.* (Debrecen), 2, 1951, 1–25.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

The Bequest of the Greeks. By Tobias Dantzig. Charles Scribner's Sons, New York, 1955. 191 pages. $3.95.

Ever since the appearance of Professor Dantzig's famous book *Number: The Language of Science*, other works by this gifted interpreter of the world of mathematics have been eagerly purchased and read by both the professional mathematician and the layman. In this new book, which is the first volume of a projected trilogy, Professor Dantzig again takes up his favorite study of the evolution of mathematical ideas. Here he is concerned with those "problems, principles and procedures which modern mathematics has inherited from Greek antiquity."

The book is divided into two parts. Part One, entitled "The Stage and the Cast," is designed for the general reader and makes practically no demands upon the reader's technical skill in mathematics. Part Two, entitled "In Retrospect," is an anthology of the Greek bequest, and is definitely more technical in character.

Part One opens with three chapters devoted to the historical mystery of the Greek achievement, to the founders of deductive mathematics, and to the origins of geometry. Here we find one of those relatively rare instances in which Professor Dantzig foregoes his general reluctance to offer his personal speculations on historically uncertain matters, and in it Thales emerges considerably stronger, and Pythagoras considerably weaker, than in traditional accounts of this period. Next follows a pair of interesting chapters devoted to Thales' measurement of an Egyptian pyramid and to the occult developments of the

golden ratio. If one is to consider seriously the alleged feat of Thales in measuring the height of a pyramid, one must then look into some of the practical difficulties involved, such as the determination of the distance from the center of the floor of the pyramid to the apex of the pyramid's shadow. These matters have received little attention in most accounts of Thales' accomplishment, and Professor Dantzig explains an historically plausible conjecture as to how Thales may have circumvented the difficulties. The chapter on occult fringes of mathematics carries the mystical golden section from the early Greeks up through the schizophrenic writings of Johann Kepler. Part One concludes with a chapter called "The Pseudomath," which many of us read when it appeared separately in *The Scientific American*, and one entitled "The Interdiction," devoted to the Greek decree that geometers limit themselves in their constructions to the straightedge and compass; the subsequent history of mathematics owes much to this deep-rooted prohibition.

The anthology of Part Two is made up of eight topics—the Pythagorean theorem, Pythagorean triples, the lunes of Hippocrates, the quadratrix of Hippias, the Euclidean algorithm, Archimedes' quadrature of the parabola, Hero's formula for the area of a triangle, and Greek trigonometry. Each of these issues has survived the passage of time, and in some form or other is still alive today. Some of the items have only recently reached full fruition, while others still remain largely incomplete. Professor Dantzig handles these items with consummate skill and interest. For example, the chapter on Pythagorean triples carries the story from the Greek period, through the work of Fibonacci in the Middle Ages, to some of the associated problems of modern times to which partial answers, solutions, and supplements have been given by Fermat, Euler, Lagrange, Legendre, and Gauss. One's conclusion is that, after having engaged the attention of top mathematicians for twenty-five hundred years, the Pythagorean equation cannot yet be said to be exhaustively solved. As regards the Pythagorean triples, Professor Dantzig strangely says nothing about the very probable ancient Babylonian treatment as substantiated by the remarkable tablet *Plimpton 322*. To the present reviewer, the high spot of Part Two is the chapter devoted to the lunes of Hippocrates; the recent work on these lunes described by Professor Dantzig is not too widely known.

Professor Dantzig's anthology contains only eight topics. There are certainly other items that come to mind, such as the Eudoxian theory of proportion and its culmination in modern times in the Dedekind theory of irrational numbers, and Euclid's parallel postulate and the subsequent liberation of geometry and modern development of axiomatics. But, as appears in a statement in the Epilogue to the book, Professor Dantzig intends to deal with other items in the next volume of the projected trilogy. We all certainly look forward to the forthcoming volumes of the trilogy.

There is much that is scholarly in Professor Dantzig's book, and there are many instances of the author's skilful use of the written word. It seems a pity to have to add a few words of a derogatory nature. But these derogatory

comments are almost all of a superficial sort. It would seem inexcusable that a work which otherwise is so well executed should have been so poorly proofread. One of the greatest sins in this respect is the frequent and confusing disagreement between the lettering of a figure and the description of the figure in the accompanying text. This fault is found, for example, in Figures 8*a* and 8*b* on p. 62, Figure 24 on p. 140, Figure 27 on p. 161, and Figure 29*a* on p. 171. On p. 63 we find "*witch'es foot*"; on p. 67 some quotation marks are omitted; on p. 95 we find the bad hyphenation "quadr-"; in the last line on p. 100 we find $ACUUV$ instead of $ACUV$; on p. 102 we find a' in Figure 15 where a should appear; on p. 130, line 15, the u should be a v, and the last displayed equation on this page should read "$u:v=a:b$"; on p. 132, line 5, we should read "(3,2)" for "(3.2)", and similarly on this same page in line 1 of section 10 we should read "(3,1)" for "(3.1)"; in section 10 on p. 132 we should interchange u and v; on p. 133 we again find the incorrect "(3.1)" and "(3.2)", and this same oversight is perpetuated on pp. 134, 135, and 136; on p. 145 there is a word missing in line 2 of section 1; the first sentence of section 2 on p. 146 is at fault; on p. 155 c is *not* always a better approximation than either a or b; on p. 156 equation (13.6) should commence with u_{n+1}; on p. 162 the definition of Heronian triples is carelessly stated; on p. 163, line 3 from the bottom, we should read "Figure 28" for "Figure 31"; on p. 164, lines 8 and 9, the language "it is greater, equal or less than 1" is faulty; on p. 166, line 3, we should read "$3y^2$" for "$3y$"; on p. 174 Figure 31*a* should be labelled Figure 31*b*, and Figure 31*b* should be labelled Figure 31*a*; on p. 175, line 5, we should read "$\cos \frac{1}{2}\theta$" instead of "$\cos \theta$," and the same correction should also be made in the accompanying Figure 32. Finally, mathematical equations are parts of sentences, and should therefore be properly punctuated. There are many omissions of such expected punctuation in this book. See, for example, pp. 126, 130, 141, 144, 156, 157.

HOWARD EVES
University of Maine

Principles of Mathematics. By. C. B. Allendoerfer and C. O. Oakley. New York, McGraw-Hill Book Company, 1955. xv+448 pages. $5.00.*

In their new book *Principles of Mathematics*, Professors Allendoerfer and Oakley have made a distinct contribution to the undergraduate curriculum. There is almost universal awareness among college mathematics teachers that something fundamental is happening to our beginning courses and there is now more widespread concern over these courses than at any time in the past fifty years. This concern is reflected in the outpouring of genuinely new textbooks.

* Due to the interest in this book, the Book Review Editor felt that two reviews were warranted, and takes full responsibility for both. E.P.V.

The editor-in-chief is both pleased and embarrassed by the attention shown this book. He has followed his usual policy of giving a free hand to Professor Vance and consequently is printing both reviews. C.B.A.

which he can scarcely believe are significant. This makes the problem of motivation extremely difficult.

The criticisms expressed here naturally suggest the question of whether or not this text might be suitable for carefully selected groups of freshmen who already know what used to be considered as high school mathematics. Though not wanting to deny that possibilities along this line may exist, this reviewer is strong in the belief that much of the contents of this text should be deferred till a junior-senior course based perhaps on the well-known text of Birkhoff and MacLane. In preparation for such a course, the elements of matrix theory could be included in freshman algebra and analytic geometry.

W. R. MANN
University of North Carolina

Topological Transformation Groups. By D. Montgomery and L. Zippin. New York, Interscience Publishers, Inc., 1955. 282 pages.

The book under review is a carefully written study of topological groups giving special emphasis to locally compact topological groups and transformation groups. It is substantially more readable than most books dealing with topics of roughly the same level of abstraction and should, therefore, be very useful to all kinds of mathematicians, from first year graduate students to mature mathematicians whose primary interest is in some other branch of mathematics.

The book consists of three parts. The first part (chapter I) introduces the reader to the basic topological and group theoretical notions. Topological groups, local groups, transformation groups, manifolds, as well as covering spaces are defined in this first chapter. It is one of the pleasant features of the book that whenever any of the basic concepts is used for the first time in a new context in a later chapter, a reference is made to the appropriate page of the first chapter.

The second part of the book (chapters II to IV) is primarily concerned with locally compact groups and their relation to Lie groups. Besides reviewing the more important classical work in the field, these chapters summarize the important work which has recently been done on Hilbert's fifth problem, work which until now could be found in papers only. It is best to state this problem in the authors' own words (page 70). Before doing this we note that a transformation group is said to act effectively on a manifold, M, if the identity element of the group is the only element mapping each point of M into itself. The authors, then, discuss the key question of this section:

"Let us now consider the following questions, the second and third of which are asked by Hilbert:

If a locally compact group acts effectively on a manifold M then

(1) is G necessarily locally euclidean,

(2) if the group G is locally euclidean, is it a Lie group in some appropriate coordinate,

(3) if G is a Lie group can coordinates be chosen in G and M so that the transformation functions are analytic?

The answer to (2) · · · is yes · · · . The answer to (1) is unknown except in some special cases · · · . The answer to (3) is no."

In the process of developing these theories use is made of the work of Gleason and Yamabe in proving the equivalence of the concepts of a Lie group and of a locally compact group without small subgroups (a group is said to have no small subgroups if there exists a neighborhood of the identity which contains no subgroups except the identity itself). Finally, the question of approximating compact and locally compact groups by Lie groups is extensively investigated in these chapters.

The last part of the book (chapters V and VI) presupposes a certain amount of background in algebraic topology. On the other hand, it is probably the most rewarding part from the point of view of the mathematician who is primarily interested in the still unsolved problems of mathematics. In this last part of the book, the authors discuss transformation groups deriving some and merely stating some of the other results while clearly indicating some of the existing gaps. The following is a partial list of the topics which are treated: differentiability properties of the transformations, conjugacy of subgroups of a Lie group, orbits and cross sections (the orbit of a point is the totality of its images under the transformation group G, a cross section is a 1:1 image of the orbits in the neighborhood of some point) periodic and pointwise periodic transformations, and finally an analysis of the action of a compact connected group on 3-space.

Summarizing, this is a very interesting book from just about every point of view. The organization and motivation are unusually good. Most potential readers can look forward to an enjoyable experience.

F. HAAS
University of Connecticut

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley and Los Angeles, 1956. Twenty-five per cent discount to members of the Mathematical Association of America.

VOLUME I. *Contributions to the Theory of Statistics*

Contributors: Joseph Berkson; Z. W. Birnbaum; Herman Chernoff and Herman Rubin; Aryeh Dvoretzky; Sylvain Ehrenfeld; G. Elfving; Ulf Grenander and Murray Rosenblatt; J. L. Hodges, Jr. and E. L. Lehmann; Wassily Hoeffding; Samuel Karlin; L. Le Cam; Herbert Robbins; Murray Rosenblatt; Charles Stein; B. L. Van der Waerden.

Cloth. List Price $6.00. Publication Date: August 18, 1956.

VOLUME II. *Contributions to Probability Theory*

Contributors: David Blackwell; Salomon Bochner; K. L. Chung; A. H. Copeland, Sr.; J. L. Doob; Robert Fortet; J. M. Hammersley; T. E. Harris; Kiyosi Itô; Paul Lévy; Michel Loève; Eugene Lukacs; Karl Menger; Edith Mourier; R. Salem and A. Zygmund.

Cloth. List Price $6.50. Publication Date: October 10, 1956.

VOLUME III. *Contributions to Astronomy and Physics*

Contributors: Olin J. Eggen: Jesse L. Greenstein; Harold L. Johnson; Gerald E. Kron; Bengt Strömgren; G. C. McVittie; Jerzy Neyman; Elizabeth L. Scott and C. D. Shane; F. Zwicky; André Blanc-Lapierre and Albert Tortrat; M. Kac; J. Kampé de Fériet; Elliott Montroll; Norbert Wiener.

Cloth. List Price $6.25. Publication Date: June 1, 1956.

VOLUME IV. *Contributions to Biology and Problems of Health*

Contributors: James F. Crow and Motoo Kimura; Everett R. Dempster; Jerzy Neyman, Thomas Park and Elizabeth L. Scott; M. S. Bartlett; A. T. Bharucha-Reid; Chin L. Chiang, J. L. Hodges, Jr. and J. Yerushalmy; Jerome Cornfield; David Kendall; William F. Taylor.

Cloth. List Price $5.75. Publication Date: June 5, 1956.

VOLUME V. *Contributions to Econometrics, Industrial Research, and Psychometry*

Contributors: Kenneth J. Arrow and Leonid Hurwicz; Edward W. Barankin; C. West Churchman; Patrick Suppes; Albert H. Bowker; Cuthbert Daniel; Milton Sobel; T. W. Anderson and Herman Rubin; Frederick Mosteller; Herbert Solomon.

Cloth. List Price $5.75. Publication Date: June 30, 1956.

By a special arrangement, the members of the Association may purchase any or all of the five volumes at a 25 percent discount. To obtain this discount, orders must be sent to the Statistical Laboratory, University of California, Berkeley 4, California. They must refer to membership in the Association and be accompanied by checks for the correct amount. The checks should be made out to the University of California Press.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNUAL MEETING OF ASEE

The annual meeting of the American Society for Engineering Education will be held on June 25–29, 1956 at Iowa State College, Ames, Iowa. The following program will be presented by the Mathematics Division of the Society.

Joint Conference with Iowa Section of the Mathematical Association of America, June 25, 1956, 2 P.M.: *Analog Computers and Research in Mathematics*, J. W. Cell, North Carolina State College; *Statistics in Engineering Education and Research*, Bernard Ostle, Montana State College; *Worktext Teaching Aid in Advanced Applied Mathematics*, G. I. Cohn, Illinois Institute of Technology; *Colleges, High Schools, and the Mathematics Curriculum*, Henry Van Engen, Iowa State Teachers College; *Mathematical Problems in Industry*, Morris Ostrofsky, Westinghouse Electric Company; *The Industrial Research Mathematician*, S. L. Levy, Midwest Research Institute.

Luncheon and Business Meeting, June 26, 1956, 12 Noon: *Engineering for Mathematicians*, C. O. Oakley, Haverford College (retiring Chairman of the Division).

Conference on Trends in Applied Mathematics, June 26, 1956, 2 P.M.: *Introductory Remarks* by Chairman, F. A. Ficken, University of Tennessee; *Fluid Mechanics*, M. H. Martin, University of Maryland; *Electro-magnetic Theory* (speaker to be announced); *Numerical Methods*, A. S. Householder, Oak Ridge National Laboratory; *Elasticity*, (speaker to be announced).

Joint Conference with Engineering Drawing, June 27, 1956, 2 P.M.: *Operational Symbolism for Graphical Processes*, S. A. Coons, Massachusetts Institute of Technology; *Graphical Field Mapping*, J. F. Calvert, University of Pittsburgh; *Graphical Analogs of Mathematical Processes*, J. F. Twigg, Massachusetts Institute of Technology.

Joint Conference with Industrial Engineering Division on Operations Research, June 28, 1956, 2 P.M.: *Development of Operations Research*, D. W. Stebbins, Iowa State College; *Tentative General Principles of the Design of Air Defense Systems*, Richard Blythe, Air Defense Command, U.S.A.F.; *Determination of Optimum Reserve Generating Capacity in an Electric Utility Industry*, E. L. Arnoff, Case Institute of Technology; *Engineering Applications on Queuing Theory*, L. D. Findley and T. A. Mortenson, Midwest Research Institute.

ASSOCIATION FOR COMPUTING MACHINERY

The annual meeting of the Association for Computing Machinery will be held at the University of California, Westwood Campus, Los Angeles, August 27–29, 1956. For information write G. W. King, Box 3251, Olympic Station, Beverly Hills, California.

CONFERENCE ON APPLIED MATHEMATICS

The University of Buffalo will hold a Conference on Applied Mathematics during the two-week period from June 11 to June 22, 1956. The principal lecturer will be Professor R. V. Churchill of the University of Michigan. The conference is sponsored by the Union Carbide and Carbon Corporation in order to further promote the use of mathematical methods in industry in the Niagara Frontier.

INSTITUTE OF NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS

The eighth annual Institute for Teachers and Professors of Mathematics, sponsored by the Association of Teachers of Mathematics in New England, will be held at Williams College, Williamstown, Massachusetts, August 16–23, 1956. Mrs. Frances L. Allen of Newton High School, Massachusetts, is General Chairman.

The program consists of morning and evening lectures and three periods daily for discussion groups (four choices each period) with topics of interest to teachers from the seventh grade up. Social hours and other recreational activities are also planned.

Some of the topics to be discussed are *Enrichment of the Teaching of Junior High School Mathematics*, *How to Integrate Algebra, Geometry, and Trigonometry with Conventional Textbooks*, *Creative Imagination in the Teaching of Mathematics*, *Number Systems*. In addition to the junior and senior high school laboratories, Professor E. A. Hoadley will be back with both beginners' and advanced courses in *Geometry Applied to Artistic Design*.

For complete program and other information, please write to Miss Barbara B. Betts, 138 Norfolk Avenue, Swampscott, Massachusetts.

ILLINOIS JOURNAL OF MATHEMATICS

The University of Illinois announces the forthcoming publication of the Illinois Journal of Mathematics, a quarterly devoted to the publication of basic research in pure and applied mathematics. The first number will appear early in 1957, and it is planned that each number will contain about 150 pages.

The present board of editors consists of Reinhold Baer, J. L. Doob, and A. H. Taub, University of Illinois; George W. Whitehead, Massachusetts Institute of Technology; and Oscar Zariski, Harvard University.

The Illinois Journal will publish papers in English, French, German, and Italian. Manuscripts should be sent to the Illinois Journal of Mathematics,

Department of Mathematics, University of Illinois, Urbana, Illinois, or to one of the editors.

The Illinois Journal of Mathematics is made possible by a grant from the bequest to the University of Illinois by the late Professor G. A. Miller, and by an initial subvention from the American Mathematical Society.

The subscription price is $9.00 a volume (four numbers), with a special rate of $5.00 for individual members of the American Mathematical Society. Subscriptions should be sent to the University Press, 207 Administration East, University of Illinois, Urbana, Illinois, U.S.A.

PERSONAL ITEMS

Lebanon Valley College announces: Associate Professor B. H. Bissinger has been elected to the status of member in the London Mathematical Society; also, the College has inaugurated, in the Harrisburg area, a curriculum in Operations Research designed for management personnel as well as engineers and scientists. The following courses have been given: Operations research for management; Linear programming and game theory in industry.

Dr. S. I. Askovitz, consultant in mathematics and staff physician at the Albert Einstein Medical Center and at the Wills Eye Hospital, Philadelphia, Pennsylvania, has been appointed Chief of the Tumor Registry at the Hospital of the University of Pennsylvania, as biostatistician in the field of cancer.

Mr. Souren Babikian has been appointed to an instructorship at Los Angeles City College.

Dr. R. W. Bagley, formerly a senior aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas, has accepted a position as associate research scientist on the applied mathematics staff of Lockheed Aircraft Corporation, Van Nuys, California.

Mr. C. J. Biggerstaff, previously of the Flight Computer Group, Firestone Company, Los Angeles, California, has a position as an electronics engineer at the Research and Development Laboratories, Rheem Manufacturing Company, Downey, California.

Associate Professor Leonard Bristow of Wisconsin State College, Oshkosh, has been promoted to a professorship.

Professor Harold Chatland, Dean of the College of Arts and Sciences of Montana State University, is now Acting Dean of the Faculty.

Mr. M. M. Chirico, previously a mathematician at the Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland, has a position as a scientist for the Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Associate Professor C. F. Christ of Johns Hopkins University has been appointed Associate Professor of Economics at the University of Chicago.

Assistant Professor H. F. DeBaggis of the University of Notre Dame has been appointed Head of the Department of Mathematics of Carroll College, Montana.

Dr. D. L. Fuller of the American Cyanamid Company has accepted a position as Director of Research with the Grace Chemical Research and Development Company, New York, New York.

Dr. George Gioumousis of the University of Wisconsin has accepted a position as a theoretical chemist with the Shell Development Company, Emeryville, California.

Dr. Stanley Katz of the Electro Data Corporation has accepted a position with the American Cyanamid Company, New York, New York.

Mr. M. L. Keedy, instructor of physics at the University of Nebraska, will serve as a science counselor for the coming year on a pilot study being conducted in Nebraska to determine ways and means of raising the level of mathematics and science teaching in the secondary schools. This study is sponsored by the American Association for the Advancement of Science and has received a grant from the Carnegie Foundation for this purpose.

Assistant Professor P. J. Kelly of the University of California, Santa Barbara College, has been promoted to an associate professorship.

Associate Professor E. S. Kennedy of the American University, Beirut, Lebanon, has been promoted to a professorship.

Mr. P. E. Klebe, Jr., has been appointed to an instructorship at the Polytechnic Institute of Brooklyn.

Professor A. S. Merrill, who has been both Dean of the Faculty and Vice-President of Montana State University, will continue as Vice-President.

Dr. W. A. Michael, Jr., of Northwestern University has accepted a position as applied science representative with International Business Machines Corporation, San Francisco, California.

Dr. John Raleigh of the University of Pennsylvania has been appointed to an assistant professorship at Lafayette College.

Mr. P. C. Rapp, formerly a dynamicist at Bell Aircraft Corporation, Niagara Falls, New York, is now an engineer at Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. R. B. Rice, previously a senior physicist for the Phillips Petroleum Company, Bartlesville, Oklahoma, has a position as a senior research physicist for the Ohio Oil Company, Denver, Colorado.

Dr. G. F. Rose, recently a mathematician with the Sandia Corporation, Albuquerque, New Mexico, is a staff member of the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. J. A. Standerfer, formerly a teaching assistant at the University of Wisconsin, has a position as an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. C. E. Stewart, previously a stress analyst for the Boeing Airplane Company, Seattle, Washington, is now a research engineer for the Armour Research Foundation, Illinois Institute of Technology.

Mr. A. V. Sylwester, formerly an instructor in the U. S. Air Force Weather Forecasting School, Chanute Air Force Base, Illinois, has been released from

active duty to accept a position as a mathematician at the Naval Ordnance Laboratory, Corona, California.

Mr. B. F. Whisler, previously a mathematician with the Army Map Service, Washington, D. C., has a position as a mathematical analyst with the Lockheed Aircraft Corporation, Burbank, California.

Professor C. R. White of the State Colored Normal, Industrial, Agricultural and Mechanical College of South Carolina has accepted a position as mathematician with the Weapon Systems Laboratory, Aberdeen Proving Ground, Maryland.

Assistant Professor R. L. Wilson of the University of Tennessee has a position as a senior aerophysics engineer with the Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dean Fredrick Wood of the University of Nevada has retired with the title of Dean Emeritus.

Mr. B. H. Youell, Jr., previously a graduate assistant at West Virginia University, has been appointed to an assistant professorship at Waynesburg College.

Mr. J. S. Ayars, Jr., graduate student at Stanford University, died on February 3, 1956.

Assistant Professor Emeritus Harriet E. Glazier of the University of California at Los Angeles died on November 7, 1955. She was a charter member of the Association.

Professor Emeritus H. B. Leonard of the University of Arizona died on December 19, 1955. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NEW EDITOR-IN-CHIEF

The Board of Governors of the Association has elected Professor Ralph D. James of the University of British Columbia as Editor-in-Chief of the AMERICAN MATHEMATICAL MONTHLY for a five-year period beginning January 1, 1957.

Professor James has served as a member of the Board of Governors from 1952 to 1955 as a Governor from the Pacific Northwest Section. He has also served as Vice-Chairman and Chairman of the Pacific Northwest Section. He is

an editor of the *Canadian Journal of Mathematics* and of the *Pacific Journal of Mathematics* and has served as an editor of the *American Journal of Mathematics*. He is a member of the Council of the American Mathematical Society and is a Vice-President of the Canadian Mathematical Congress.

After July 1, 1956, articles intended for publication in the MONTHLY should be sent to Professor James at this address: University of British Columbia, Vancouver 8, B.C., Canada.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 113 persons have been elected to membership by the Board of Governors on applications duly certified.

F. B. ALLEN, M.S. (Iowa) Math. Teacher, Lyons Township High School, LaGrange, Ill.

A. T. ANASTASIO, Student, Fordham University.

R. C. ANDERSON, SR., M.S. (Mississippi) Asst. Professor, McNeese State College.

B. H. ARNOLD, Ph.D. (Princeton) Asso. Professor, Oregon State College.

RIC ARUFFO, B.A. (Temple) Math., Philco Corp., Philadelphia, Pa.

W. F. BAIRD, JR., B.A. (Texas Christian) Tech. Analyst, Consolidated-Vultee Aircraft Corp., Fort Worth, Texas.

A. F. ANN BAKER, B.S. (Kentucky S.C.) Grad. Fellow, Fisk University.

WILFRED E. BARNES, Ph.D. (British Columbia) Instr., State College of Washington.

MRS. LIDA K. BARRETT, Ph.D. (Pennsylvania) Instr., University of Connecticut at Waterbury.

MRS. VIRGINIA L. BARTELL, M.Ed. (Oregon) Math. Teacher, Grant High School, Portland, Ore.

E. L. BATTISTE, B.S. (Illinois State Normal) Applied Science Representative, International Business Machines Corp., Oklahoma City, Okla.

T. E. BAYNHAM, JR., B.S. (Austin Peay) Grad. Asst., University of Tennessee.

JACQUES BAZINET, B.S. (Montreal) Grad. Student, University of Montreal.

LT. (J.G.) F. T. BIRTEL, M.S. (Notre Dame) Math. Instr., U. S. Navy Nuclear Power School, New London, Conn.

P. K. BLACKWELL, B.A. (Chicago) Milwaukee School of Engineering.

S. S. BLAKNEY, M.S. (Illinois) Instr., Grambling College.

A. H. BOWKER, Ph.D. (Columbia) Professor, Stanford University.

NEAL BRADLEY, Student, Georgia Institute of Technology.

R. L. BROCK, Ph.D. (Oregon S.C.) Res. Engr., Boeing Airplane Co., Seattle, Wash.

M. F. BRYN, B.S. (North Dakota Agric. C.) Instr., North Dakota Agricultural College.

J. F. BURRIS, B.S. (Southeastern S.C.) Grad. Asst., Oklahoma Agricultural and Mechanical College.

J. E. CALVERT, Student, San Jose State College.

D. F. CLAPP, S.B. (M.I.T.) Grad. Student, University of Technology.

E. N. CLARK, M.S. in E.E. (Minnesota) Electrical Engr., Pioneer Service and Engineering Co., Chicago, Ill.

W. R. COOPER, M.A. (Columbia) Teacher, Northeastern Academy, New York, N. Y.

L. J. CORIO, Student, City College of the City of New York.

M. D. COX, Student, Tennessee Polytechnic Institute.

T. H. M. CRAMPTON, Ph.D. (Indiana) Instr., Mount Holyoke College.

WARD CROWLEY, S.M. (Brown) Asst. Professor, State College of Washington.

W. S. CURRIE, M.A. (Mississippi) Math., Texas Company Research Lab., Bellaire, Texas.

H. J. CURTIS, Ph.D. (Illinois I.T.) Asst. Professor, University of Illinois, Chicago.

LT. COL. H. T. DARRACOTT, M.S. (Arkansas) Asst. Director of Research, Signal Corps Engineering Labs., Fort Monmouth, N. J.

M. D. DAVIS, M.A. (California) Grad. Student, University of California.

LUTHER DAWES, B.S. (Oklahoma City U.) Math. and Science Teacher, Jarman Junior High School, Midwest City, Okla.

W. S. DORN, Ph.D. (Carnegie) Supervisor, General Electric Co., Cincinnati, Ohio.

D. E. EDMONDSON, Ph.D. (Calif. I.T.) Asst. Professor, Southern Methodist University.

J. W. ELLIS, Ph.D. (Tulane) Asst. Professor, Florida State University.

H. J. ELLSWORTH, Partner, Fankhanel-Ellsworth Properties, Minneapolis, Minn.

ANGELINE W. EVANS, Student, Agnes Scott College.

H. C. FILGO, JR., Ph.D. (Rice) Asst. Professor, University of Alabama.

G. D. FINDLAY, M.A. (Cambridge) Lecturer, McGill University.

C. P. GADSDEN, M.S. (Tulane) Instr., Tulane University.

Z. T. GALLION, Ph.D. (Louisiana S.U.) Professor, Southwestern Louisiana Institute.

B. N. GOULET, M.S. (Catholic) Math., Ballistic Research Labs., Aberdeen, Md.

J. R. GRAY, M.A. (Loyola, Ill.) Teacher, Wright Junior College, Chicago, Ill.

W. R. HARLOW, M.A. (Cincinnati) Asst. Professor, University of Cincinnati.

R. J. HARVEY, A.B. (California) Grad. Student, University of California.

T. P. HIGGINS, Ph.D. (Texas) Res. Engr., Boeing Airplane Co., Seattle, Wash.

J. B. F. HOCEPIED, Ingenieur-Architecte (Ghent) Consulting Engr., Bruges, Belgium.

KLAUS HÖECHSMANN, Student, University of British Columbia.

BERNARD HOWARD, B.S. (William and Mary) Grad. Asst., Brown University.

W. J. KAMMERER, M.S. (Wisconsin) Grad. Student, University of Wisconsin.

NAOKI KIMURA, M.S. (Osaka U.) Grad. Fellow, Tulane University.

R. D. KING, Student, Georgia Institute of Technology.

TADASHI KISHI, M.A. (California) Math., University of California Radiation Lab., Livermore, Calif.

W. R. KNIGHT, B.A. (British Columbia) Grad. Asst., University of British Columbia.

CLARENCE LANE, M.S. (Purdue) Instr., Purdue University.

L. L. LASSEN, M.A. (Arizona S.C.) Asso. Professor, Arlington State College.

BERNADINE C. LAW, B.S. (Mount St. Scholastica) Grad. Fellow, St. Louis University.

LOUIS LEMIEUX, Professor, College Jean de Brebeuf, Montreal.

COL. HENRY LINSERT, M.A. (Pennsylvania S.U.) Instr., East Tennessee State College.

L. I. LOKOMOWITZ, B.E.E. (C.C.N.Y.) Jr. Engr., Philco Corp., Philadelphia, Pa.

E. F. LOW, JR., Ph.D. (Florida) Asst. Professor, University of Miami.

MRS. GRETA S. MACK, A.B. (Duke) Asst. Instr., University of Kansas.

BERNARD MALINA, M.S. (Northwestern) Instr., Wright Junior College, Chicago, Ill.

L. L. MCNAIR, M.A. (Georgia) Instr., University of Georgia.

STELLA R. MIZELL, A.B. (North Carolina) Professor, Chowan College.

P. R. MONSON, B.S. (Utah State Agric.) Instr., Utah State Agricultural College.

AUGUSTA H. MORSE, M.A. (Radcliffe) Math. Teacher, Anna Head School, Berkeley, Calif.

H. A. MORTENSEN, Student, Utah State Agricultural College.

ABRAHAM NEMETH, M.A. (Columbia) Instr., University of Detroit.

C. C. NIELSEN, B.S. (Utah State Agric.) Instr., Utah State Agricultural College.

F. G. O'BRIEN, M.S. (Fordham) Asst. Professor, Fordham University.

R. O. PAN, Student, Case Institute of Technology.

EMANUEL PAPADOPULOS, B.A. (Sacramento S.C.) Jr. Engr., Boeing Airplane Co., Seattle, Wash.

D. W. PATTERSON, A.B. (Sacramento S.C.) Physicist, University of California Radiation Lab., Livermore, Calif.

ELAINE B. PAVELKA, B.A. (Northwestern) Grad. Student, Northwestern University.

R. Q. PETERSEN, Student, San Jose State College.

F. R. PLACHY, M.A. (Wyoming) Teacher, Center High School, Colo.

H. E. POKORNY, B.S. (Marquette) Instr., Milwaukee School of Engineering.

STANLEY PREISER, M.S. (N.Y.U.) Math., Nuclear Development Corp. of America, White Plains, N. Y.

R. F. QUANSTROM, JR., Student, Eastern Nazarene College.

A. T. RICE, A.M. (Boston U.) Math., United States Air Force, Washington, D. C.

D. W. ROBINSON, M.A. (Utah) Instr., Case Institute of Technology.

J. D. ROBINSON, Student, Siena College.

H. L. ROLF, M.A. (Vanderbilt) Instr., Vanderbilt University.

R. E. K. ROURKE, A.M. (Harvard) Head, Department of Mathematics, Kent School, Conn.

L. A. RUBEL, Ph.D. (Wisconsin) Instr., Cornell University.

H. J. SCHNELKER, Student, Purdue University, Fort Wayne, Ind.

P. A. SCHWEITZER, Student, Holy Cross College.

T. R. SHERROW, B.S. (Tulsa) Senior Clerk, Sinclair Oil & Gas Co., Tulsa, Okla.

P. C. SHIELDS, Student, Colorado College.

W. A. SILLARS, B.A. (Pennsylvania S.U.) Asst., Pennsylvania State University.

RONALD SIRCY, Student, Tennessee Polytechnic Institute.

SISTER MARY ALBERTA, A.M. (Loyola, Ill.) Head, Department of Mathematics, Central Catholic High School, Toledo, Ohio.

SISTER MARY CAROLINE WALLACE, M.S. (Marquette) Instr., St. Xavier College, Chicago, Ill.

SISTER MARY MATTHEW DONOVAN, Ed.D. (Houston) Asst. Professor, Sacred Heart Dominican College, Houston, Texas.

G. A. SOPP, M.S. in M.E. (Southern California) Res. Engr., Rocketdyne, Woodland Hills, Calif.

D. A. STEINBERG, B.A. (Kansas) Math., University of California Radiation Lab., Livermore, Calif.

E. J. STEWART, Ph.D. (Rice) Asso. Professor, U. S. Naval Postgraduate School.

RALPH SURASKY, B.S. (South Carolina) Grad. Asst., University of South Carolina.

A. M. SUTTON, B.A. (Harvard) Actuarial Student, Equitable Life Assurance Society, New York, N. Y.

PATRICIA A. TULLEY, A.B. (Vassar) Teacher, Kent School, Englewood, Colo.

P. J. USAVAGE, B.S. (Alliance) Instr., Chester High School, Pa.

R. J. WAGNER, B.S. (Lebanon Valley) Teaching Asst., Rutgers University.

J. H. WALTER, Ph.D. (Michigan) Instr., University of Washington.

R. L. WEBSTER, B.A. (Harvard) Field Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

CHARLES WEISS, Student, Siena College.

ROSEMARY F. WILEY, A.M. (Chicago) Instr., University of Illinois, Chicago.

WALTER L. WILLIAMS, Student, University of Chicago.

G. L. WILSON, Student, University of California.

J. B. WILSON, M.S. (Cornell) Grad. Asst., University of Florida.

LT. H. A. YOUNG, A.B. (Harvard) Asst. Consultant Meteorologist, Air Research and Development Command, Baltimore, Md.

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The eighteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at Stanford University, Stanford, on January 14, 1956. Professor C. C. Torrance, Chairman of the Section, presided at both the morning and the afternoon sessions.

There were 105 persons in attendance at the meeting including 65 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor H. L. Alder, University of California, Davis; Vice-Chairman, Professor Harley Flanders, University of California, Berkeley; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College, Fresno.

By invitation of the section, Professor Ivan Niven, University of Oregon and University of California, delivered an address at the morning session entitled *Irrational Numbers*. Abstract of this address follows:

This expository paper treated two parts of the subject: (1) a review of the classification, rational and irrational, algebraic and transcendental, of the commonly occurring numbers of mathematics, culminating in the generalized Lindemann theorem and the Gelfond-Schneider theorem; (2) the approximation of irrational numbers by rationals, concluding with the recent result of K. F. Roth on the best possible approximation of algebraic numbers by rationals. A new proof of the irrationality of the trigonometric functions for non-zero rational values of the arguments was given. The talk consisted of selections from the speaker's forthcoming Carus Monograph on the subject.

The following papers were presented:

1. *A curious trigonometric identity*, by Professor Raphael M. Robinson, University of California, Berkeley.

The formula $|\sin(x+iy)| = |\sin x + \sin iy|$ is valid for real x and y. This functional equation essentially characterizes the sine function.

2. *Calculations with random digits*, by Professor C. L. Perry, United States Naval Postgraduate School, Monterey.

Procedures for obtaining estimates to integrals by calculations using sequences of random digits were reviewed. This process, called the Monte Carlo method, is described in Applied Mathematics Series No. 12 (National Bureau of Standards), and in textbooks such as Householder's *Principles of Numerical Methods*, McGraw-Hill, 1953. Derivation of the expected value and variance for the Monte Carlo estimate for the value of a double integral was used for illustration of the method.

3. *Closed level curves*, by Professor H. A. Arnold, University of California, Davis.

In teaching complex variable, the level curves, $f(z)=k$, are usually assumed to be closed, although counter-examples such as $|f(z)| = |e^z| = e^x = k$ (a straight line here) are common. It is pointed out that if $f(z)$ is continuous on the unit disc $|z| \leqq 1$ and holomorphic on its interior, $f(z)=0$ if and only if $z=0$; and if $f'(0) \neq 0$, then $|f(z)| = k$ is a simple closed analytic curve for z in $|z| \leqq 1$ and $0 < k < \min f(z)$ on $|z| = 1$. It is pointed out that by considering $f(z)$ merely a local homeomorphism, similar conclusions may be drawn.

4. *A remark on the axiom of continuity*, by Professor C. M. Fulton, University of California, Davis.

Hilbert's plane axioms of incidence and the axioms of order are assumed as well as the postulate of Dedekind. It is shown that there exists at least one line through a given point which does not intersect a given line. If the axiom of continuity is omitted, all lines may intersect.

5. *Psychological factors in attitudes towards mathematics*, by Professors D. A. Norton and T. M. Poffenberger, University of California, Davis.

During the summer of 1955 at the University of California at Davis a preliminary survey was made of the factors contributing to an individual's attitude towards mathematics. The principal factors tentatively identified are the following: parents determine the initial attitudes of their children towards mathematics; the parents' expectation of the child's performance influences his achievement; the mathematics teachers have a strong effect on his further development.

6. *Formation of probability concepts*, by Professor C. C. Torrance, United States Naval Postgraduate School, Monterey.

This paper presented arguments for the following issues. (1) The basic addition and multiplication laws should be given *ab initio* for the general cases of dependent events and overlapping classifications. (Tests show that, even among trained mathematicians and scientists, the special forms of these laws are habitually misapplied to general situations because of the usual overemphasis of these special forms.) (2) The usual simple proofs of the basic laws can easily be made rigorous and general by introducing the concept of a *representative* sample of trials. (3) Standard functional notation should be used for conditional probabilities instead of the classical notation $Pr(E|F)$. This minimizes the difficulty of the transition from summation to integration.

7. *Some heuristic opportunities in the teaching of analytic geometry*, by Mr. C. M. Larsen, San Jose State College.

The point of view was advanced that heuristic teaching (as illustrated by G. Pólya in his book *How To Solve It*, Princeton University Press, Princeton, N. J., 1945) is desirable, but not easy to work into typically crowded courses. A few instances were then discussed where some major heuristic points can be conveniently worked into analytic geometry and calculus courses. These points included symmetry, generalization and specialization, and, especially, the sequence of leading questions: Do you know a related problem? Can you use its method? Can you use its result?

8. *Regular polygons with an odd number of sides*, by Professor V. E. Hoggatt, San Jose State College.

The question, as to whether or not three diagonals of a regular odd-sided polygon can be concurrent except at the vertices, was answered in the negative for $2n+1$ where its prime factors are greater than 11. This is a partial solution to problem 4444 of the advanced problem section of the MONTHLY (page 422, vol. 58, no. 6, June–July, 1952).

9. *The solution of simultaneous algebraic equations on an automatic digital computer*, by Professor B. J. Lockhart, United States Naval Postgraduate School, Monterey.

The solution of a system of four fourth order polynomial equations by various iterative methods on a CRC 102A digital computer is described. The practical requirement of obtaining convergence in an economically feasible length of time is the basic problem considered. The techniques described are well known as the gradient, composite gradient, successive searching and inversion of the Jacobian matrix methods.

10. *Unknowns, variables, and indeterminates*, by Professor Leonard Tornheim, University of California, Berkeley.

In the theory of polynomials and rational functions the symbol "x" plays three roles—unknown, variable, and indeterminate (or transcendental). Confusion occurs when these roles are not distinguished in topics such as cancellation and the remainder theorem.

C. D. OLDS, *Secretary*

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-third annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at McNeese State College, Lake Charles, Louisiana, on February 17 and 18, 1956. The Friday afternoon meeting was held in two concurrent sessions. Dean W. H. Bradford and Professor B. O. VanHook presided. Professor T. L. Reynolds presided at the evening session and at the Saturday morning session.

There were 124 persons registered including 51 members of the Association.

The following officers were elected for the coming year: Chairman, Dean W. H. Bradford, McNeese State College; Vice-Chairman for Mississippi, Professor R. D. Sheffield, Jr., University of Mississippi; Vice-Chairman for Louisiana, Professor Elsie T. Church, Northwestern State College; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

The invited speaker for the meeting was Professor J. C. Eaves of the University of Kentucky. His address on Friday evening was entitled "One, Two, Three, . . . ", and at the Saturday morning session, "A Matrix and a Number."

The following papers were presented:

1. *Report on the Summer Institute for Teachers of Collegiate Mathematics*, by Professor H. F. Schroeder, Louisiana Polytechnic Institute.

The speaker discussed the programs of the Summer Institutes for Teachers of Collegiate Mathematics and Teachers of Secondary Mathematics which were held at Oklahoma Agricultural and Mechanical College on June 13–July 23, 1955. These programs were sponsored by the National Science Foundation.

2. *Freshman mathematics programs*, by Professor F. A. Rickey, Louisiana State University.

The speaker presented a summary of the problems facing planners of courses in freshman mathematics with illustrations from the program at Louisiana State University.

3. *Mathematics in general education*, by Professor W. H. Spragens, University of Mississippi.

The speaker, who spent the year 1954–55 at the College of the University of Chicago as a Carnegie Intern in General Education, described the mathematics course in the curriculum there.

4. *Cubic inversion*, by Professor Elsie T. Church, Northwestern Louisiana College.

In this paper the speaker developed inversion using a cubic as the base curve and discussed the following: the inverses of various curves and points; illustrations of the effects of cubic inversion on the multiple points of a n-ic; invariants under the transformation; comparisons between cubic inversion and quadric inversion.

5. *The names of the curve of Agnesi*, by Reverend T. F. Mulcrone, St. Charles College.

The speaker quoted the pertinent passages from G. Grandi's note in the *Opere di Galileo Galilei* and his *Quadratura Circuli*, and M. G. Agnesi's *Instituzioni Analitiche*, and showed how the locus associated with the expression 'sinui verso'—the projection of a (circular) arc upon the diameter of the circle,—came eventually to be so badly designated, from mathematical and

etymological considerations, by the term "Witch" of Agnesi. It was noted also that the term "Witch" was a singularly inept one, because of its sinister implications, to predicate anything intended to honor a most honorable woman—the noted linguist, philosopher, mathematics writer, and hospital nun, Maria Gaetana Agnesi (1718–1799). The speaker recommended that English and American authors adopt the practice of the French (courbe d'Agnesi) and the Germans (Agnesische Kurve), writing simply "the curve of Agnesi."

6. *A construction relevant to the catenary and tractrix*, by Professor B. Ernest Mitchell, University of Mississippi.

The catenary and the tractrix are transcendental curves, but due to the fact that some of their elements are related algebraically certain constructions relative to the curves can be made, namely to draw their tangents at a given point and to find their radii of curvature there.

7. *Circles and the normal line: spheres and the normal plane*, by Professor V. B. Temple, Louisiana College.

The rectangular equations of two circles (spheres), each of radius r tangent to each other, and each tangent, one externally and the other internally, to the circle (sphere) of radius a with center at the origin, are respectively,

(1) $[x - (a \pm r)\cos\theta]^2 + [y - (a \pm r)\sin\theta]^2 = r^2$, (rolling circles),

(2) $[x - (a \pm r)\cos\alpha]^2 + [y - (a \pm r)\cos\beta]^2 + [z - (a \pm r)\cos\gamma]^2 = r^2$, (rolling spheres),

where θ and α, β, γ are the direction angles of the line of centers. Subtracting equations (1) and (2) independently, we have

$$x\cos\theta + y\sin\theta = a, \quad \text{and} \quad x\cos\alpha + y\cos\beta + z\cos\gamma = a,$$

the normal line and the normal plane respectively.

8. *Points symmetric with respect to a fixed line in a plane*, by Professor J. T. Humphrey, Grambling College.

The case considered discusses the equations of a particular transformation. The properties of the transformation are exhibited as well as their application to certain curves. The discussion has been limited to cases most commonly met and to those which prove interesting.

The general line, the lines $x=0$, $y=0$, $x=y$, and $x=-y$ are chosen as axes of reflexion. Here we note that a point reflected through $y=0(Tx)$ then $x=0(Ty)$ is identical to its reflexion through the origin (T_0), *i.e.*, $T_yT_x=T_0$, similarly $T_xT_y=T_0$, $T_0T_y=T_x$, $T_yT_0=T_x$, $T_0T_x=T_y$, and $T_xT_0=T_y$.

The order in which the various topics are discussed is as follows: (1) definition; (2) co-ordinates of point pairs associated with the general case; (3) the lines $y=-x$, $y=x$, $y=0$, and $x=0$; (4) theorem: straight lines subjected to the transformation remain straight lines; (5) theorem: conics subjected to T_r remain invariant with respect to type; (6) theorem: if a curve in the plane is subjected to any two of the transformations T_x, T_y, T_0 the result will have the effect of the third in each case.

9. *Solution of heat equation on the electronic analog computer*, by Professor Margaret M. LaSalle, Southwestern Louisiana Institute.

The heat equation for a homogeneous medium of two dimensions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is solved by use of the electronic analog computer using the difference method. The error introduced by such a method is computed in a problem with nine stations. Considering symmetry, the set-up for the analog computer involving six amplifiers is given and the physical significance of the graphs so obtained is discussed. This problem was solved during a course at University of Michigan.

10. *A Pascal triangle for the coefficients of a polynomial*, by Professor R. A. Miller, University of Mississippi.

If $S(r_i, j, k)$ denotes the sum of the products, taken j at a time, of the first k elements of the set of n elements, r_i, with $S(r_i, 0, k)=1$ and $S(r_i, j, k)=0$, for $k<j$, it is observed that a simple recursion formula holds: $S(r_i, j, k)=S(r_i, j, k-1)+rkS(r_i, j-1, k-1)$ for $j>1$, $k>1$. If $S(r_i, j, k)$ is the element in the jth row and kth column of a matrix and the elements of the jth row are multiplied by $(-1)^j$, the resulting elements of the kth column will be the coefficients of a polynomial with zeros $r_1, r_2, \cdots, r_k$. A Pascal triangle for the binomial coefficients results when the zeros are all equal to -1.

11. *On the matrix equation* $PX=Q$, by Professor T. J. Pignani, Loyola University.

This paper discusses conditions that must be imposed on P and Q so that the solution, $X=S$, of $PX=Q$ be a non-singular matrix, where P and Q are matrices with m rows and n columns, whose elements are real numbers. The paper contains preliminary results on the matrix equation $AYB=C$. Further it shows when certain conditions, to obtain the above results, are relaxed, then S is of rank not greater than a number determined by the rank of the coefficient matrices. The paper points out the application of the above results to interface problems.

12. *A formal derivation of finite sums*, by Professor S. R. Knox, Millsaps College.

By using the definition of the definite integral,

$$\int_a^b f(x)dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(\xi_i)(X_i - X_{i-1}),$$

a formal method of obtaining formulas for certain finite sums is demonstrated.

13. *The rise of a liquid*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

The paper discussed certain functions derived from the rotation of a liquid in a smooth container. A general type container was first considered. The surface of such a container was obtained by revolving the curve $x^2=f(y)$ about the y axis. In this function $x^2-f(0)=0$ was assumed to have two real, non-zero roots. Further, it was assumed that $dy/dx>0$ for all values of y. A fundamental relationship was derived which gave the angular velocity of the rotative fluid as a function of the initial fluid depth, σ, the rise, λ, caused by a given angular velocity, and the function $f(\lambda)$. Then several conclusions were drawn from this fundamental equation.

14. *Some probabilities*, by Professor T. A. Bickerstaff, University of Mississippi.

From an infinite universe $\{x\}$ with continuous density function $f(x)$, with no parameters known, a sample (x_i), $i=1, 2, \cdots, n$ was taken. To the order statistics, x_i, the probability integral transform $p_a=\int_{-\infty}^{x_a} f(x)dx$ was applied. With order preserved, the well-known Beta distribution of the order statistic, p_a, was pointed out and the super Beta function of the joint distribution of two order statistics was derived. These yielded a variety of probabilities and confidence limits for order parameters. Methods of evaluating the incomplete integrals were indicated.

Z. L. LOFLIN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Wash ington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Concordia College, Moorhead, October 6, 1956.
MISSOURI
NEBRASKA
NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 26, 1956.
PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.
PHILADELPHIA, November 24, 1956.
ROCKY MOUNTAIN
SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.
SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.
SOUTHWESTERN
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No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.
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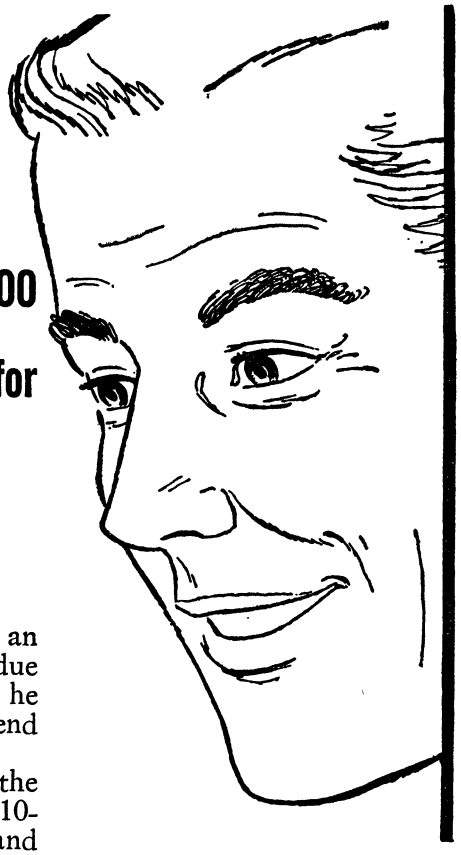
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SYSTEMS OF DISTINCT REPRESENTATIVES AND LINEAR PROGRAMMING*

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1. Introduction. In a recent discussion [5] in this MONTHLY of extensions and applications of P. Hall's theorem on distinct representatives of subsets, H. B. Mann and H. J. Ryser presented a sufficient condition that a system of representatives include a prescribed set of elements ([5], Theorems 2.4 and 2.5). The purposes of this note are

(i) to replace the Mann-Ryser condition by a necessary and sufficient condition, and

(ii) to exhibit in the proof two applications of the theory of linear inequalities—specifically, the duality theorem of linear programming [3] and a property of the solutions to transportation problems [4]—to a combinatorial problem.

Although the interested reader can easily concoct a simple, strictly combinatorial proof modeled on the demonstration of Hall's theorem given in [2], it seems worthwhile to call attention to this new method of establishing certain types of combinatorial results. Several theorems that fall into this pattern are treated by various authors in [6].

2. Statement. Let $\mathcal{S} = \{S_1, \cdots, S_n\}$ be a finite collection of subsets of a given set S. A set $R = \{a_1, \cdots, a_n\}$ of n distinct elements of S such that $a_j \in S_j$, for $j = 1, \cdots, n$, is called a *system of distinct representatives for* $\mathcal{S}$ (abbreviated SDR). Hall's theorem asserts that the following condition is necessary and sufficient for the existence of an SDR:

(A) For each $q = 1, \cdots, n$, the union of any selection of q sets from $\mathcal{S}$ contains at least q distinct elements of S.

Mann and Ryser asked for further restrictions on $\mathcal{S}$ that would insure the existence of an SDR containing l prescribed elements $E = \{e_1, \cdots, e_l\}$—called *marginal elements*. If $t > 0$ is the largest number of marginal elements in any S_j, they found that the following condition, combined with (A), suffices for such an SDR:

(B) Every marginal element appears at least t times among the S_j.

To see that this condition is not necessary, consider $S = \{1, 2\}$, $S_1 = \{1, 2\}$, $S_2 = \{2\}$, $E = \{1, 2\}$. Then $t = 2$, (B) is not satisfied, but $R = \{1, 2\}$ is an SDR containing E. Thus (B) is too restrictive; we shall introduce the weaker condition:

(C) For each $p = 1, \cdots, l$ and any selection $E' = \{e_{i_1}, \cdots, e_{i_p}\}$ of p marginal elements, the number of sets S_j that meet E' is at least p.

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Condition (C) is clearly necessary: if there is an SDR containing E, by associating to any p marginal elements the sets that they represent, we have immediately the p sets S_j required. It is also obvious that (B) implies (C). Indeed, suppose that some selection E' of p marginal elements meets but p' of the S_j, with $p' < p$. Then the largest possible number of occurrences of marginal elements in these S_j is $p't$. By condition (B), the smallest possible number of such occurrences is pt. Hence $p't \geqq pt$, contradicting $p' < p$.

THEOREM. *Conditions* (A) *and* (C) *are necessary and sufficient for the existence of an SDR containing the prescribed marginal elements* E.

3. Proof of sufficiency. Note that only finite sets S need be considered by restricting S to the finite set of elements needed to verify (A) and (C). List the elements of S as $a_1, \cdots, a_m$ with $a_1 = e_1, \cdots, a_l = e_l$; clearly, $m \geqq n$ by (A) and $n \geqq l$ by (C). It is then convenient to regard the collection $\mathfrak{S}$ as given by an m by n incidence matrix:

$$C = (c_{ij}) \quad \text{with} \quad c_{ij} = \begin{cases} 1 \text{ if } a_i \in S_j \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

We define an (l, m, n) *permutation matrix* (for $l \leqq n \leqq m$) to be an m by n matrix of 0's and 1's with

(2) exactly one 1 in each of the first l rows;
(3) no more than one 1 in each of the last $m - l$ rows;
(4) exactly one 1 in each of the n columns.

Observe that each SDR for $\mathfrak{S}$ containing E can be identified with an (l, m, n) permutation matrix $P = (p_{ij})$ such that $p_{ij} = 1$ implies $c_{ij} = 1$ for all i and j. Conversely, every (l, m, n) permutation matrix (p_{ij}) supplies an SDR containing E if $p_{ij} = 1$ implies $c_{ij} = 1$.

Next, consider the polyhedral convex set $\mathfrak{X}$ of all m by n matrices $X = (x_{ij})$ such that

$$x_{ij} \geqq 0 \quad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n) \tag{5}$$

$$\sum_j x_{ij} = 1 \qquad (i = 1, \cdots, l) \tag{6}$$

$$\sum_j x_{ij} \leqq 1 \qquad (i = l + 1, \cdots, m) \tag{7}$$

$$\sum_i x_{ij} = 1 \qquad (j = 1, \cdots, n). \tag{8}$$

The following lemma relates the convex set $\mathfrak{X}$ to the combinatorial problem.

LEMMA. *The linear form* $\sum_{i,j} c_{ij}x_{ij}$ *defined on* $\mathfrak{X}$ *attains its maximum at an* (l, m, n) *permutation matrix.*

Proof. We shall present two proofs of this lemma. The first depends on the fact that a linear form, defined on a compact convex set, achieves its maximum

at an extreme point of that set. (An *extreme point* is any point not expressible as the midpoint of two distinct points of the set.) Clearly, the (l, m, n) permutation matrices, containing only 0's and 1's, are extreme in $\mathfrak{X}$. To show that they are the only extreme points, we need only show that every point in $\mathfrak{X}$ is a convex combination of (l, m, n) permutation matrices. Any $X \in \mathfrak{X}$ can be completed to a square m by m matrix X' by the following rules:

$$x'_{ij} = 0 \qquad (i = 1, \cdots, l \text{ and } j = n+1, \cdots, m) \tag{9}$$

$$x'_{ij} = \frac{1 - \sum_{j=1}^{n} x_{ij}}{m - n} \qquad (i = l+1, \cdots, m \text{ and } j = n+1, \cdots, m) \tag{10}$$

$$x'_{ij} = x_{ij} \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n). \tag{11}$$

The matrix X' is *doubly-stochastic*: non-negative entries with row and column sums equal to one. As Birkhoff [7] has observed, a theorem* of König [1] (which is, curiously enough, equivalent to Hall's original theorem) implies that X' is the convex combination of *permutation matrices*: matrices composed of 0's and 1's, with exactly one 1 in each row and column. Since X' has only 0's in the first l rows and last $m - n$ columns, the permutation matrices used must also share this property. As a result, the first n columns are (l, m, n) permutation matrices and the desired expression for X has been found.

For a second proof of the lemma, let $\overline{C} = (\bar{c}_{ij})$ be the m by $n+1$ matrix defined by

$$\bar{c}_{ij} = c_{ij} \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n) \tag{1'}$$

$$\bar{c}_{i,n+1} = w \qquad (i = 1, \cdots, l), \tag{12}$$

where w is a negative number large in absolute value,

$$\bar{c}_{i,n+1} = 0 \qquad (i = l+1, \cdots, m). \tag{13}$$

Let $\overline{\mathfrak{X}}$ be the set of matrices $\overline{X} = (\bar{x}_{ij})$ satisfying

$$\bar{x}_{ij} \geqq 0 \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n+1) \tag{5'}$$

$$\sum_i \bar{x}_{ij} = 1 \qquad (i = 1, \cdots, m) \tag{6'}$$

$$\sum_i \bar{x}_{ij} = 1 \qquad (j = 1, \cdots, n) \tag{7'}$$

$$\sum_i \bar{x}_{i,n+1} = m - n. \tag{7''}$$

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* Namely, every term of the expansion of the determinant of an n by n matrix vanishes if and only if the common part of some set of p rows and $n+1-p$ columns contains only zeros.

Consider the problem of maximizing the linear form

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{c}_{ij}\bar{x}_{ij} \tag{14}$$

on the set $\bar{\mathfrak{X}}$. It is clear from (1'), (12), and (13) that the first n columns of the maximizing $\bar{X}$ will be in $\bar{\mathfrak{X}}$ and will maximize $\sum_{i,j} c_{ij}x_{ij}$. However, the problem of maximizing (14) on the set $\bar{\mathfrak{X}}$ is a special case of the so-called "transportation problem" prominent in linear programming, and it is known [4] that there is always a solution, every entry of which is an integer. Therefore, (5'), (6'), and (7') imply that every entry of the solution is either 0 or 1. Hence the first n columns will be an (l, m, n) permutation matrix. This completes the second proof of the lemma.

Now we are in a position to apply the duality theorem of linear programming; a complete discussion of this theorem can be found in [3]. The version needed here relates the following two problems built out of the same data, namely, an M by N matrix $A=(a_{\mu\nu})$, an M-vector $B=(b_\mu)$ and an N-vector $C=(c_\nu)$:

Maximum Problem	*Minimum Problem*	
Maximize $\sum c_\nu\xi_\nu$ for	Minimize $\sum b_\mu\eta_\mu$ for	
$\xi_\nu \geqq 0$	$\sum \eta_\mu a_{\mu\nu} \geqq c_\nu$	$(\nu = 1, \cdots, N)$
$\sum a_{\mu\nu}\xi_\nu \begin{cases} \leqq b_\mu \text{ or} \\ = b_\mu \end{cases}$	$\begin{cases} \eta_\mu \geqq 0 \text{ or} \\ \eta_\mu \text{ unrestricted.} \end{cases}$	$(\mu = 1, \cdots, M)$

The duality theorem then asserts that, if the maximum problem is solvable, then the minimum problem is also solvable and the constrained maximum of $\sum c_\nu\xi_\nu$ equals the constrained minimum of $\sum b_\mu\eta_\mu$.

Resuming the proof of the theorem, the maximum of $\sum_{i,j} c_{ij}x_{ij}$ on $\mathfrak{X}$ (that is, subject to (5)–(8)) is some integer $t \leqq n$. The duality theorem establishes that the minimum of

$$\sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j \tag{15}$$

for

$$u_i + v_j \geqq c_{ij} \qquad (i = 1, \cdots, m \text{ and } j = 1, \cdots, n) \tag{16}$$

and

$$u_i \geqq 0 \qquad (i = l+1, \cdots, m) \tag{17}$$

is also equal to $t \leqq n$.

Let $\bar{u}_1, \cdots, \bar{u}_m, \bar{v}_1, \cdots, \bar{v}_n$ yield a minimum for (15) and have the property that among all solutions it contains the smallest number of non-integral $\bar{u}_i$ and $\bar{v}_j$. We shall show that this implies that the $\bar{u}_i$ and $\bar{v}_j$ are *all* integers. If some are

not integers, let us say that the number of non-integral $\bar{u}_i$ is not less than the number of non-integral $\bar{v}_j$ (if the reverse is true, the argument is changed in an obvious way). Let $\delta = \min(\bar{u}_i - [\bar{u}_i])$ for non-integral $\bar{u}_i$ and define new values by:

$$\bar{u}_i' = \begin{cases} \bar{u}_i & \text{if } \bar{u}_i \text{ is an integer,} \\ \bar{u}_i - \delta & \text{otherwise,} \end{cases} \tag{18}$$

$$\bar{v}_j' = \begin{cases} \bar{v}_j & \text{if } \bar{v}_j \text{ is an integer,} \\ \bar{v}_j + \delta & \text{otherwise.} \end{cases} \tag{19}$$

Clearly $\sum \bar{u}_i' + \sum \bar{v}_j' \leqq \sum \bar{u}_i + \sum \bar{v}_j$. Furthermore, the new values satisfy (16) and (17); the only possibility for violating (16) occurs for a pair of indices (i, j) such that $\bar{v}_j$ is an integer and $\bar{u}_i$ is not. Then

$$\bar{u}_i' + \bar{v}_j' = \bar{u}_i - \delta + \bar{v}_j \geqq [\bar{u}_i] + \bar{v}_j \geqq c_{ij}. \tag{20}$$

Thus, the new values satisfy (16) and (17), do not yield a larger value for (15), but contain fewer non-integral values. This contradiction proves that the chosen $\bar{u}_i$ and $\bar{v}_j$ are all integers.

Next, we shall show that one may take all of the $\bar{u}_i$ and $\bar{v}_j$ to be non-negative. Assume first that some $\bar{v}_j$ is negative. Then, by (16) and (1), it follows that all the $\bar{u}_i$ are positive. Hence, if all of the $\bar{v}_j$ are increased by one and all of the $\bar{u}_i$ are decreased by one, then (16) and (17) will still hold and the value of (15) will not be increased since $n \leqq m$. This process is continued until all of the $\bar{v}_j$ are non-negative.

Assume next that $\min_i \bar{u}_i = -r \leqq -2$. Then, by (16), $\bar{v}_j \geqq r \geqq 2$ for $j = 1, \cdots, n$. If we increase all of the $\bar{u}_i$ equal to $-r$ by one, and decrease all of the $\bar{v}_j$ by one, then (16) and (17) will still hold and the value of (15) will not be increased since $l \leqq n$. Hence any negative $\bar{u}_i$ can be taken equal to -1. Let there be p of these, say $\bar{u}_{i_1} = \cdots = \bar{u}_{i_p} = -1$. Since the $\bar{u}_i$ and $\bar{v}_j$ minimize (15), it follows that a certain number of the $\bar{v}_j$ are equal to 2, and the remainder are 1. Precisely, $\bar{v}_j = 2$ if and only if S_j contains one of the elements $e_{i_1}, \cdots, e_{i_p}$ and there are at least p such by condition (C). If we decrease these $\bar{v}_j$ by one and increase $\bar{u}_{i_1}, \cdots, \bar{u}_{i_p}$ by one, then (16) and (17) will still hold and the value of (15) will not be increased.

Hence we may assume that all of the $\bar{u}_i$ and $\bar{v}_j$ are non-negative integers which, by (16) and (1), are either 0 or 1. Let $\bar{v}_{j_1} = \cdots = \bar{v}_{j_q} = 1$ and the remaining $\bar{v}_j = 0$. Since the $\bar{u}_i$ and $\bar{v}_j$ minimize (15), it follows that $\bar{u}_i = 1$ if and only if a_i is an element of some S_j for which $\bar{v}_j = 0$. By condition (A) there are at least $n - q$ such elements and

$$t = \sum_i \bar{u}_i + \sum_j \bar{v}_j \geqq (n - q) + q = n. \tag{21}$$

As previously remarked, the value t of the maximum program is not greater

than n. Hence it is exactly n and any (l, m, n) permutation matrix that achieves the maximum defines an SDR containing E.

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A GENERAL EDUCATION COURSE IN PURE MATHEMATICS

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A description follows ot a course given at Harvard in the spring semester of 1955; the course was sponsored jointly by the Mathematics Department and the Committee on General Education as an elective course not normally open to freshmen. The course appeared to be quite successful. There has been considerable interest in such courses in recent years and although none of the material presented was new to such efforts, there seemed to be enough that was distinctive in arrangement and emphasis to warrant a brief report.*

Enrollment was limited generally, though not inflexibly, to students with an honors average. The only prerequisite was a high school mathematics background that included synthetic geometry. Students with college analysis courses were also admitted, but those with courses in modern algebra or projective geometry were not. The course had 35 regular students and about 15 auditors. About half had had no college mathematics. More than half were concentrators in Humanities or Social Sciences. About half were seniors, the rest juniors and sophomores.

The course was meant to explain, within obvious limitations, something of

* The course was first offered in 1954. It is in a continuing state of development in both form and content. It was given in 1955 by the author and in the preceding year by Professor A. M. Gleason, to whom the author is indebted for valuable suggestions.

the spirit and method of modern pure mathematics. In the arrangement of material for the course, a number of basic preliminary problems were faced. The first was that of deciding how much of the course should be presentation of axiomatic method as method and how much should be communication of substantive mathematical subject matter. It was felt that method was more important for the course, but it was also recognized that standards of significance and beauty in mathematics are largely determined by the extant body of subject matter. The problem was resolved by largely devoting lectures to the axiomatic method, while adopting Courant and Robbins' *What is Mathematics?* as the principal reading text. Each student also wrote a term paper reporting on some substantive topic. A second problem was that of choosing between a qualitative survey and a careful, close exposition of material. This was resolved by varying the lecture treatment from topic to topic; some topics were presented in careful detail, others in looser historical summary. A third problem was that of arranging the course so that the student could discover good reasons for being interested in pure mathematics. Two approaches seemed possible: emphasis could be allowed to rest on the intrinsic intellectual and aesthetic interest of abstract mathematics; or an attempt could be made to show the student how the tools of modern applied mathematics are not naively visualizable, and require an understanding of rigor and the axiomatic method. While these approaches were not incompatible, it seemed wisest, in view of purposes and limited time, to choose topics with the first of these approaches in the foreground. The existence of these three problems and the overall limitations of the course were emphasized to the students from the start. It was pointed out that they could examine only a very small part of an active and changing discipline, and they were urged to remain flexible and uncommitted in their initial attitudes toward it.

The course occupied 37 one hour lectures. There was a mid-term hour test and a three hour final examination. There were weekly homework exercises and occasional more difficult extra-credit problems for students of more advanced background or ability. At intervals through the term, some of the better logical "brain-teasers" were also given as unofficial problems. A term paper was required in the form of a report on outside reading of the student's choice. The student conferred with the teacher about this choice, and permissible topics depended on the student's background. Several students with no analysis undertook to learn elementary calculus from the later chapters of Courant and Robbins and did so with considerable success. Others ranged in their subjects from the real number system to the Theory of Games and elementary theorems in Artin's work on braids.

Courant and Robbins was adopted as the principal text. While parts of it are rather sophisticated for a course at this level, and while the net official reading assignment was consequently small, it was nevertheless felt to be admirably suited to student browsing. A list of suggested books for supplementary reading was given out at the beginning of the term. (This is given below.) It was thought

important to give the student definite warning and guidance in this regard as there is enormous variation of quality and purpose among the usual library volumes in the introductory and "semi-popular" areas.

Before a brief outline of the lecture topics is given, several omissions deserve special mention. No logical symbolism, as such, was presented. Matters of normative linguistic convention quickly arose anyway and were dealt with as such. It was felt that this was more efficient in the time available and that a separate treatment of logic might be more distraction than help in achieving the purposes of the lectures. Much can, of course, be said against this view. Conventions such as vacuous implication, the empty set, and existential import, had to be made clear, and the logical quantifiers and "if . . . then . . . " arrows were occasionally used as abbreviations; nevertheless natural language remained the basic logical medium throughout the term. No rigorous treatment was given of the real number system. Examples from algebra and geometry seemed better suited to introductory illustrations of rigorous argument. Furthermore, in an initial exposure to the axiomatic method, a student's partial knowledge of the real numbers may somewhat obscure for him the necessity and productivity of the axiomatic approach. No topics from analysis were given. This was partly because of varied background among the students and partly because of the relatively greater logical complexity of elementary notions of analysis. The central position of analysis in the body of mathematics was strongly emphasized to the students, and supplementary reading was recommended. The omitted areas of logic and analysis were urged as possible sources of material for term papers, and a number of students chose to work in them.

An outline of the lecture subject matter follows.

I. The first two lectures were devoted to presenting, slowly and carefully, proofs of the irrationality of $\sqrt{2}$, the infinitude of the primes, and the unique prime factorization of the integers. The proofs were based on "known" properties of the integers rather than on explicitly listed axiomatic assumptions. The theorems served two purposes: 1) they showed the students the degree of logical facility that would be expected of them during the term (as was intended, this prompted several to transfer out of the course), and 2) the theorems gave vivid examples of the productivity of rigorous proof, even in dealing with familiar and supposedly well understood objects. This was most valuable, as many of the students seemed to have come with the feeling that mathematics is a matter of ingenious definition followed by relatively straightforward application and that proof is a kind of scholastic exercise whereby one "justifies" what one already sees to be true. The theorems also served as an illustration of the subtleties of indirect proof. The proof of unique factorization was the elegant and seldom used proof that appears in Courant and Robbins. No preliminary divisibility lemma is necessary, and mathematical induction is disguised and compressed into an initial choice of the least integer not possessing unique factorization.

II. One lecture was then given over to a qualitative preview discussion of the axiomatic method. It seemed worthwhile to expose the students to such a

preliminary summary, even though, as was announced to them, many were expected not to follow the lecture entirely.

III. Several lectures were then spent on a critique of the proofs of Euclidean geometry as based on Euclid's original axioms. Emphasis was placed on the grosser tacit assumptions rather than on the finer logical inadequacies. The more celebrated gaps were pointed out and some of the traditional "false" proofs ("every triangle is isosceles") were also given. (We remark that a good example of a tacit assumption occurs in Euclid I, 1, the construction of an equilateral triangle on a given line segment. Here Euclid assumes that the arcs drawn from each end of the segment will intersect.)

IV. Axioms for synthetic finite affine geometry were then presented as the first major topic of the term. This material is well known as a highly successful pedagogical illustration of axiomatic development. It goes back to Young, *Lectures on Fundamental Concepts of Algebra and Geometry* (1911) and has appeared in a number of more recent texts. It well exhibits the relation between a theory and its specific models and the possibility—new to many of the students—of having a non-categorical theory. Once the existence of the various non-intuitive models is understood, the necessity for a careful axiomatic process becomes clear. The lectures presented the material slowly, first using nonsense words for "point" and "line" and then using those words themselves with the observation that they were to function like the nonsense words. A particular virtue of this material is that theorems of considerable generality can soon be reached—from a proof that every two lines have the same number of points (where the notion of one-one correspondence must be used) to a sequence of non-trivial structure theorems (every finite model has n^2 points, n^2+n lines, *etc.*). The structure theorems were presented as such to the student and it was pointed out that they provided a good example in microcosm of much modern mathematical activity. In addition, a number of unsolved problems could be stated (existence of a 10 point line, *etc.*), and it could be remarked that there was current research going on in the area. The development was also used to provide clear illustrations of the process of definition, the use of counter-examples, and the notions of consistency, independence and incompleteness of an axiom system, (the use of finite geometry in this respect is well described in Wilder, *Introduction to the Foundations of Mathematics*).

The material also furnished a good opportunity for discussion of such matters as: the motives that may exist for examining a particular system; the unexpected fruitfulness and generality that may appear; and the overall flexibility and control inherent in the axiomatic approach. Axioms for finite projective geometry were also presented and the corresponding structure theory derived. Final theorems relating this to the affine theory were proved. The subject of finite geometry occupied seven lectures. It was treated, for the most part, in full detail.

V. Four lectures were devoted to the Hilbert axiomatization of Euclidean geometry, with the purpose of conveying an understanding of the axioms them-

selves and then giving a brief indication of how the usual theorems grow out of them. One lecture was given on abstract relations (preparation for "between" and "congruent") and then the axioms were presented in detail along with proofs of a few of the earlier and simpler theorems. The presentation in the 1930 German edition of Hilbert, *Grundlagen der Geometrie*, was used as a source. This is somewhat better than in the 1899 edition, which appears in English translation. Material from Hilbert together with additional comment was distributed to the students in mimeograph form.

VI. Six lectures were used to give an outline of the origins and discovery of non-Euclidean geometry. Material was presented in roughly historical sequence. The mathematical treatment was qualitative to the extent that synthetic proofs at about the level of school geometry were given. Wolfe, *Non-Euclidean Geometry*, proved to be a good source and reference. Attention was directed first to propositions equivalent to the fifth postulate and then to the use of tacit assumptions or bad logic in various historical false proofs of the postulate. The elements of non-Euclidean geometry itself were then described and the proof of consistency with Euclidean geometry outlined. Two matters that were naturally emphasized were: 1) the importance of keeping theory and model distinct; and 2) the fact that much excellent mathematics can historically precede full axiomatization, since adequate conventional standards can tacitly appear among good mathematicians, (*e.g.*, by 1800 the false proofs of the fifth postulate were recognized as such).

VII. Seven lectures then gave a careful treatment of elementary material from algebra. Axioms for a commutative ring were given and various examples of rings supplied. A few of the simplest theorems were proved. Specializations to integral domain, field and Boolean ring were briefly investigated, and the notion of group was described.

VIII. The subject of transfinite cardinal numbers was considered next. While not thought to have the basic relevance of some of the other material in the course, it furnished good logical exercise, had a special metaphysical interest for many of the students and paved the way for a final discussion of foundational matters. Treatment was non-axiomatic and proceeded by a combination of illustration, logical argument and statement of results. Relevant properties of the finite cardinals were first examined and then modified appropriately to lead to the transfinite case. The relation of cardinal equivalence was made the central concept for as long as possible, and then the transfinite numbers themselves were introduced. Cantor's theorem was proved, both in decimal-diagonal and in its general form. Examples were given of various enumerable and non-enumerable sets. Addition and multiplication were defined, and the algebraic properties of the arithmetic of cardinals were discussed. The subject occupied five lectures.

IX. The subject of cardinal numbers had emphasized the central importance of the concept of *collection*. The students had also seen its basic role in their study of geometry. It was thus natural and easy to raise the question of what

constituted legitimate argument about collections. Furthermore, the counter-intuitive quality of the results of the transfinite theory had aroused student interest in finding an axiom system for that theory. The final three lectures of the term were therefore devoted to a qualitative discussion of foundations and of axiomatic set theory. The earlier proof of Cantor's theorem made possible a statement of Cantor's "greatest cardinal" paradox. Russell's paradox was stated as well, and the general history of the foundational crisis was described. The nature and purposes of logical formalization were outlined, and various formal means for avoiding the paradoxes were briefly indicated. The role of an axiomatic set theory as a common standard of argument rather than common means of expression was pointed out. It was observed (as had been done several times earlier in the term) that all of usual mathematics can be put into the symbolism of such a theory. The existence of a variety of such theories was mentioned, together with Gödel's results on incompleteness and on impossibility of a consistency proof. While these last lectures were highly qualitative and necessarily inadequate, they seemed to give the students a fairly satisfactory understanding of the main issues considered.

The course omitted much standard material, but it was felt at the conclusion that some feeling had been communicated for the blending of rigor and intuition that makes up contemporary pure mathematics, and that interested students would have a fair, if rudimentary, idea of what to expect in further exploration of the subject.

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Survey of Modern Algebra, Birkhoff and MacLane
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FOURIER ANALYSIS OF ENGINE UNBALANCE BY CONTOUR INTEGRATION*

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1. Introduction. A Fourier analysis of the dynamic unbalance of a reciprocating engine by contour integration may be of pedagogic interest in teaching the engineering applications of complex variable theory. Figure 1 gives the notation for the displacement variables s, θ and ϕ, which are the piston displacement, the crank angle, and the connecting rod angle, respectively. The ratio of the crank length r to the connecting rod length l is denoted by e. The complete analysis of dynamic unbalance follows directly from the Fourier series for s, $\dot{\phi}$, $\dot{\phi}^2$ and $\dot{s}^2$. Each of these is treated here in turn.

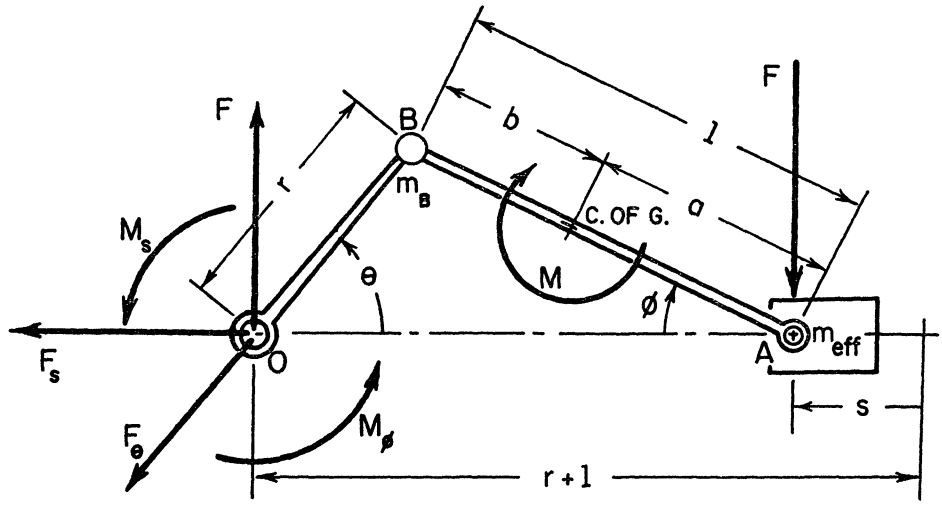


FIG. 1

2. Fourier Series for s. The piston displacement s is expressed in terms of the crank angle θ by

$$s = r[1 - \cos\theta + 1/e - \sqrt{1 - e^2\sin^2\theta}/e]. \tag{1}$$

The Fourier series for s is

$$s = r\left[a_0 - \cos\theta + \sum_{n=1}^{\infty} a_{2n}\cos 2n\theta\right]. \tag{2}$$

The value of a_0, the average value of s/r, is found by integration to be

$$a_0 = 1 + 1/e - 2E(e)/\pi e, \tag{3}$$

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where $E(e)$ is the complete elliptic integral of the second kind and modulus e. The coefficient a_{2n} for $n>0$ is

$$a_{2n} = -\int_0^{2\pi} \sqrt{1 - e^2 \sin^2\theta}(\cos\theta + j\sin\theta)^{2n} d\theta/\pi e, \tag{4}$$

using De Moivre's theorem for $\cos 2n\theta + j \sin 2n\theta$, and noting that the imaginary part of the integral vanishes since the radical is an even function of θ. Reference to [1] shows that the integral for a_{2n} may be converted to a contour integral by the imaginary exponential substitution

$$z = e^{j\theta} = \cos\theta + j\sin\theta. \tag{5}$$

The result is

$$a_{2n} = j\int_{ABC} z^{2n-2}\sqrt{(z^2 + \alpha^2)(z^2 + \alpha^{-2})}\, dz/2\pi, \tag{6}$$

where

$$\alpha = (1 - \sqrt{1 - e^2})/e, \tag{7}$$

and the path of integration is counterclockwise around the unit circle $|z| = 1$ in the complex plane, *i.e.*, the path ABC in Figure 2. In order that the integrand in (6) may be analytic in a suitable region, introduce branch cuts, indicated by heavy lines in Figure 2, along the imaginary axis from $-j\infty$ to $-j/\alpha$, from $-j\alpha$ to $+j\alpha$, and from $+j/\alpha$ to $+j\infty$, and consider the branch of the radical which is defined to have the value $\sqrt{r_1 r_2 r_3 r_4}\exp[j(\phi_1+\phi_2+\phi_3+\phi_4)/2]$, where $-\pi/2 \leqq \phi_1 < 3\pi/2$, $-\pi/2 \leqq \phi_2 < 3\pi/2$, $3\pi/2 \leqq \phi_3 < 7\pi/2$ and $\pi/2 \leqq \phi_4 < 5\pi/2$. The symbols in this definition are given in Figure 2.

By use of the branch thus defined, the path of integration may be deformed to the path ADC along the two sides of the imaginary axis and around the branch point $+j\alpha$, as shown in Figure 2. After making the substitution $z=jy$, the expression for a_{2n} becomes

$$a_{2n} = 2(-1)^n \int_0^{\alpha} y^{2n-2}\sqrt{(\alpha^2 - y^2)(\alpha^{-2} - y^2)}\, dy/\pi. \tag{8}$$

The further substitution $y=\alpha u$ gives

$$a_{2n} = 2(-1)^n \alpha^{2n-1}[\alpha^4 Y_{2n+2} - (1 + \alpha^4) Y_{2n} + Y_{2n-2}]/\pi, \tag{9}$$

where Y_{2n} is the elliptic integral

$$Y_{2n} = \int_0^1 u^{2n} du/\sqrt{(1 - u^2)(1 - \alpha^4 u^2)}. \tag{10}$$

The usual three-term recurrence formula for Y_{2n} is obtained from the differentiation

$$(d/du)u^{2n-1}\sqrt{(1-u^2)(1-\alpha^4u^2)}$$

$$(11)\qquad = \frac{(2n+1)\alpha^4u^{2n+2} - 2n(1+\alpha^4)u^{2n} + (2n-1)u^{2n-2}}{\sqrt{(1-u^2)(1-\alpha^4u^2)}}.$$

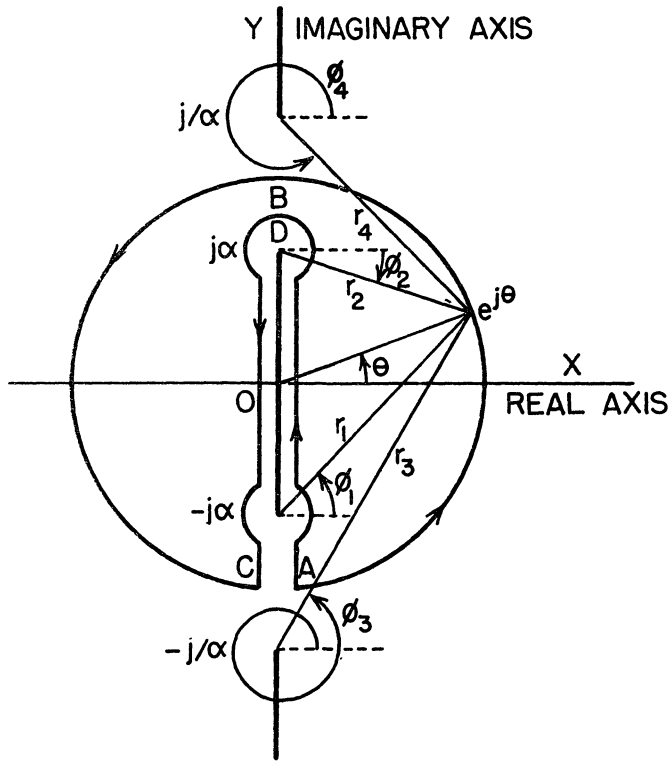


FIG. 2

Integration of (11) from $u=0$ to $u=1$ furnishes the recurrence formula

$$(12)\qquad (2n+1)\alpha^4Y_{2n+2} - 2n(1+\alpha^4)Y_{2n} + (2n-1)Y_{2n-2} = 0.$$

The initial values required in the use of (12) are

$$Y_0 = K(\alpha^2) \quad \text{and} \quad Y_2 = [K(\alpha^2) - E(\alpha^2)]/\alpha^4,$$

where $K(\alpha^2)$ and $E(\alpha^2)$ are the complete elliptic integrals of the first and second kinds, respectively, for modulus α^2. Useful tables of Y_0 and Y_2 are given in [2] on pages 85 and 83, respectively. Use of (9) and (12) makes possible the computation of a_{2n} for all values of n. Thus, for $n=1$,

$$(13)\qquad a_2 = -2[(1+\alpha^4)E(\alpha^2) - (1-\alpha^4)K(\alpha^2)]/3\alpha^3\pi.$$

Lengthy formulae could be given here for other a_{2n} coefficients, but they are calculated more conveniently by the numerical use of (9) and (12).

3. Fourier series for $\dot{\phi}$. The value of $\dot{\phi}$ is $\dot{\phi}=e\dot{\theta}\cos\theta/\sqrt{1-e^2\sin^2\theta}$, and its Fourier series is

$$\dot{\phi} = \dot{\theta}\sum_{n=0}^{\infty} b_{2n+1}\cos(2n+1)\theta. \tag{14}$$

The imaginary exponential substitution of (5) and the deformed contour ADC of Figure 2 may be used to give

$$b_{2n+1} = \int_{ADC} z^{2n}(z^2+1)dz/\pi j\sqrt{(z^2+\alpha^2)(z^2+\alpha^{-2})}. \tag{15}$$

The integrals on the infinitesimal circles surrounding the poles $z=\pm j\alpha$ may be shown to vanish. The successive substitutions $z=jy$ and $y=\alpha u$ then give

$$\begin{aligned} b_{2n+1} &= 4(-1)^n\alpha^{2n+1}\int_0^1 (u^{2n}-\alpha^2u^{2n+2})du/\pi\sqrt{(1-u^2)(1-\alpha^4u^2)} \\ &= 4(-1)^n\alpha^{2n+1}(Y_{2n}-\alpha^2Y_{2n+2})/\pi. \end{aligned} \tag{16}$$

Thus, for $n=0$,

$$b_1 = 4[E(\alpha^2)-(1-\alpha^2)K(\alpha^2)]/\pi, \tag{17}$$

and for $n=1$,

$$b_3 = 4[(\alpha^4-3\alpha^2+2)K(\alpha^2)-(2\alpha^4-3\alpha^2+2)E(\alpha^2)]/3\pi\alpha^3. \tag{18}$$

4. Fourier series for $\dot{\phi}^2$. The Fourier series for $\dot{\phi}^2=e^2\dot{\theta}^2\cos^2\theta/(1-e^2\sin^2\theta)$ is

$$\dot{\phi}^2 = \dot{\theta}^2\left[c_0/2+\sum_{n=1}^{\infty} c_{2n}\cos 2n\theta\right]. \tag{19}$$

The imaginary exponential substitution of (5) and a path of integration around the unit circle $|z|=1$ gives

$$c_{2n} = \int_{ABC} z^{2n-1}(z^2+1)^2dz/\pi j(z^2+\alpha^2)(z^2+\alpha^{-2}). \tag{20}$$

For $n=0$ there are simple poles within the contour at the origin and $\pm j\alpha$. Calculation of the residue sum gives

$$c_0 = (1+\alpha^2)e^2. \tag{21}$$

For $n>0$ there are simple poles at $\pm j\alpha$ only, and calculation of the residue sum gives

$$c_{2n} = 2(-1)^{n-1}\alpha^{2n}\sqrt{1-e^2}. \tag{22}$$

5. Fourier series for $\dot{s}^2$. Since $\dot{s}=r(\dot{\theta}+\dot{\phi})\sin\theta$, it follows that

$$\dot{s}^2 = r^2(1-\cos 2\theta)(\dot{\theta}^2+2\dot{\theta}\dot{\phi}+\dot{\phi}^2)/2. \tag{23}$$

Substituting in (23) for $\dot{\phi}$ and $\dot{\phi}^2$ from (14) and (19), respectively, and using the addition theorem for cosines, gives

$$s^2 = (r^2\dot{\theta}^2/2)\sum_{k=0}^{\infty} d_k \cos k\theta, \tag{24}$$

where

$$\begin{aligned} d_0 &= 1 + (c_0 - c_2)/2, \\ d_1 &= b_1 - b_3, \\ d_2 &= c_2 - 1 - (c_0 + c_4)/2, \\ d_{2n+1} &= 2b_{2n+1} - b_{2n-1} - b_{2n+3} \text{ for } n > 0, \end{aligned} \tag{25}$$

and

$$d_{2n} = c_{2n} - (c_{2n-2} + c_{2n+2})/2 \text{ for } n > 1.$$

Substitution from (16), (21) and (22) into (25) gives

$$\begin{aligned} d_0 &= 1 + \alpha^2, \\ d_1 &= -4a_2 > 0, \\ d_2 &= -1 - \alpha^4, \\ d_{2n+1} &= 4(-1)^n\alpha^{2n-1}(Y_{2n-2} + \alpha^2 Y_{2n} - \alpha^4 Y_{2n+2} - \alpha^6 Y_{2n+4})/\pi \text{ for } n > 0, \\ &= 2a_{2n} + 2a_{2n+2} + 4b_{2n+1}/e^2, \end{aligned} \tag{26}$$

and

$$d_{2n} = (-1)^{n-1}\alpha^{2n-2}(1 - \alpha^4) \text{ for } n > 1.$$

Note, that, for customary values of e, d_2 is greater in magnitude than d_1 and of opposite sign.

6. **Dynamic unbalance components.** In the analysis of dynamic unbalance it is customary, *e.g.* [3], because of balancing considerations, to replace the connecting rod by a statically and dynamically equivalent connecting rod having its mass concentrated at the two ends of the rod. To preserve static and translational dynamic equivalence, all that is necessary is to maintain the center of gravity invariant. To this end the portions of the mass m of the connecting rod concentrated at the crank pin B and wrist pin A of Figure 1 are $m_B = ma/l$ and $m_A = mb/l$, respectively, where the distances of wrist pin and crank pin from the center of gravity are a and b, respectively. This substitution may also be regarded as yielding an equivalent engine having a massless connecting rod with a piston of effective mass m_{eff} greater than its true mass by the amount $m_A = mb/l$, and a crank of effective mass greater than its true mass by the amount $m_B = ma/l$. To achieve angular dynamic equivalence, it is necessary to apply to the connecting rod an auxiliary external torque $M = (mab - \bar{I})\ddot{\phi}$ where $\bar{I}$ is the moment of inertia of the rod about its center of gravity. This additional

torque is required in order to accelerate the increase in the moment of inertia caused by concentrating the mass at the ends of the connecting rod. A free body of the equivalent engine is given in Figure 1, the symbols for the various forces and torques being explained in the following paragraphs. The free body omits static and frictional forces. To simplify the expressions for dynamic unbalance given below, the customary assumption of constant crank angular velocity $\dot{\theta}$ is made, although this assumption is not necessary in the present method of analysis.

The Fourier components of dynamic unbalance may now be obtained in a very direct fashion by time differentiation of the Fourier kinematic series previously obtained. The shaking force F_s on the crank shaft due to the motion of the effective piston mass m_{eff} is found by differentiating (2) to be

$$\begin{aligned} F_s &= m_{\text{eff}}\ddot{s} \\ &= m_{\text{eff}} r\dot{\theta}^2 \left(\cos\theta - \sum_{n=1}^{\infty} 4n^2 a_{2n} \cos 2n\theta \right). \end{aligned} \tag{27}$$

The torque M_s about the crank shaft, due to the motion of the effective piston mass, is found from the work and energy relationship

$$M_s \dot{\theta} dt = (d/dt)(m_{\text{eff}} \dot{s}^2/2) dt \tag{28}$$

and (24) to be

$$M_s = -(m_{\text{eff}} r^2 \dot{\theta}^2/4) \sum_{k=1}^{\infty} k d_k \sin k\theta. \tag{29}$$

The torque M_ϕ about the crank shaft, due to the auxiliary torque M acting on the equivalent connecting rod, is found from the virtual displacement relationship

$$\begin{aligned} M_\phi \dot{\theta} dt &= -M\dot{\phi} dt \\ &= -(d/dt)[(mab - \bar{I})\dot{\phi}^2/2] dt, \end{aligned} \tag{30}$$

and (19) to be

$$M_\phi = (mab - \bar{I}) \sum_{n=1}^{\infty} n c_{2n} \sin 2n\theta. \tag{31}$$

The forces F, perpendicular to the cylinder center line, may be obtained by noting that, for constant crank angular velocity, the moment of momentum of the equivalent engine about the crank shaft O of Figure 1 is constant. Hence the resultant torque about O must vanish, giving

$$M_s + M_\phi - M - (r + l - s)F = 0. \tag{32}$$

It is unnecessary to discuss the rotating force F_θ, 180° out of phase with the crank angle θ, due to the centripetal acceleration of the effective crank mass,

since a Fourier analysis is not involved.

It is beyond the scope of the present paper to discuss dynamic unbalance in multicylinder engines, where the net effect of certain Fourier components may be eliminated.

References

1. Churchill, R. V., Introduction to Complex Variables and Applications, McGraw-Hill Book Co., New York, 1948, p. 130.

2. Jahnke, E. and Emde, F., Tables of Functions with Formulae and Curves, Dover Publications, New York, 1945.

3. Root, R. E., Dynamics of Engine and Shaft, John Wiley and Sons, Inc., New York, 1932, p. 7.

A SIMPLE MATRIX APPROACH TO LINEAR DIFFERENTIAL EQUATIONS

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In texts on elementary differential equations the subject of systems of linear differential equations with constant coefficients is frequently dismissed with little more than an example or two. From time to time suggestions have been made for improving the presentation through the use of ideas from the elementary theory of matrices,† but to date these seem to have had little effect on the books available for use in the classroom. This is perhaps because such discussions have been based on the canonical form

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n, \qquad i = 1, 2, \cdots, n,$$

which can be obtained from a system of differential equations as they actually arise only through a considerable amount of preliminary algebra. It is the purpose of this note to suggest a more practical matrix approach to the solution of a system of equations in their original form

$$\begin{aligned} a_{11}(D)x_1 + \cdots + a_{1n}(D)x_n &= f_1(t) \\ &\cdots\cdots\cdots\cdots \\ a_{n1}(D)x_1 + \cdots + a_{nn}(D)x_n &= f_n(t), \end{aligned} \tag{1}$$

where the a_{ij} are polynomials in $D(\equiv d/dt)$ with constant coefficients. In our

† C. M. Cramlet, Linear differential equations with constant coefficients, this MONTHLY, vol. 45, 1938, p. 162.

J. S. Frame, On the explicit solution of simultaneous linear differential equations with constant coefficients, this MONTHLY, vol. 47, 1940, p. 35.

See also Richard Bellman, Stability Theory of Differential Equations, p. 12, McGraw-Hill Book Co., New York, 1953.

work we shall actually use little more than the matrix notation; specifically, we need only a knowledge of how to add, multiply, and differentiate matrices.

Introducing the matrices

$$A(D) = \begin{Vmatrix} a_{11}(D) & \cdots & a_{1n}(D) \\ \cdot & \cdots & \cdot \\ a_{n1}(D) & \cdots & a_{nn}(D) \end{Vmatrix}, \quad X = \begin{Vmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{Vmatrix}, \quad F = \begin{Vmatrix} f_1(t) \\ \cdot \\ \cdot \\ f_n(t) \end{Vmatrix},$$

(1) can be written in the form

$$AX = F. \tag{2}$$

By analogy with a single differential equation we are led to consider the related homogeneous equation

$$AX = 0 \tag{3}$$

and to attempt to find solutions of it of the form

$$X = Ce^{mt} \tag{4}$$

where

$$C = \begin{Vmatrix} c_1 \\ \cdot \\ \cdot \\ c_n \end{Vmatrix}$$

is a constant, non-trivial coefficient matrix. Substitution of (4) in (3) yields the equation

$$A(m)Ce^{mt} = 0$$

which requires that

$$A(m)C = 0 \tag{5}$$

or finally

$$|A(m)| = 0. \tag{6}$$

This, of course, is the usual determinantal characteristic equation. It is a polynomial equation in m of degree k, say, each of whose roots, m_j, gives rise to a solution of (3) of the form (4) provided C is suitably determined. For each m_j this can be done by applying familiar algebraic procedures to (5). The resulting C_j is of course not a unique matrix but a matrix determined only to within an arbitrary scalar multiplier b_j. Since linear combinations of particular solutions are also solutions,

$$X = \sum_j b_j C_j e^{m_j t} \equiv \sum_j b_j X_j$$

is a solution of (3) for all values of the arbitrary scalar multipliers b_j. If the roots

of (6) are all distinct this is the complete solution of (3) and the complementary function of (2).

As in the case of a single differential equation, if the set $\{m_j\}$ includes one or more pairs of conjugate complex roots it is desirable to reduce the corresponding complex exponential solutions to a purely real form. To see how this can be accomplished let $m_1, m_2 = p \pm iq$ be a pair of conjugate complex roots of (6) and let C_1 be a particular coefficient matrix corresponding to $m_1 = p + iq$. Then by taking conjugates throughout (5) it is clear that $C_2 = \overline{C}_1$ is a coefficient matrix corresponding to $m_2 = p - iq$. Hence we have the two particular solutions

$$C_1 e^{(p+iq)t} \quad \text{and} \quad \overline{C}_1 e^{(p-iq)t}.$$

By combining these as follows and applying the Euler formulas we obtain two independent real particular solutions:

$$\begin{aligned} \tfrac{1}{2}[C_1 e^{(p+iq)t} + \overline{C}_1 e^{(p-iq)t}] &= e^{pt}[\tfrac{1}{2}(C_1 + \overline{C}_1) \cos qt + \tfrac{1}{2}i(C_1 - \overline{C}_1) \sin qt] \\ &= e^{pt}[\mathfrak{R}(C_1) \cos qt - \mathfrak{I}(C_1) \sin qt], \\ -\tfrac{1}{2}i[C_1 e^{(p+iq)t} - \overline{C}_1 e^{(p-iq)t}] &= e^{pt}[-\tfrac{1}{2}i(C_1 - \overline{C}_1) \cos qt + \tfrac{1}{2}(C_1 + \overline{C}_1) \sin qt] \\ &= e^{pt}[\mathfrak{I}(C_1) \cos qt + \mathfrak{R}(C_1) \sin qt]. \end{aligned} \tag{7}$$

In many cases this method of determining the necessary relations among the coefficients of solutions of (3) of the form

$$x_j = a_j e^{pt} \cos qt + b_j e^{pt} \sin qt$$

is simpler than the usual process of substituting these expressions into the original differential equations, collecting terms, and equating the resulting coefficients to zero.

If $|A(m)| = 0$ has a double root, say $m = m_1$, we proceed very much as in the case of a single differential equation. Of course

$$X_1 = C_1 e^{m_1 t}$$

is one solution of (3). However as a second independent solution we must try not $C_2 t e^{m_1 t}$, as strict analogy with the solution-process for a single differential equation would dictate, but rather†

$$X_2 = C_2 e^{m_1 t} + C_3 t e^{m_1 t}.$$

Substituting this into (3), observing that for any polynomial operator $P(D)$

$$P(D) t e^{mt} = P(m) t e^{mt} + P'(m) e^{mt},$$

and recalling the definition of the derivative of a matrix, we obtain

$$A(m_1) C_2 e^{m_1 t} + A(m_1) C_3 t e^{m_1 t} + A'(m_1) C_3 e^{m_1 t} = 0.$$

This requires that

† The term $C_2 e^{m_1 t}$ must be retained because in general C_2 will not be a scalar multiple of C_1 and hence in constructing the general solution the term $C_2 e^{m_1 t}$ cannot be absorbed in the term $C_1 e^{m_1 t}$ as is necessarily the case for a single differential equation.

$$A(m_1)C_3 = 0 \quad \text{and} \quad A(m_1)C_2 = -A'(m_1)C_3. \tag{8}$$

The first of these equations shows that to within an arbitrary scalar factor C_3 is the same as C_1. The second constitutes a system of nonhomogeneous linear algebraic equations from which the elements of C_2 can be found by familiar processes. Its solution is of the form

$$C_2 = \lambda C_1 + {}^*C_2$$

where ${}^*C_2$ is a "particular integral" of the (algebraic) system. Hence, omitting the term $\lambda C_1 e^{m_1 t}$ from X_2 since it can obviously be absorbed in X_1, and taking $C_3 = C_1$, we obtain

$$X_2 = {}^*C_2 e^{m_1 t} + C_1 t e^{m_1 t}$$

as a second independent particular solution of (3) corresponding to the double root m_1 of (6).

Roots of the characteristic equation of higher multiplicity are handled in essentially the same way. For instance if $|A(m)| = 0$ has a triple root $m = m_1$, then not only are

$$C_1 e^{m_1 t} \quad \text{and} \quad {}^*C_2 e^{m_1 t} + C_1 t e^{m_1 t}$$

solutions of (3), but

$$C_4 e^{m_1 t} + C_5 t e^{m_1 t} + C_6 t^2 e^{m_1 t}$$

is also a solution for properly determined coefficient matrices C_4, C_5, and C_6. Moreover, using the result

$$P(D)t^k e^{mt} = P(m)t^k e^{mt} + \binom{k}{1} P'(m)t^{k-1}e^{mt} + \binom{k}{2} P''(m)t^{k-2}e^{mt} + \cdots,$$

we find that the equations that determine C_4, C_5, and C_6 are

$$A(m_1)C_6 = 0, \quad A(m_1)C_5 = -2A'(m_1)C_6, \quad \text{and} \quad A(m_1)C_4 = -A'(m_1)C_5 - A''(m)C_6.$$

In particular, the first of these shows that like C_3 in the case of a double root, C_6 is also proportional to C_1.

Example. For the system

$$\begin{aligned} (D^2 + D + 8)x_1 + (D^2 + 6D + 3)x_2 &= 0 \\ (D + 1)x_1 + (D^2 + 1)x_2 &= 0 \end{aligned}$$

we have the characteristic equation $m^4 + 2m^2 - 8m + 5 = 0$ whose roots are $1, 1, -1 \pm 2i$. For the root $-1 + 2i$, (5) becomes

$$\begin{Vmatrix} (4 - 2i) & (-6 + 8i) \\ (2i) & (-2 - 4i) \end{Vmatrix} \cdot \begin{Vmatrix} c_1 \\ c_2 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix},$$

which is equivalent to the single scalar equation

$$2ic_1 + (-2 - 4i)c_2 = 0.$$

Hence we may take

$$C = \left\| \begin{matrix} c_1 \\ c_2 \end{matrix} \right\| = \left\| \begin{matrix} 1 + 2i \\ i \end{matrix} \right\|.$$

Then from (7) we obtain immediately the two particular solutions

$$X_1 = \left\| \begin{matrix} e^{-t}(\cos 2t - 2 \sin 2t) \\ -e^{-t} \sin 2t \end{matrix} \right\| \quad \text{and} \quad X_2 = \left\| \begin{matrix} e^{-t}(2 \cos 2t + \sin 2t) \\ e^{-t} \cos 2t \end{matrix} \right\|.$$

For the repeated root 1 we have one solution of the form C_1e^t where, from (5)

$$\left\| \begin{matrix} 10 & 10 \\ 2 & 2 \end{matrix} \right\| \cdot C_1 = 0, \text{ so that we may take } C_1 = \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\|.$$

As a second independent particular solution we have

$$*C_2e^t + C_1te^t$$

where, from (8), $*C_2$ is any particular solution of the system

$$\left\| \begin{matrix} 10 & 10 \\ 2 & 2 \end{matrix} \right\| \cdot C_2 = - \left\| \begin{matrix} 3 & 8 \\ 1 & 2 \end{matrix} \right\| \cdot C_1 \equiv \left\| \begin{matrix} 5 \\ 1 \end{matrix} \right\|.$$

Hence we may take

$$*C_2 = \left\| \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right\|.$$

Thus for the double root $m = 1$ we have the two particular solutions

$$X_3 = \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\| e^t \equiv \left\| \begin{matrix} e^t \\ -e^t \end{matrix} \right\| \text{ and } X_4 = \left\| \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right\| e^t + \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\| te^t \equiv \left\| \begin{matrix} te^t \\ \frac{1}{2}e^t - te^t \end{matrix} \right\|.$$

To complete the solution of the nonhomogenous equation (2) it is now necessary to find a particular integral. When the elements of F are functions having only a finite number of independent derivatives the method of undetermined coefficients can be employed very much as in the case of a single equation. For instance, corresponding to matrices F of the forms

$$B,\ Be^{at},\ B_1 \sin at + B_2 \cos at,\ B_1 + B_2t + \cdots + B_nt^n$$

we would assume trial particular integrals of the respective forms

$$K,\ \Upsilon e^{at},\ K_1 \sin at + K_2 \cos at,\ K_1 + K_2t + \cdots + K_nt^n.$$

Substituting these into (2) and equating coefficients we obtain systems of linear algebraic equations from which the elements of the coefficient matrices can be found at once provided no term, P_1, in the normal choice of a trial particular integral P duplicates a term in the complementary function. Should such duplication occur, then as in the case of a single equation, the usual choice of a particular integral must be modified. However for a system of equations not only must we multiply by t all those terms in P arising from the terms of F associated with P_1, but we must also retain the duplicating terms as well. Thus if Be^{at} is a term of F which is duplicated by a term of the complementary function, the appropriate choice for a particular integral corresponding to such a term is not just Kte^{at} but rather

$$K_1e^{at} + K_2te^{at}.$$

In this case substitution in (2) yields the equation

$$A(a)K_1e^{at} + A(a)K_2te^{at} + A'(a)K_2e^{at} = Be^{at}.$$

Hence we must have

$$A(a)K_2 = 0 \quad \text{and} \quad A(a)K_1 = B - A'(a)K_2.$$

In particular, the first of these shows that the matrix K_2 is proportional to the matrix coefficient of e^{at} in the complementary function.

The method of variation of parameters also carries over in very much the same fashion to systems of linear differential equations. To employ it, we interpret the scalar coefficients in the general solution of the homogeneous equation (3),

$$X = b_1X_1 + b_2X_2 + \cdots + b_kX_k,$$

not as constants but as functions of t, and attempt to determine them so that X will satisfy the nonhomogeneous system (2). To explore the matter further let (2) be written in the form

$$(A_0D^r + A_1D^{r-1} + \cdots + A_{r-1}D + A_r)X = F. \tag{9}$$

Clearly, in order to substitute X into this equation we will need to know its first r derivatives. Differentiating successively and, as in the case of a single equation, simplifying at each of the first $(r-1)$ stages by equating to zero the terms involving the first derivatives of the b_i we obtain the $(r-1)$ equations:

$$b_1'X_1^{(j)} + b_2'X_2^{(j)} + \cdots + b_k'X_k^{(j)} = 0, \qquad j = 0, 1, 2, \cdots, (r-2). \tag{10}$$

Substituting the various derivatives of X (as simplified) into (9) we obtain

$$\begin{aligned} A_0[(b_1X_1^{(r)} + \cdots + b_kX_k^{(r)}) + (b_1'X_1^{(r-1)} + \cdots + b_k'X_k^{(r-1)})] \\ + A_1[b_1X_1^{(r-1)} + \cdots + b_kX_k^{(r-1)}] + \cdots + A_{r-1}[b_1X_1' + \cdots + b_kX_k'] \\ + A_r[b_1X_1 + \cdots + b_kX_k] = F. \end{aligned}$$

Since each X_i is a solution of the homogeneous system, the last equation reduces to

$$A_0[b_1'X_1^{(r-1)} + \cdots + b_k'X_k^{(r-1)}] = F. \tag{11}$$

Equations (10) and (11) are in effect a system of linear equations from which the b_i can be found without difficulty provided the rank of the augmented matrix of the subsystem (11) is equal to the rank of the matrix of coefficients of this system. This will always be the case if the matrix A_0 is nonsingular. If A_0 is singular then in general the method of variation of parameters cannot be applied directly to (2), though it can be used on any nonsingular system of differential equations to which (2) can be reduced. Of course when the b_i' are determined, the b_i themselves can be found at once by integration.

Example. For the system

$$(D^2 - 7)x_1 - (D^2 - 3)x_2 = f_1$$
$$(D^2 + 2)x_1 + (D^2 - 2)x_2 = f_2$$

we find easily the complementary function

$$X = b_1\begin{Vmatrix} e^t \\ 3e^t \end{Vmatrix} + b_2\begin{Vmatrix} e^{-t} \\ 3e^{-t} \end{Vmatrix} + b_3\begin{Vmatrix} e^{2t} \\ -3e^{2t} \end{Vmatrix} + b_4\begin{Vmatrix} e^{-2t} \\ -3e^{-2t} \end{Vmatrix}.$$

The method of variation of parameters now yields the two matrix equations

$$b_1'\begin{Vmatrix} e^t \\ 3e^t \end{Vmatrix} + b_2'\begin{Vmatrix} e^{-t} \\ 3e^{-t} \end{Vmatrix} + b_3'\begin{Vmatrix} e^{2t} \\ -3e^{2t} \end{Vmatrix} + b_4'\begin{Vmatrix} e^{-2t} \\ -3e^{-2t} \end{Vmatrix} = 0$$

$$\begin{Vmatrix} 1 & -1 \\ 1 & 1 \end{Vmatrix}\left\{b_1'\begin{Vmatrix} e^t \\ 3e^t \end{Vmatrix} + b_2'\begin{Vmatrix} -e^{-t} \\ -3e^{-t} \end{Vmatrix} + b_3'\begin{Vmatrix} 2e^{2t} \\ -6e^{2t} \end{Vmatrix} + b_4'\begin{Vmatrix} -2e^{-2t} \\ 6e^{-2t} \end{Vmatrix}\right\} = \begin{Vmatrix} f_1 \\ f_2 \end{Vmatrix}.$$

These are equivalent to the four scalar equations

$$e^tb_1' + e^{-t}b_2' + e^{2t}b_3' + e^{-2t}b_4' = 0$$
$$3e^tb_1' + 3e^{-t}b_2' - 3e^{2t}b_3' - 3e^{-2t}b_4' = 0$$
$$-2e^tb_1' + 2e^{-t}b_2' + 8e^{2t}b_3' - 8e^{-2t}b_4' = f_1$$
$$4e^tb_1' - 4e^{-t}b_2' - 4e^{2t}b_3' + 4e^{-2t}b_4' = f_2,$$

from which we find without difficulty

$$b_1 = \int \frac{f_1 + 2f_2}{12} e^{-t}dt, \qquad b_2 = -\int \frac{f_1 + 2f_2}{12} e^{t}dt,$$

$$b_3 = \int \frac{2f_1 + f_2}{24} e^{-2t}dt, \qquad b_4 = -\int \frac{2f_1 + f_2}{24} e^{2t}dt.$$

MATHEMATICAL NOTES

EDITED by F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A NOTE ON NORMAL MATRICES

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A matrix is said to be *normal* if it commutes with its transposed conjugate. The product of two normal matrices is not necessarily normal but the following result, proved some years ago by Wiegmann [1], is available.

THEOREM. *If A, B, and AB are normal matrices, then BA is also normal.*

The aim of the present note is to give a very simple proof of Wiegmann's theorem. We denote by X^* the transposed conjugate of the matrix X, and write

$$f(X) = \text{trace } (X^*X), \qquad g(X) = |\lambda_1|^2 + \cdots + |\lambda_n|^2,$$

where $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of X. Our proof is based upon the following well-known criterion of Schur [2]:

(1) X is normal if and only if $f(X) = g(X)$.

Since A, B are normal matrices and since, moreover,

$$\text{trace } (XY) = \text{trace } (YX),$$

we have

$$\begin{aligned} f(AB) &= \text{trace } (B^*A^*AB) = \text{trace } (BB^*A^*A) \\ &= \text{trace } (B^*BAA^*) = \text{trace } (A^*B^*BA), \end{aligned}$$

so that

(2) $$f(AB) = f(BA).$$

Now, AB and BA have the same characteristic roots. Therefore

(3) $$g(AB) = g(BA).$$

Again, since AB is normal, we have by (1)

(4) $$f(AB) = g(AB).$$

From (2), (3), and (4) we at once infer that $f(BA) = g(BA)$; and the required conclusion now follows by (1).

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AN APPLICATION OF ESTIMATION THEORY TO DETERMINANTAL INEQUALITIES

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The inequalities presented below are generalizations of the results obtained by the writer in [1]. In [2] Ky Fan and John Todd first present an algebraic proof of the results obtained in [1] by statistical theory, and then, by an extension of their algebraic argument, derive their determinantal inequality. The determinantal inequalities given below are derived by an extension of the statistical proof used in [1], and, except for the case in which $m=2$, the following results are distinct from those given in [2].*

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1m} \\ a_{21} & a_{22} \cdots a_{2m} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{n1} & a_{n2} \cdots a_{nm} \end{pmatrix}$$

be a matrix of real elements a_{ij} with $n>m\geqq 2$ and such that all square sub-matrices of order m are non-singular; let $G=A'A$ where A' is the transpose of A. Define $A^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}$ as a square matrix consisting of m rows of A including the ith row of A, and preserving their order; let $|G|=\det G$ and let $|G_{j1}|$ be the cofactor of g_{j1} in G where $g_{j1}=\sum_{i=1}^{n} a_{ij}a_{i1}$; let $|A_{i1}^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}|$ be the cofactor of a_{i1} in $A^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}$. Define

$$\lambda_i = \sum_{\substack{\alpha_1<\alpha_2\cdots<\alpha_{m-1} \\ (\alpha_r\neq i;\, r=1,\cdots,m-1)}}^{n-m+1} \cdots \sum^{n} \frac{|A_{i1}^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}|}{|A^{(\alpha_1,\alpha_2,\cdots,i,\cdots,\alpha_{m-2},\alpha_{m-1})}|}$$

where each λ_i consists of $C_{n-1,m-1}$ terms.
Then

$$\frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}|\right\}^2 \leqq C_{n,m}^{-2}\sum_{i=1}^{n}\lambda_i^2. \tag{1}$$

Proof: Let $\epsilon_i=z_i-a_{i1}\xi_1-a_{i2}\xi_2-\cdots-a_{im}\xi_m (i=1, 2, \cdots, n)$ represent n random independent deviates with $E(\epsilon_i)=0$, $E(\epsilon_i^2)=1$, random components z_i, known parameters a_{ij} subject only to the restrictions stated above, and with unknown parameters $\xi=(\xi_1, \xi_2, \cdots, \xi_m)$.

The classical least-squares estimate of the vector ξ is determined by selecting the point $\hat{\xi}=(\hat{\xi}_1, \hat{\xi}_2, \cdots, \hat{\xi}_m)$ which minimizes the sum of squares from it to

* For the case of $m=2$, the inequalities stated herein as (1) and (3) are each equivalent to Theorem 1 in the Fan-Todd results, while (2) and (4) are equivalent to their Theorem 2, with the exception that the results given in the Fan-Todd paper extend to the unitary n-space U^n, while the derivations in this paper limit the results to the Euclidean hyperspace.

each of the respective hyperplanes $\epsilon_i = 0$. Each value, $\hat{\xi}_l$, thus obtained is an unbiased estimate of the corresponding parameter component ξ_l and in accordance with the generalized Markoff theorem on least squares [3] each component of $\hat{\xi}$ is the minimum variance unbiased estimate of the corresponding component of ξ in the class of unbiased linear estimates. This holds in particular for $\hat{\xi}_1$ and its variance is well known to be identical to the left side of the inequality stated above [4].

To outline first the remainder of the proof, consider the point of intersection of any m of the n hyperplanes $\epsilon_i = 0$. The coordinates of this point yield unbiased estimates of the corresponding coordinates of the parameter point, ξ. It follows that the arithmetic average of corresponding coordinates of each of the possible $C_{n,m}$ intersections also provides an unbiased estimate of ξ. If this average is labelled $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m)$, then by the aforementioned minimum variance property of $\hat{\xi}$ it follows that $\sigma^2(\hat{\xi}_i) \leqq \sigma^2(\bar{x}_i)$ for all i, and therefore $\sigma^2(\hat{\xi}_1) \leqq \sigma^2(\bar{x}_1)$. It will be shown that

$$\sigma^2(\bar{x}_1) = C_{n,m}^{-2} \sum_{i=1}^{n} \lambda_i^2.$$

To show that

$$x_1^{(1)} = \frac{\begin{vmatrix} z_1 & a_{12} & \cdots & a_{1m} \\ z_2 & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_m & a_{m2} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}}$$

is an unbiased estimate of $\hat{\xi}_1$, we note that

$$E(x_1^{(1)}) = \frac{\begin{vmatrix} E(z_1) & a_{12} & \cdots & a_{1m} \\ E(z_2) & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ E(z_n) & a_{m2} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}};$$

from $\epsilon_i = z_i - a_{i1}\xi_1 - \cdots - a_{im}\xi_m$, and $E(\epsilon_i) = 0$ it follows that $E(z_i) = a_{i1}\xi_1 + a_{i2}\xi_2 + \cdots + a_{im}\xi_m$, and therefore

$$E(x_1^{(1)}) = \frac{\begin{vmatrix} a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1m}\xi_m & a_{12} \cdots a_{1m} \\ \cdots & \cdots \\ a_{m1}\xi_1 + a_{m2}\xi_2 + \cdots + a_{mm}\xi_m & a_{m2} \cdots a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} \cdots a_{1m} \\ \cdots \\ a_{m1} \cdots a_{mm} \end{vmatrix}} = \xi_1.$$

Thus $x_1^{(1)}$ is an unbiased estimate of ξ_1 and it follows that $\bar{x}_1 = C_{n,m} \sum_j x_1^{(j)}$ also is.

Next, denote the square sub-matrix consisting of rows $r_1, r_2, \cdots, r_m$ of A by $A^{(r_1, r_2, \cdots, r_m)} \equiv A^{(r)}$. Corresponding to each of the $C_{n,m}$ sets of such submatrices we obtain from $A^{(r)}x^{(r)} \equiv z^{(r)}$,

$$x_1^{(r)} = x_1^{(r_1, r_2, \cdots r_m)} = \frac{\begin{vmatrix} z_{r_1} & a_{r_1 2} \cdots a_{r_1 m} \\ \cdots & \cdots \\ z_{r_m} & a_{r_m 2} \cdots a_{r_m m} \end{vmatrix}}{|A^{(r)}|}$$

$$= \frac{1}{|A^{(r)}|}\{z_{r_1}|A_{r_1 1}^{(r)}| + z_{r_2}|A_{r_2 1}^{(r)}| + \cdots + z_{r_m}|A_{r_m 1}^{(r)}|\},$$

where $|A_{r_j 1}^{(r)}|$ is the cofactor of $a_{r_j 1}$ in $A^{(r)}$. Then the coefficient of z_i in

$$\sum_r x_1^{(r)} = \sum_{r_1 < r_2 \cdots < r_n} \cdots \sum x_1^{(r_1, r_2, \cdots, r_m)}$$

becomes

$$\lambda_i = \sum_{\substack{\alpha_1 < \alpha_2 \cdots < \alpha_{m-1} \\ (\alpha_r \neq i;\, r = 1, \cdots, m-1)}}^{n-m+1} \cdots \sum^{n} \frac{|A_{i1}^{(\alpha_1, \alpha_2, \cdots, i, \cdots, \alpha_{m-2}, \alpha_{m-1})}|}{|A^{(\alpha_1, \alpha_2, \cdots, i, \cdots, \alpha_{m-2}, \alpha_{m-1})}|}.$$

It follows from the independence of the z_i and their unit variances that

$$\sigma^2\left(\sum x_1^{(r)}\right) = \sum_{i=1}^{n} \lambda_i^2.$$

Therefore

$$\sigma^2(\bar{x}_1) = C_{n,m}^{-2} \sum_{i=1}^{n} \lambda_i^2,$$

thus establishing (1).

To obtain a weighted generalization of (1), let w be a real vector in $C_{n,m}$-space, restricted to the hyperplane $\sum_{\alpha=1}^{C_{n,m}} w^{(\alpha)} = 1$. Then,

$$(2)\qquad \frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}|\right\}^2 \leqq \sum_{i=1}^{n}\left\{\sum_{\alpha_1<\cdots<\alpha_{m-1}}\cdots\sum w^{(\alpha_1,\cdots,i,\cdots,\alpha_{m-1})}\frac{|A_{i1}^{(\alpha_1,\cdots,i,\cdots,\alpha_{m-1})}|}{|A^{(\alpha_1,\cdots,i,\cdots,\alpha_{m-1})}|}\right\}^2.$$

The proof of (2) is identical to that of (1), with the exception that, while the right hand term of the latter represents the variance of a uniformly weighted arithmetic average of unbiased estimates, the right-hand member of the inequality (2) represents the variance of an unbiased weighted average of the same unbiased estimates, so that Markoff's theorem again applies.

The restriction that each m-order determinant from A be non-singular can be dropped with respect to (2), provided at least one such determinant remains non-singular. When this is the case all of the terms with zero denominators in the right side of (2) may be permitted to vanish, provided that the weights of the non-vanishing terms sum to unity.

Partition the $n\times m$ matrix A into a set of r or $r+1$ submatrices, as follows. Let the ith sub-matrix consist of n_i rows and m columns, with $n_i\geqq m$ for $i=1, 2, \cdots, r$. Let

$$A^{(n_1)} \equiv A^{(1,2,\cdots,n_1)}$$

be the submatrix which contains the first n_1 rows of A,

$$A^{(n_2)} \equiv A^{(n_1+1,n_1+2,\cdots,n_1+n_2)}$$

be that which is comprised of the next n_2 rows,

$$\vdots$$

$$A^{(n_r)} \equiv A^{(n_1+n_2+\cdots+n_r-1+1,\cdots,n_1+n_2+\cdots+n_r-1+n_r)},$$

the rth submatrix, and containing the last n_r rows of A if $\sum_{i=1}^{r}n_i=n$; if $\sum_{i=1}^{r}n_i<n$, then $n_{r+1}<m$ and the partitioning divides A into $r+1$ instead of r submatrices. Define $G^{(n_l)}=A^{(n_l)'}A^{(n_l)}$ for $l=1, 2, \cdots, r$. Then

$$(3)\qquad \frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}|\right\}^2 \leqq \left(\frac{1}{r}\right)^2\sum_{l=1}^{r}\left[\frac{1}{|G^{(n_l)}|^2}\sum_{i=n_{l-1}+1}^{N_l}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}^{(n_l)}|\right\}^2\right]$$

where

$$N_k = \sum_{i=1}^{k} n_i.$$

As proof of (3) it need only be stated that the right side represents the variance

of a uniformly weighted arithmetic average of r unbiased least squares estimates of ξ_1.

As a weighted generalization of (3) there follows, subject to $\sum_{l=1}^{r} w_l = 1$,

$$(4) \qquad \frac{1}{|G|^2}\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}|\right\}^2 \leqq \sum_{l=1}^{r}\left[\frac{w_l^2}{|G^{(n_l)}|^2}\sum_{i=N_{l-1}+1}^{N_l}\left\{\sum_{j=1}^{m} a_{ij}|G_{j1}^{(n_l)}|\right\}^2\right].$$

In this case the greater side is the variance of a weighted average of unbiased least squares estimates.

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$\Gamma(z)$ AT MINUS INFINITY

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The behavior of $\Gamma(z)$ in the distant portions of the complex plane outside any fixed angle containing the negative real axis is given by Stirling's formula

$$(1) \qquad \Gamma(z) \sim \sqrt{2\pi}z^{z-1/2}e^{-z},$$

which is valid as $|z|\to\infty$, $-\pi+\delta\leqq\arg z\leqq\pi-\delta$ for any δ, $0<\delta<\pi$. That the behavior inside the exceptional angle follows from this formula and the identity

$$(2) \qquad \Gamma(z)\cdot\Gamma(1-z) = \pi \csc \pi z$$

seems not to be widely known. Let z be any fixed complex number, not an integer. Then $(z-n)$, for large integer n, is in the critical angle, but bounded away from the poles. Furthermore $(1-z+n)$ is exterior to this angle so that (1) applies and we have, as n tends toward infinity:

$$\begin{aligned}\Gamma(z-n) &= \frac{\pi \csc \pi(z-n)}{\Gamma(1+n-z)} = \frac{(-1)^n\pi \csc \pi z}{\Gamma(1+n-z)} \\ &\sim \frac{(-1)^n\pi \csc \pi z}{\sqrt{2\pi}}(1+n-z)^{z-n-1/2}e^{1+n-z} \\ &\sim (-1)^n\sqrt{\frac{\pi}{2}}\,n^{z-n-1/2}e^n \csc \pi z.\end{aligned}$$

This formula can, of course, be improved by using more terms in the complete asymptotic expansion of which (1) gives the first term. More labor would then yield more terms in the resulting asymptotic expression for $\Gamma(z-n)$.

CLASSROOM NOTES

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Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON TABLES OF FACTORS AND PRIMES

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The object of this paper is to describe a method making use of "key numbers and possible factors" which will increase the usefulness of any table of factors and primes approximately tenfold.* This method will enable one, by inspection alone, to determine whether a given number up to about 10 times the largest entry given in a table of primes is prime or not, and if not prime what its factors are.

The following symbols will be used:

N is the number whose factors, if any, are to be determined.

A is N with the last digit dropped. Thus if $N=17{,}873$, $A=1787$.

R is the range and is the integral part of the square root of N with its last two digits dropped. Thus R of $17{,}873=13$

K is the key number and is found in the table below.

P_1 is a possible factor corresponding to K.

The twin series are given in the table for the sake of completeness but are not written down in actual practice.

There are two steps as outlined below.

STEP I

N	K	P_1	Series	Procedure
Any number ending in 1, 3, 7, or 9 within limits stated above	$A-R$	$10R+$last digit	$K+n$; P_1-10n $n=0, 1, \cdots, 2R+1$	Beginning with K read Table of Factors and Primes from left to right and look for corresponding possible factors.

STEP II

N ending in	K	P_1	Series	Procedure
1	$(A-2)-3R$	$10R+7$	$K+3n$; P_1-10n $n=0, 1, \cdots, 2R+1$	Same as in Step I except that only every *third* entry in Table of Primes and Factors is examined for corresponding possible factors.
3	$A-3R$	$10R+1$	$n=0, 1, \cdots, 2R+1$	
7	$(A-2)-3R$	$10R+9$	$n=0, 1, \cdots, 2R+1$	
9	$A-3R$	$10R+3$	$n=0, 1, \cdots, 2R+1$	

* A table of factors and primes having 2009 entries will suffice to determine any number up to 19,949.

As an example consider 7519. We shall use mathematical tables from the *Handbook of Chemistry and Physics.*

Step I

From our table $N=7519$; $A=751$; $R=8$; $K=743$; $P_1=89$ and the series obtained:

743	89
744	79
745	69
746	59
751	9
752	1
753	11
760	81

Examining the Table of Factors and Primes beginning with 743 we note whether 743 has 89 as a factor; whether 744 has 79 as a factor, *etc.* We continue to read from left to right keeping in mind the possible factors 69, 59 $\cdots$ 81.

Since none are found we conclude that 7519 has no factors within the given range ending in 9 or 1.

Step II

From our table $K=727$ and $P_1=83$; the series obtained is:

727	83
730	73
733	63
751	3
754	7
778	87

We proceed exactly as in Step I. Beginning with 727 we note whether it has 83 as a factor. Thereafter we examine every *third* entry for the remaining corresponding possible factors 73, 63 $\cdots$ 3, 7, $\cdots$ 87. Since 730 yields the factor 73 we conclude that 73 is a factor of 7519. By division $7519=73\times103$. Had no factor been found in either Step I or Step II we would have concluded that the number under consideration was prime.

Proof for Step I

Let N end in 1. Then, if N has a factor ending in 1, we can write

$$(10a+1)(10b+1) = N = 10A+1,$$

$$10ab + a + b = A$$

or

$$b(10a + 1) = (A - a).$$

This proves that any factor $10a+1$ of N is also a factor of $A-a$.

If we put $a=(R-n)$, then $(10a+1)$ becomes $(10R+1)-10n$ and $(A-a)$ becomes $(A-R)+n$, *i.e.*, P_1-10n and $K+n$ respectively as given in the table. Since P_1-10n becomes a number ending in 9 when $P_1<10n$, all possible factors ending in 9 are also provided for.

The proof for Step II is similar. For possible factors ending in 3 or 7 we write

$$(10a + 7)(10b + 3) = N = 10A + 1.$$

By identical reasoning this results in P_1-10n and $K+3n$ respectively given in the table. This completes the proof for numbers ending in 1.

Identical reasoning establishes the key numbers and series for numbers ending in 3, 7, or 9.

A METHOD FOR SOLVING SCHLÖMILCH'S INTEGRAL EQUATION

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In chapter 11 of the fourth edition of their work *A Course of Modern Analysis*, Whittaker and Watson devote sections 11.8–11.81 to the integral equations of Abel and Schlömilch. After reading this reference, however, the student is not likely to observe the fact that these equations are closely related. This is perhaps due to the manner in which the equations are solved. In this note we offer an alternative method for the solution of Schlömilch's equation

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} g(x \sin \theta)d\theta, \qquad f, g \in C'([-\pi, \pi]) \tag{1}$$

which not only seems simpler and more direct, but, as a consequence of Abel's equation, gives what is believed to be a new expression for the solution.

The substitutions $x \sin \theta = \sqrt{\xi}$, $x^2 = t$ readily transform (1) to the ordinary Abel equation

$$f(\sqrt{t}) = \int_0^t (t - \xi)^{-1/2} G(\xi)d\xi, \tag{2}$$

where $G(\xi)$ denotes the function $\pi^{-1}\xi^{-1/2}g(\sqrt{\xi})$. But the solution of (2) is

$$G(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t (t - \xi)^{-1/2} f(\sqrt{\xi})d\xi, \tag{3}$$

and therefore

$$g(x) = x\frac{dx}{dt}\frac{d}{dx}\int_0^{\pi/2} (x^2 - x^2\sin^2\theta)^{-1/2} f(x\sin\theta)\, x^2 \sin 2\theta\, d\theta$$

$$= \frac{d}{dx}\int_0^{\pi/2} (x\sin\theta) f(x\sin\theta)\, d\theta \tag{4}$$

or equivalently,

$$g(x) = f(0) + x\int_0^{\pi/2} f'(x\sin\theta)\, d\theta. \tag{5}$$

This follows from (4) by differentiating the integral and then integrating by parts. For

$$g(x) = \int_0^{\pi/2} [\sin\theta f(x\sin\theta) + x\sin^2\theta f'(x\sin\theta)]\, d\theta$$

$$= -f(x\sin\theta)\cos\theta\Big|_0^{\pi/2} + x\int_0^{\pi/2} \cos^2\theta f'(x\sin\theta)\, d\theta$$

$$+ x\int_0^{\pi/2} \sin^2\theta f'(x\sin\theta)\, d\theta \tag{6}$$

$$= f(0) + x\int_0^{\pi/2} f'(x\sin\theta)\, d\theta.$$

DERIVING THE EQUATIONS OF THE SECTIONS OF A CONE

LUISE LANGE, Chicago Teachers College

In a recent note G. B. Huff* called attention to the gap which the traditional presentation in Analytic Geometry leaves between the two sets of definitions of the "conic sections," as plane sections of a right circular cone on the one side, and as the several geometrical loci on the other. He closed this gap by showing geometrically (or drawing attention to proofs given elsewhere) that the variously inclined sections of a cone satisfy the respective locus conditions.

In the following the equivalence of the two sets of definitions is demonstrated analytically. That is, we show—without using analytic space geometry—that the equations of the curves of intersection of a right circular cone with variously inclined planes are the same as those obtained from the locus definitions; the coefficients moreover reveal the relation between the parameters characterizing the sections of the cone and those characterizing the loci.

Let VAB (Fig. 1) be a section through the axis of a right circular cone, and Γ a plane perpendicular to the axis at C, intersecting the cone in a circle of radius r. Let α be the angle between the elements of the cone and Γ.

Intersect the cone with a second plane Δ through C forming the dihedral angle $\phi = ACD$ with Γ. On the intersection of Δ with the cone take an arbitrary

* G. B. Huff, On defining conic sections, this MONTHLY, vol. 62, 1955, pp. 250–251.

point P. Draw $CP=\rho$, forming angle $\theta=DCP$ with CD. Then (ρ, θ) are the polar coordinates of P in plane Δ, with C as origin and the line through CD as polar axis.

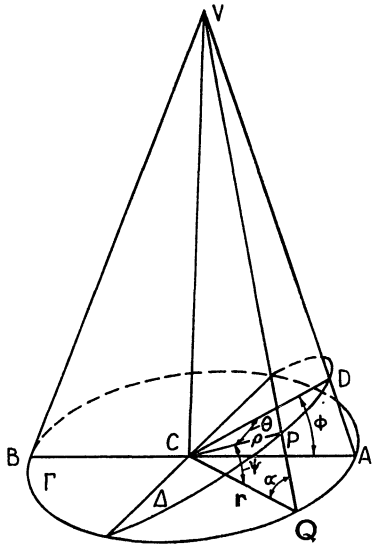


FIG. 1

Draw the element VP intersecting the circle in plane Γ at Q. Connect Q with C. Then the angle $\psi=PCQ$, which ρ forms with plane Γ, is given by the relation

$$\sin \psi = \sin \phi \cdot \cos \theta. \tag{1}$$

Plane Δ is inclined towards Γ by angle ϕ; CP forms angle θ with the "line of steepest ascent" CD; ψ is the inclination of CP towards Γ. This leads readily to (1). (See Fig. 2.)

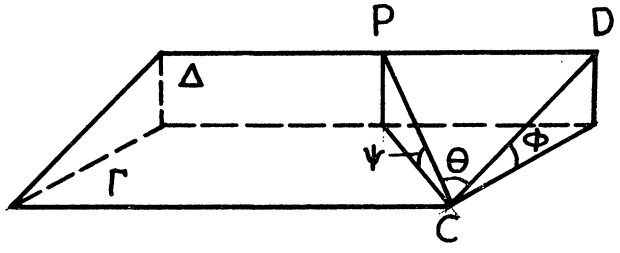


FIG. 2

We next derive the relation between ρ and θ from triangle CPQ in which $CP=\rho$; $CQ=r$; angle $Q=\alpha$; angle $C=\psi$. Hence $\rho/\sin \alpha=r/\sin (\alpha+\psi)$. Expanding $\sin (\alpha+\psi)$, and expressing $\sin \psi$ and $\cos \psi$ in terms of ϕ and θ by virtue of (1),

one obtains the polar equation of the curve of intersection:

$$(2) \quad f(\rho, \theta) = (\sin^2 \alpha - \sin^2 \phi \cdot \cos^2 \theta) \cdot \rho^2 + r \cdot \sin 2\alpha \cdot \sin \phi \cdot \rho \cos \theta - r^2 \sin^2 \alpha = 0.$$

Transforming to rectangular coordinates in plane Δ, $x = \rho \cdot \cos \theta$, $y = \rho \cdot \sin \theta$, one obtains

$$(3) \quad F(x, y) = (\sin^2 \alpha - \sin^2 \phi) \cdot x^2 + \sin^2 \alpha \cdot y^2 + r \cdot \sin 2\alpha \cdot \sin \phi \cdot x - r^2 \sin^2 \alpha = 0.$$

This equation is the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ obtained from the locus definitions of the "conic sections."

The conditions derived from these for the elliptical, parabolic, and hyperbolic case respectively, $AC \gtreqless 0$, take on the form $|\sin \phi| \lesseqgtr |\sin \alpha|$ or $\phi \lesseqgtr \alpha$ (since only first-quadrant angles need to be considered in this context).

For the eccentricity one obtains from (3) $e = \sin \phi / \sin \alpha$. This shows that the greatest eccentricity for a section of a given cone is obtained for $\phi = 90°$, $e = 1/\sin \alpha$. This is at once seen to be the eccentricity of a hyperbola whose asymptote angle is equal to the vertex angle of the cone.

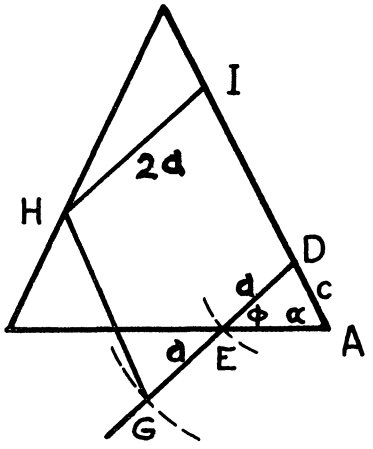


FIG. 3

Equation (3)—which was derived by assuming the plane Δ to pass through a point C on the axis of the cone, which point was further taken as the origin of the coordinate system,—only permits to approach such hyperbolic sections as the limit for $\phi \to 90°$. Placing $\phi = 90°$ gives, indeed, the "degenerate" axial section the two straight lines intersecting at V, (3) becoming factorable in the form $(x \cos \alpha - r \sin \alpha)^2 - y^2 \sin^2 \alpha = 0$. But the equations of the sections parallel to the axis ($\phi = 90°$) are easily derived separately. Passing plane Δ through a point D on the cone parallel to the axis at distance k, one readily obtains for the rectangular coordinates of a point P on the curve of intersection, with respect to D as origin,

$$\frac{(x+h)^2}{h^2} - \frac{y^2}{k^2} = 1,$$

where $h = k \cdot \tan \alpha$, and hence $(-h, -k)$ are the coordinates of vertex V of the cone. (Proof omitted.) These hyperbolas are seen to have the eccentricity $e = 1/\sin \alpha$.

We still point out that the relation $e = c/a = \sin \phi / \sin \alpha$ furnishes a simple construction of a central conic of specified axis $2a$ and focal distance $2c$ to be cut from a given cone (Fig. 3). From an arbitrary point A on the cone lay off $AD = c$ along an element. From D strike an arc of radius a intersecting the diameter of the circular section through A at E. DE then forms with EA the proper angle ϕ. Construct $DG = 2a$. The indicated parallel construction gives the specified axis $HI = 2a$.

This construction again implies the limiting condition, $a \geqq c \cdot \sin \alpha$, that is, $e \leqq 1/\sin \alpha$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1226. *Proposed by J. V. Pennington, Houston, Texas*

A rancher bought a white cow, and in the following year a red one. Each succeeding year he duplicated his purchases of the preceding two years, buying the same number of cows, of the same colors and in the same order. Thus, in the third year, he bought a white and then a red cow; in the fourth year, a red, then a white, and then a red cow; and so on. What was the color of the nth cow?

E 1227. *Proposed by R. L. Helmbold, Carnegie Institute of Technology*

Find all values of $a \geqq 1$ such that $a^x \geqq x^a$ holds for all values $x \geqq 0$.

E 1228. *Proposed by Viktors Linis, University of Ottawa*

Let $n(P)$ be the number of distinct lines L through a point P such that L divides the area of a given triangle in two equal parts. Show that the locus of all points P with $n(P) \geqq 2$ is a region the ratio of whose area to the area of the given triangle is an absolute constant.

E 1229. *Proposed by M. P. Drazin, Trinity College, Cambridge, England*

Given any point O in the plane of a triangle $\Delta \equiv ABC$, let the sides a, b, c of Δ subtend angles A', B', C' at O, and let the distances from O to the vertices of Δ be a', b', c'. Show that the triangle with sides aa', bb', cc' has angles $A'-A$, $B'-B$, $C'-C$, and find the sextic polynomial relation connecting a, b, c, a', b', c'.

E 1230. *Proposed by Vassili Daiev, Sea Cliff, Long Island*

Find the nth term of the sequence of ordered positive integers prime to 30.

SOLUTIONS

Permutations and Permutable Transpositions

E 1118 [1954, 345; 1955, 43]. *Proposed by Joel Brenner, State College of Washington*

Let us say a permutation is of type T if it is a product of mutually permutable transpositions. Is every permutation p a product of two permutations of type T?

Addendum by the Proposer. Every permutation is a product of disjoint cycles. In [1955, 43], a proof was given for finite cycles. For infinite cycles the answer is also yes, for

$$(\cdots, -4, -3, -2, -1, 0, 1, 2, 3, \cdots) = R \cdot S,$$

where

$$R = (0, -1)(1, -2)(2, -3)\cdots, \qquad S = (-1, 1)(-2, 2)(-3, 3)\cdots.$$

The 2319th Digit in 1000!

E 1180 [1955, 493; 1956, 189]. *Proposed by M. S. Klamkin and Alex Kraus, Polytechnic Institute of Brooklyn*

Determine the 2319th digit in the expansion of 1000!.

Editorial Note. H. S. Uhler has calculated the exact value of 1000!. See his article "Exact values of 996!, and 1000!, with skeleton tables of antecedent constants," *Scripta Mathematica*, vol. XXI, pp. 261–268. The 2319th digit in the expansion of 1000! appears as 2, as was shown in [1956, 189].

An Incomplete Assignment

E 1196 [1956, 39]. *Proposed by T. S. Ferguson and C. H. Kraft, University of California*

A student awoke at the end of a class in algebra one morning just in time to hear his teacher say, " . . . and I will give you the hint that all the roots are real and positive." Looking at the blackboard he discovered a 20th degree equation to be solved for home work, which he hastily tried to copy down. He succeeded in getting only the first two terms, $x^{20}-20x^{19}$, before his teacher erased the board completely; however, he did remember that the constant term was $+1$. Can you help our hero by not only solving the equation, but also showing uniqueness

of the solution?

Solution by D. S. Greenstein, University of Pennsylvania. The roots are positive and have their arithmetic and geometric means both equal to 1. It follows that the roots are all equal to 1.

Also solved by W. E. Baxter, G. E. Bills, Louis Brickman, P. L. Chessin, A. E. Danese, J. M. Elkin, Michael Goldberg, A. J. Goldman, Virginia Hanly, Vern Hoggatt, F. A. Homann, A. R. Hyde, Lawrence Isenecker, Edgar Karst, M. S. Klamkin, D. C. B. Marsh, Morris Morduchow, J. B. Muskat, Herbert Nadler, Walter Penney, L. L. Pennisi, C. F. Pinzka, M. A. Rashid, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., E. D. Schell, J. P. Scholz, R. R. Seeber, Jr., R. E. Shafer, O. E. Stanaitis, Art Steger, Chih-yi Wang, David Zeitlin, and the proposers. Late solution by Alan Wayne.

Editorial Note. For a proof of the fact that the arithmetic and geometric means of n positive numbers are equal if and only if the n numbers are all equal, see *e.g.*, G. Chrystal, *Algebra*, Part II, pp. 46–47, or Hardy, Littlewood and Pólya, *Inequalities*, p. 17. As was pointed out by Wang, if the word "positive' is omitted from the statement of the problem, then uniqueness cannot be shown. Thus we might also take the roots to be eight 2's, four 1's, four 1/2's, and four $-1/2$'s.

A Rich Configuration

E 1197 [1956, 39]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be a right triangle and CH the altitude on the hypotenuse AB. Show that the sum of the radii of the inscribed circles of triangles ABC, HCA, HCB is equal to CH.

Solution by Leon Bankoff, Los Angeles, Calif. I. The diameter of the circle inscribed in a right triangle is equal to the sum of the legs minus the hypotenuse. Applying this relation in triangles ABC, HCA, HCB, we get

$$\frac{(AC + CB - AB) + (AH + CH - AC) + (CH + HB - CB)}{2} = CH.$$

II. In similar right triangles, the ratios of inradius to hypotenuse are equal. We may therefore write

$$r/c = r_1/b = r_2/a = (r + r_1 + r_2)/(a + b + c),$$

where r, r_1, r_2 are the inradii of triangles ABC, HCA, HCB. Since

$$r/c = CH/(a + b + c)$$

it follows that $r+r_1+r_2=CH$.

Additional selected properties of the configuration. R, S, T are the incenters of triangles AHC, CHB, ABC, respectively, and H, R', S', T' the orthogonal projections of C, R, S, T on AB. Let r, r_1, r_2 denote the inradii of triangles ABC,

AHC, CHB. P and Q are the feet of the cevians CR and CS. RS cuts AC in U and CB in V. K is the intersection of PS and RQ.

(1) T' is the circumcenter of triangle RST; Q, S, T, R, P are concyclic; $PT'=TT'=T'Q$.

(2) T is the orthocenter of triangle CRS and the circumcenter of triangle CPQ.

(3) $r_1^2+r_2^2=r^2$.

(4) $RS=CT=PT=QT=r\sqrt{2}$.

(5) Triangles HSR, ABC, AHC, HCB are similar.

(6) A, R, S, B are concyclic. Triangles RST and ABT are inversely similar.

(7) R, T', H, S are concyclic. (RS is a diameter of the circle.)

(8) $T'S$ is parallel to AC; RT' is parallel to CB.

(9) Triangles $RR'T'$ and $T'S'S$ are congruent (and similar to triangle ABC).

(10) Triangles ATB, BCS, ARC are similar.

(11) Angle ATB = angle BSC = angle $ARC=135°$.

(12) $AC=AQ$; $PB=CB$.

(13) $CU=CV$; CT is the perpendicular bisector of UV.

(14) PS, RQ, CH are concurrent at K, the orthocenter of triangle CPQ.

(15) PS is parallel to AT; RQ is parallel to TB; triangles PBS and CSB are congruent; triangles AQR and ARC are congruent.

(16) The midpoint of RS is the nine point center of triangle CPQ.

(17) The circumcircle of triangle HSR is the nine point circle of triangle CPQ.

(18) The circumcircles of triangles ARC and CSB are tangent at C, and CT is their common internal tangent.

(19) $RT=KS=SQ$; $RP=RK=TS$; triangles PKR and KQS are isosceles right triangles. (Also triangle $RT'S$.)

(20) A, P, T, C are concyclic; B, Q, T, C are concyclic.

(21) The perimeter of triangle $T'S'S$ = perimeter of triangle $RR'T=CH$ (since $SS'=r_2, T'S'=r_1$, $T'S=r$).

(22) Area of triangle $RST=(a+b-c)^3/8c=r^3/c$.

(23) Area of pentagon $PQSTR=2r^3/c+r^2$.

(24) Area of triangle $CPQ=abr/c$.

Also solved by W. A. Al-Salam, L. C. Barrett, Robert Bart, G. E. Bills, R. L. Caskey, G. B. Charlesworth, N. A. Childress, T. Y. Chow, Mary Constable, R. J. Cormier, K. W. Crain, A. E. Danese, D. E. D'Atri, G. W. Day, Hazel Evans, Herta Freitag, Michael Goldberg, A. J. Goldman, Peter Gould, Cornelius Groenewoud, D. J. Hansen, Vern Hoggatt, R. T. Hood, Roger Hou, J. P. Hoyt, Raymond Huck, Louise Hutchinson, A. R. Hyde, P. W. M. John, Edgar Karst, M. S. Klamkin, W. G. Koellner, Sam Kravitz, M. A. Laframboise, L. E. Laird, B. R. Leeds, L. I. Lokomowitz, Robert Lynch, D. C. B. Marsh, Beckham Martin, C. N. Mills, C. S. Ogilvy, Margaret Olmsted, M. J. Pascual, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), C. F. Pinzka, P. W. A. Raine, M. A. Rachid, L. A. Ringenberg, Azriel Rosenfeld, Donald Rubin, C. M.

Sandwick, Sr., E. D. Schell, G. J. Simmons, Bernard Smilowitz, Sister M. Stephanie, A. V. Sylwester, W. R. Talbot, Chih-yi Wang, R. M. Warten, Dale Woods, Roscoe Woods, André Yandl, David Zeitlin, and the proposer. Late solutions by Paul Herzberg and Alan Wayne.

It was pointed out that this problem appears in N. A. Court, *College Geometry*, 2nd ed., p. 93, ex. 19b, and in *Scripta Mathematica*, vol. 16 (1950), p. 167.

A Coin Tossing Procedure

E 1198 [1956, 39]. *Proposed by E. T. Frankel, U. S. Dept. of Health, Education and Welfare, New York, N. Y.*

Describe a coin tossing procedure to determine a random integer, such that all integers from 1 to n have equal probabilities of being selected.

Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J. A simple but inefficient procedure is to label n coins $1, \cdots, n$ and let the choice of the integer k correspond to the event in which all coins but the kth fall tails.

A more efficient procedure is to toss s labeled coins, where s is the smallest integer such that $2^s \geqq n$, and let $1, \cdots, n$ correspond to n of the 2^s possible outcomes.

Also solved by J. L. Alperin, J. L. Baker, G. B. Charlesworth, Michael Goldberg, Vern Hoggatt, A. R. Hyde, Lawrence Isenecker, P. W. M. John, M. S. Klamkin, L. E. Laird, D. C. B. Marsh, J. B. Muskat, Herbert Nadler, C. S. Ogilvy, M. J. Pascual, Walter Penney, Ronald Pyke and S. C. Saunders (jointly), C. M. Sandwick, Sr., E. D. Schell, G. J. Simmons, A. V. Sylwester, W. R. Van Voorhis, and the proposer. Late solutions by Paul Herzberg and Alan Wayne.

Maximum Area Between a Hanging Chain and Its Chord

E 1199 [1956, 39]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

A flexible chain of length L is suspended from its endpoints. Determine the maximum area between the chord joining the endpoints and the hanging curve.

Solution by C. M. Sandwick, Sr., Easton High School, Easton, Pa. Let $-x+w$ and $x+w$ be the abscissas of two points on the curve having equation $y=\cosh x$, such that the figure formed by the arc and the chord joining the two points is similar to the chain of length L and the chord joining its points of suspension. Let A be the plane area bounded by the chain and its chord, and let z be the length of the chord. Then

$$z = L(x^2 + \sinh^2 w \sinh^2 x)^{1/2}/\cosh w \sinh x,$$

$$A = L^2(x \cosh x - \sinh x)/2 \cosh w \sinh^2 x.$$

If w is held constant, $dA/dz=0$ when

$$x = 2(\sinh x \cosh x)/(1 + \cosh^2 x) = 1.6062, \text{ approximately.}$$

For any x, A is a maximum when $w=0$, which occurs when and only when the chord is horizontal. Then

$$A = L^2(x \cosh x - \sinh x)/2 \sinh^2 x \quad \text{and} \quad z = Lx/\sinh x;$$

so the maximum area is approximately $0.1549L^2$, the points of suspension being joined by a horizontal chord whose length is approximately $0.6716L$.

Also solved by Hüseyin Demir, Michael Goldberg, Vern Hoggatt, A. R. Hyde, Lawrence Isenecker, D. C. B. Marsh, J. W. Ross, and Ernst Trost. Late solution by A. J. Goldman.

Editorial Note. Eliminating x from

$$A = L^2(x \cosh x - \sinh x)/2 \sinh^2 x \text{ and } L = 2 \sinh x$$

we find

$$A_{\max} = (L^2 + 4)^{1/2} \sinh^{-1} (L/2) - L.$$

Sum of an Infinite Series

E 1200 [1956, 39]. *Proposed by Sidney Weiner, Washington, D. C.*

Evaluate the convergent infinite series

$$S = \sum_{k=1}^{\infty} \left(\sum_{p=1}^{2k-1} 1/p \right) \Big/ 2k(2k+2).$$

Solution by W. A. Al-Salam, Duke University. We have

$$\begin{aligned} 4S &= \sum_{k=1}^{\infty} \left(\sum_{p=1}^{2k-1} 1/p \right) \Big/ k(k+1) \\ &= \sum_{k=1}^{\infty} 1/k(k+1) + (1/2 + 1/3) \sum_{k=2}^{\infty} 1/k(k+1) \\ &\quad + (1/4 + 1/5) \sum_{k=3}^{\infty} 1/k(k+1) + \cdots . \end{aligned}$$

But

$$\sum_{k=m}^{\infty} 1/k(k+1) = 1/m.$$

Therefore

$$\begin{aligned} 4S &= 1 + \sum_{k=2}^{\infty} [1/(2k-2) + 1/(2k-1)]/k \\ &= 1 + (1/2) \sum_{k=2}^{\infty} 1/k(k-1) + \sum_{k=2}^{\infty} 1/k(2k-1) \end{aligned}$$

$$= 1 + 1/2 + 2\sum_{k=2}^{\infty} [1/(2k-1) - 1/2k]$$
$$= 3/2 + 2[1/3 - 1/4 + 1/5 - 1/6 + \cdots]$$
$$= 3/2 + 2[\ln 2 - (1 - 1/2)]$$
$$= 1/2 + 2 \ln 2$$

and

$$S = 1/8 + (\ln 2)/2.$$

Also solved by G. B. Charlesworth, T. Y. Chow, A. E. Danese, Nathaniel Grossman, Emil Grosswald, Vern Hoggatt, A. R. Hyde, M. S. Klamkin, P. K. Maloof, D. C. B. Marsh, R. M. Meisel, C. S. Ogilvy, Walter Penney, L. L. Pennisi, M. A. Rashid, L. A. Ringenberg, David Rosen, E. D. Schell, R. E. Shafer, O. E. Stanaitis, Chih-yi Wang, G. N. Wollan, David Zeitlin, and the proposer. Late solutions by A. J. Goldman, Virginia Hanly, and Alan Wayne

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4698. *Proposed by H. F. Sandham, Dublin Institute for Advanced Studies, Ireland*

Prove that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m a_n}{m+n+1/6} \leqq 2\pi \sum_{r=0}^{\infty} a_r^2,$$

provided the right-hand side converges.

4699. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let $f(x)$ be a frequency function of finite variance σ^2 and such that

$f(x)\cdot\log f(x)$ is summable ($x \log x$ is defined to be $=0$ if $x=0$). Show that

$$\int_{-\infty}^{\infty} f(x)\log f(x)dx \geqq \log\frac{1}{\sigma\sqrt{2\pi e}}$$

with equality if and only if $f(x)$ is equal almost everywhere to a normal function

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-m)^2/2\sigma^2}.$$

4700. *Proposed by J. W. Clawson, Ursinus College*

It is well known that the midpoints of the diagonals of a 4-line lie on a straight line which has been called the *Newtonian* of the 4-line. It is also known that the Newtonians of the five 4-lines obtained by omitting in turn each of the sides of a 5-line concur in a point which we may call the *Newtonian point* of the 5-line.

Prove (1) that the Newtonian points of the six 5-lines obtained by omitting in turn each of the sides of a 6-line lie on a conic which may be called the *Newtonian conic* of the 6-line; (2) that the Newtonian conics of the seven 6-lines obtained by omitting in turn each of the sides of a 7-line concur in three points, two of which may be imaginary.

4701. *Proposed by Paul Erdös, Technion Mathematics Department, Haifa, Israel*

Let α be complex, $|\alpha|<1$. Prove that there exists a power series $\sum a_k z^k$ with rational integral coefficients, $|a_k| \leqq |1/\alpha^2|$, such that $\sum a_k\alpha^k=0$.

4702. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Stratford, Conn., and W. E. Weissblum, Massachusetts Institute of Technology*

Let

$$\begin{matrix} A_{11}, & A_{12}, & A_{13}, \cdots \\ A_{21}, & A_{22}, & A_{23}, \cdots \\ \cdots & \cdots & \cdots \end{matrix}$$

be an infinite square array of sets of real numbers in (0, 1). If $A_{kn}\to 0$ for each fixed k, does it follow that there exists a diagonal sequence

$$A_{1n_1}, \quad A_{2n_2}, \quad A_{3n_3}, \cdots$$

whose limit is 0?

SOLUTIONS

Prime Divisors of Fibonacci Numbers

4637 [1955, 259]. *Proposed by W. E. Briggs, University of Colorado.*

Let $f(n)$ be the smallest number such that $u_{f(n)}$ is divisible by n, where $u_1=1$, and $u_2=1$ and $u_m=u_{m-1}+u_{m-2}$. Prove (or disprove) that for any prime p greater

than 2 and for all n,

$$f(p^n) = p^{n-1}f(p).$$

Solution by V. D. Gokhale, Atlanta University. We first state four lemmas, of which the first three are obvious and the fourth is easily proved by induction on r.
Lemma 1. If $n>1$ divides u_p, it does not divide u_{p-1}.
Lemma 2.

$$u_{h+k} = u_k u_{h+1} + u_{k-1}u_h.$$

Lemma 3. If p^n divides u_k, it divides u_{h+k} if and only if it divides u_h.
Lemma 4.

$$u_{rm} = \sum_{t=0}^{r-1} {}_rC_t u_{r-t} u_m^{r-t} u_{m-1}^{t},$$

$$u_{rm-1} = \sum_{t=0}^{r-1} {}_rC_t u_{r-t-1} u_m^{r-t} u_{m-1}^{t} + u_{m-1}^{r}.$$

Since

$$u_{f(p^{n+1})}$$

is divisible by p^n, it follows from lemma 3 that $f(p^{n+1})$ is a multiple of $f(p^n)$. Set $f(p^{n+1}) = rf(p^n)$. In the first expansion of lemma 4, all terms except the last are divisible by p^n where $m = f(p^n)$. The last term is

$$r u_{f(p^n)-1}^{r-1} u_{f(p)}.$$

Now p does not divide $u_{f(kp)-1}$. Hence if p^λ is the highest power of p which divides $u_{f(p)}$, $p^{\lambda+1}$ divides

$$u_{rf(p^n)}$$

if and only if p divides r. Hence if $f(p^\lambda) = f(p^n)$, then $f(p^{\lambda+1}) = pf(p^n)$. Hence it follows that if $f(p) = f(p^k)$ then $f(p^{k+1}) = pf(p)$ and by a finite number of steps $f(p^{k+n}) = p^n f(p)$. Hence we have proved: *If for a prime $p>2$, $u_{f(p)}$ is not divisible by p^2, then $f(p^n) = p^{n-1}f(p)$ for every n.*

It does not appear whether $u_{f(p)}$ can ever be a multiple of p^2. For $p=2$, the result differs only slightly. We find $f(2)=3$, $f(2^2) = f(2^3) = 6$, $f(2^n) = 2^{n-2}f(2)$ for $n>2$.

Also solved by Leonard Carlitz, M. S. Klamkin, and Chih-yi Wang.

Editorial Note. Klamkin notes that these results were stated by Lucas (*Comptes Rendus*, Paris, 82, 1876, pp. 1303–1305) as being true also of the slightly more general sequence

$$u_n = \frac{a^n - b^n}{a - b},$$

where a and b are roots of a quadratic equation with integral coefficients. The present problem is the case $a=(1+\sqrt{5})/2$, $b=(1-\sqrt{5})/2$.

Identity

4639 [1955, 371]. *Proposed by L. B. Rall, Oregon State College*

Given a set of distinct numbers $a_0, a_1 \cdots$, such that $a_j \neq a_k$ if $j \neq k$, prove that for all positive integers n,

$$\sum_{j=0}^{n} \prod_{k=0, k\neq j}^{n} \frac{(a_k - a_{n+1})}{(a_k - a_j)} = 1.$$

Solution by Leonard Carlitz, Duke University. The stated identity is an immediate special case of the Lagrange interpolation formula (see, *e.g.*, Weisner, *Theory of Equations*, New York, 1938, p. 57 and ex. 4(b), p. 58). However, a direct proof is quite simple. Put

$$L_j(x) = \prod_{k=0, k\neq j}^{n} \frac{a_k - x}{a_k - a_j}.$$

Clearly $L_j(a_j)=1$ while $L_j(a_k)=0$ for $0 \leqq k \leqq n$, $k \neq j$. Consequently

$$\sum_{j=0}^{n} L_j(x) = 1 \tag{1}$$

for $x=a_0, a_1, \cdots, a_n$. The left member of (1) is a polynomial of degree $\leqq n$ which is equal to 1 for $n+1$ distinct values of x. Therefore (1) is an identity.

Also solved by W. J. Blundon, R. H. Breusch, N. J. Fine, A. R. Hyde, D. C. B. Marsh, W. V. Parker, Berthold Schweizer, M. F. Smiley, Chih-yi Wang, and the Proposer. Late solution by M. S. Klamkin.

Subgroups

4640 [1955, 371]. *Proposed by M. L. Keedy, University of Nebraska*

Let H be a proper subgroup of a group G and $\overline{H}$ its complement in G. Show that $H'=\overline{H}\,\overline{H}$ is a normal subgroup of G, and that it equals either G or H. Characterize those subgroups for which $H'=H$.

Solution by Virginia S. Hanly, Ohio State University. If the index of H in G is 2, then $H'=H$. This follows from the fact that for any $\bar{h} \in \overline{H}$, $\overline{H}=\bar{h}H=H\bar{h}$, so that for any $a, b \in \overline{H}$, $ab \in H$.

If the index of H in G is >2, then $H'=G$. We prove this by showing $G \subset H'$. Let g be any element of G and let a be any element of $\overline{H}$ which is not an element of gH. Such an a exists since the index of H in G exceeds 2. Then $g=ax$ has a solution $x \in \overline{H}$, since $x \in H$ implies $a \in gH$ contrary to hypothesis. In either case H' is clearly a normal subgroup.

Also solved by J. D. Baum, W. J. Blundon, G. U. Brauer, R. H. Breusch, J. R. Byrne, N. J. Fine, A. H. Frey, Jr. and R. J. Wisner, V. D. Gokhale, D. S.

Greenstein, J. Horváth, H. H. Johnson, Tung-Po Lin, D. C. B. Marsh, Katuyosi Memezawa, T. O. Moore, G. C. Preston, Azriel Rosenfeld, H. D. Ruderman, E. V. Schenkman, J. D. Thomas, J. V. Whittaker, and the Proposer. Late solution by J. B. Kelly.

Quadratic Residues

4641 [1955, 371]. *Proposed by E. P. Starke, Rutgers University*

Given N, however large, prove that there exist primes P such that each of the numbers 1, 2, 3, $\cdots$, N is a quadratic residue (mod P).

I. *Solution by Harley Flanders, University of California, Berkeley.* We shall prove, by induction on N, that there are infinitely many such primes. The proof uses (1) the Dirichlet theorem that each arithmetic progression $x \equiv a \pmod{m}$, where $(a, m) = 1$, contains an infinitude of primes, and (2) an easy consequence of the Gauss quadratic reciprocity law which asserts that if $m_1 \equiv m_2 \pmod{8ak}$, then $(a \mid m_1) = (a \mid m_2)$, where $(a \mid m)$ is the Jacobi symbol and k is arbitrary.

For $N = 1$, the result is clear. Suppose now that $(1 \mid P) = (2 \mid P) = \cdots = (N-1 \mid P) = 1$ has an infinitude of solutions, P, but that almost all of these satisfy $(N \mid P) = -1$. Take two such, P_1 and P_2. Then $(1 \mid P_1P_2) = (2 \mid P_1P_2) = \cdots = (N-1 \mid P_1P_2)$ and $(N \mid P_1P_2) = (N \mid P_1)(N \mid P_2) = 1$. Any prime Q satisfying $Q \equiv P_1P_2 \pmod{8 \cdot N!}$ will satisfy $(1 \mid Q) = \cdots = (N \mid Q) = 1$, and there are infinitely many such Q.

II. *Solution by Virginia S. Hanly, Ohio State University.* Let $p_1, p_2, \cdots, p_n$ be the odd primes $\leqq N$. The Legendre symbol $(x \mid P)$ is factorable, whence $(x \mid P) = 1$ for $x = 1, 2, \cdots, N$ if and only if $(x \mid P) = 1$ for $x = 2, p_1, p_2, \cdots, p_n$. Let S be the system $\{1 + 8p_1p_2 \cdots p_n t\}$. Let P be one of the infinitely many primes (by Dirichlet's theorem) in S. Then $P \equiv 1 \pmod{x}$ for $x = 8, p_1, p_2, \cdots, p_n$, whence $(2 \mid P) = 1$ and $(P \mid x) = 1$ for $x = p_1, p_2, \cdots, p_n$. Since $P \equiv 1 \pmod{4}$ it follows from the reciprocity law that $(x \mid P) = 1$ for $x = 2, p_1, p_2, \cdots, p_n$ so that each of the numbers 1, 2, $\cdots$, N is a quadratic residue (mod P).

Also solved by W. J. Blundon, R. H. Breusch, Leonard Carlitz, J. Horváth, D. C. B. Marsh, A. Walfisz, and Morgan Ward.

Editorial Note. Walfisz points out a more general result of V. R. Fridlender, On the least nth power non-residue, *Doklady Akad. Nauk USSR* (*N.S.*) 66, (1942), 351–352. See also *Mathematical Reviews*, 10 (1949), 684.

A Minimax Problem

4642 [1955, 371]. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Stratford, Conn.*

Let $a_2, a_3, \cdots, a_n$ be arbitrary. Prove that there must be an x in (0, 1) such that

$$| x + a_2x^2 + \cdots + a_nx^n | \geqq \frac{1}{n} \tan \frac{\pi}{4n} \cdot$$

Solution by D. S. Greenstein, University of Pennsylvania. Let

$$P(x) = x + a_2x^2 + \cdots + a_nx^n.$$

Since for all real x

$$|P(x)| \geqq |\operatorname{Re} P(x)| = |x + (\operatorname{Re} a_2)x^2 + \cdots + (\operatorname{Re} a_n)x^n|$$

it will suffice to consider the case where $P(x)$ is a real polynomial.

Suppose

$$|P(x)| < \frac{1}{n}\tan\frac{\pi}{4n} \qquad (0 \leqq x \leqq 1).$$

Consider now the polynomial

$$P^*(x) = (-1)^n\left(\frac{1}{n}\tan\frac{\pi}{4n}\right)T_n\left\{x\left(1 + \cos\frac{\pi}{2n}\right) - \cos\frac{\pi}{2n}\right\},$$

$T_n(x)$ being that polynomial of Tchebychev for which $T_n(\cos\theta) = \cos n\theta$. It is readily verified that $P^*(0) = 0$ and $P^{*\prime}(0) = 1$, so that $P^*(x)$ is of the same form as $P(x)$. Furthermore

$$|P^*(x)| \leqq \frac{1}{n}\tan\frac{\pi}{4n},$$

$$P^*(x_k) = (-1)^{n+k}\frac{1}{n}\tan\frac{\pi}{4n}, \qquad (k = 1, \cdots, n),$$

where $0 < x_1 < \cdots < x_n = 1$, and

$$x_k = \frac{\cos\dfrac{\pi}{2n} - \cos\dfrac{k\pi}{n}}{1 + \cos\dfrac{\pi}{2n}}.$$

It is now obvious from the graph that there must exist points ξ_k, $(k = 1, \cdots, n-1)$ such that $x_k > \xi_k > x_{k+1}$ and $P^*(\xi_k) = P(\xi_k)$. Since, furthermore, $P(x) - P^*(x)$ has at least a double zero at the origin, it follows that $P(x) - P^*(x)$ has at least $n+1$ zeros, hence that $P(x) = P^*(x)$. This being impossible, there must be at least one x such that $0 < x \leqq 1$ and $|P(x)| \geqq \tan(\pi/4n)/n$.

Also solved by R. H. Breusch, Gabor Szegö, Chih-yi Wang, and the Proposer.

Editorial Note. Szegö points out that the present result, and more, is implied in a paper by I. Schur, Maximum des absoluten Betrages eines Polynoms, *Math. Zeitschrift*, 4 (1919), p. 280. Wang finds an equivalent result in S. Bernstein, *Leçons sur les Propriêtés Extrêmales et la Meilleure Approximation des Fonctions Analytiques d' une Variable Rêele*, Paris, 1926, pp. 36–37.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All new material for this section should be sent to the incoming editor, Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

Elementary Statistics for Students of Social Science and Business. By R. Clay Sprowls. New York, McGraw-Hill Book Company, 1955. xiii+392 pages. $5.50.

The underlying theme of this text is well-enunciated by the author in his preface in the following four statements:

"1. Statistics is the science of making decisions based upon incomplete knowledge.

2. Implicit in making a decision is the acceptance of one of two alternative hypotheses.

3. Associated with the acceptance of either hypothesis is the risk that the one selected is incorrect.

4. The function of statistics is to aid in the formulation of the alternative hypotheses and the evaluation of the associated risks of being wrong."

In this reviewer's opinion, the author is unusually successful in maintaining this principle as the central theme of his text. Sprowls' repeated insistence on the formulation of rules of inductive behavior and the evaluation of their consequences by means of operating characteristics as the *raison d'être* of statistics is convincingly and effectively made a part of the student's belief.

Two sampling experiments introduced in the first chapter form a basis for giving the student concrete experience with the problems of inference and are used also in later sections of the text to illustrate further developments. In Chapter 3 the student is faced with the problem of applying these concepts to a personal decision problem, whose value, to quote the author, "lies in its ability to bring home to the student the basic methods of statistics and their relationship to everyday decisions, even though such decisions may not require the use of statistical data." A small research project, suggested to be started at about the middle of the course, is well-presented in the discussion of Chapter 6, "Statistical Research Methodology", which can be read profitably by the student without benefit of lectures. Again, the author's comment, "Students thus realize the complexity of apparently simple problems", seems significant.

In general, the problems throughout the text are meaningful and varied in fields of application. Except for the four chapters on Time Series and Index Numbers, the text material and exercises are in no way restricted to the business and economic aspects of statistics.

Among more specific highlights of the text are:

1. The author's use of pre-learning and anticipation. Concepts are frequently introduced briefly in one context, then later presented more formally and developed in detail.

2. The requirement of a certain amount of abstract reasoning imposed by the text. This reviewer shares the author's belief that there has been too much writing down to the level of the student. In Chapter 2, after a brief discussion of classical notions of probability, the author presents quite simply an axiomatic definition of discrete probability. (A later brief presentation of the continuous case seems less fortuitous.)

3. This is very definitely a text in the category of the so-called "non-mathematical" statistics treatments; in general, no proofs or derivations are given, but the author is usually scrupulous in presenting underlying assumptions and in making precise statements of what are essentially mathematical theorems to be employed. It is because of the author's success in this last respect, in the setting of the central theme of the text cited earlier, that this treatment avoids the frequently encountered recipe-book stigma. To this reviewer it appears that more "mathematical reasoning" has been incorporated into Sprowls' text than in other texts of this category.

An exception of the statement concerning the author's scruples in presenting underlying assumptions seems to be present in his discussion of optimal properties of various tests. It is doubtless good pedagogy to rely on the student's intuition at the outset in picking, for example, a left-hand tail region for rejecting the hypothesis $p=p_0$ in favor of the alternative hypothesis $p=p_1$, with $p_1<p_0$, for the class of binomial distributions. It would seem more in the spirit of Sprowls' philosophy, however, then to make some statement other than the dogmatic assertion (not as a theorem, but with the impact of "obviously") that such a choice in this case leads to a test with best power. (A reference is given at the end of Chapter 4 to Neyman's text.)

Finally, the author's selection of topics seems to have been made judiciously and realistically. For students in fields other than business or economics who may not be interested in the chapters on Time Series and Index Numbers, material only briefly introduced in the last chapter would furnish a springboard for study of topics such as inference problems involving two or more samples, which otherwise would have to be omitted.

The treatment of a time series from the point of view of a stochastic process, with statistical tests suggested for the detection of systematic components, is skillfully presented and, again, is consistent with the four-fold theme of the text. The student of economics and business is initiated in these chapters into some of the current controversies in these fields.

An instructor's manual furnished by the author gives useful suggestions based on his experience in teaching the course, including a suggested time-schedule based on a semester course of 45 lectures and 15 two-hour laboratory periods. The manual is also illuminating in conveying Sprowls' philosophy and enthusiasm with some prospect of contagion.

GRACE E. BATES
Mt. Holyoke College

Vector Analysis, By H. E. Newell, Jr. McGraw-Hill Book Company, Inc., New York, 1955. 216 pp. $5.50.

This new book by a well-known physicist at the Naval Research Laboratory is intended for use in a one-semester course either at the undergraduate or beginning graduate level. Part I of *Vector Analysis* treats the algebra and calculus of vectors and includes a chapter reviewing some concepts from the calculus. Part II, which occupies somewhat less than half of the book, deals with applications of the theory to mechanics and electromagnetism.

Dr. Newell, in the preface to his book, states that he is writing primarily for the engineer and physicist and hopes that his readers will make vector concepts an integral part of their physical and intuitive reasoning. He leans heavily on the use of problems to complete and enrich the text material, which is, perhaps, a little sparse for general use. Indeed, a general criticism of the book is that no freedom of choice has been left to the teacher in the selection of topics to be studied. Neither would this book be of too much value on a reference shelf, for many of the most useful results are only to be found scattered in the exercises.

The author has been quite careful in his statements of theorems and definitions, but has omitted most of the arguments necessary to make his proofs rigorous, leaving these to be filled in, as exercises, by his readers. He has, however, made the arguments plausible and has, in some cases, indicated how a rigorous proof would be carried out.

The format of the book is neat and attractive, but the figures, in many instances, could be improved visually.

S. C. LOWELL
New York University

CORRECTION

My review of Vance's *Trigonometry** (Addison-Wesley, 1954) erroneously claimed that this book contains "the first completely satisfactory derivation of composite angle formulas to be found in an elementary text." I should have added the qualifying phrase, "to the reviewer's knowledge", to that sentence. The book *Elementary Analysis*, by K. O. May, among others, contains a somewhat similar derivation. I hereby apologize to Professor May and all concerned.

J. G. HOCKING
Michigan State College

* This MONTHLY, vol. 62, 1955, pp. 458–459.

NEW BOOKS RECEIVED

How to Calculate Quickly—Rapid Methods in Basic Mathematics. By Henry Sticker. New York, Dover Publications, Inc., 1956. 256 pages. Paperbound $1.00.

Risk and Gambling, The Study of Subjective Probability. By John Cohen and Mark Hansel. New York, Philosophical Library, Inc., x+153 pages. $3.50.

Solid Geometry. By D. T. Sigley and W. T. Stratton. New York, The Dryden Press, Inc., 1956. x+197 pages. $2.60.

Integral Functions. No. 44 in Cambridge Tracts in Mathematics and Mathematical Physics. General Editors P. Hall and F. Smithies. By M. L. Cartwright. New York, Cambridge University Press, 1956. 135 pages. $3.50.

Elementare Differentialgeometrie. By W. Haack. Basel, Verlage Birkhäuser, 1955. viii+239 pages. SFr. 22.

Combinatorial Topology, Vol. I. By P. S. Aleksandrov. Rochester, New York, The Graylock Press, 1956. xvi+225 pages. $4.95.

Weather Analysis and Forecasting, Vol. I, Motion and Motion Systems, 2nd Ed. By Sverre Petterssen. New York, McGraw-Hill Book Co., 1956. xix+428 pages. $8.50.

Calculus. By Jack R. Britton. New York, Rinehart and Company, Inc., 1956. xiv+584 pages. $6.50.

Royal Society Mathematical Tables, 2, Rectangular-Polar Conversion Tables. By E. H. Neville. New York, Cambridge University Press, 1956. xxxii+109 pages. $5.50.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

LECTURES IN MATHEMATICAL STATISTICS AT CATHOLIC UNIVERSITY OF AMERICA

A lecture series in the field of mathematical statistics will be held by the Department of Mathematics of Catholic University at the beginning of the academic year 1956–57, probably in October. It is planned to invite prominent mathematical statisticians from the east coast to give addresses. The cooperation of other universities and interested organizations in the area will be sought. Enquiries concerning the lecture series may be addressed to the Department of Mathematics, Catholic University of America, Washington, D. C.

MATHEMATICAL MONOGRAPH NO. 16

A university library is interested in purchasing a copy of R. D. Carmichael, "Diophantine Analysis," published by Wiley in 1915 as Mathematical Monograph No. 16. Anyone having a copy for sale should write to the office of the Mathematical Association of America.

WILLIAM B. ORANGE MATHEMATICS PRIZE COMPETITION

The Mathematics Department of Los Angeles City College held the sixth annual William B. Orange Mathematics Prize Competition for students in Los Angeles High Schools on May 18, 1956. There were 167 students enrolled in the Competition. These students represented 35 local high schools.

Team winner was Van Nuys High School. Individual winners were Michael Raugh of Van Nuys High School, Howard Weisberg and Harold Stark of Hamilton High School, Louis Jaeckel of Van Nuys High School, and Floris Y. Tsang of Los Angeles High School.

PERSONAL ITEMS

Dean W. L. Duren, Jr., University of Virginia, will represent the Association at the inauguration of Chancellor G. C. Simpson of Mary Washington College of the University of Virginia on October 18–19, 1956.

Professor H. M. Gehman of the University of Buffalo was the delegate of the Association at the Convocation commemorating the Centennial of the Establishment of Niagara University on May 10, 1956.

The John Simon Guggenheim Memorial Foundation has announced the granting of Fellowship Awards in Mathematics and Statistics to the following for research as indicated: Professor C. A. Truesdell, III, of Indiana University, studies of the history of the mathematical theory of elasticity; Professor E. E. Moise of the University of Michigan, studies of classical problems in topology of higher dimensional euclidean manifolds; Professor J. L. Hodges, Jr., University of California, studies in the use of combinatorial analysis in mathematical statistics; Professor W. H. Fuchs of Cornell University, study of the deficient values of meromorphic functions of finite order; Professor E. W. Barankin, University of California, quantitative studies toward a theory of human behavior.

The National Science Foundation has announced the appointment of Professor E. J. McShane of the University of Virginia and Dr. Warren Weaver of the Rockefeller Foundation to the National Science Board.

Professor A. A. Albert of the University of Chicago has been appointed to membership on the Postdoctoral Fellowship Board of the National Academy of Sciences.

Professor R. P. Boas, Jr., of Northwestern University has received a two-year grant for research on trigonometric series from the National Science Foundation.

Professor Louis Brand of the University of Cincinnati has been awarded an honorary degree of Doctor of Science by the University.

Professor T. Y. Thomas of Indiana University has been promoted to Distinguished Service Professor, the University's highest academic rank.

Auckland University College, University of New Zealand, announces: Professor H. G. Forder, who has resigned as Chairman of the Department of Mathematics, has been appointed Professor Emeritus and is continuing to teach at the College; Professor Frederick Chong of Sydney University, Australia, has been appointed Chairman of the Department.

Brown University reports the following: Dr. D. A. Buchsbaum of the University of Chicago has been appointed to an assistant professorship; Dr. W. D. Barcus of Princeton University has been appointed to an instructorship; Assistant Professor M. R. Demers of the University of Nevada has been appointed Research Assistant; Dr. Jean-Pierre Meyer has been appointed a research associate; Dr. David Harrison of Princeton University has been appointed to a teaching internship; Professor Maurice Heins and Assistant Professor John Wermer are on leave of absence for 1956–57 and are at the Institute for Advanced Study.

At Cornell University: Associate Professor Harry Pollard has been promoted to a professorship; Dr. H. D. Block, Dr. C. S. Herz, and Dr. D. B. Ray have been promoted to assistant professorships; Dr. Isaac Namioka of the University of California and Dr. Wolfgang Rindler of the University of London have been appointed to instructorships; Dr. Jerome Blackman of Syracuse University has been appointed Acting Assistant Professor; Dr. Ambikeshwar Sharma of the University of Lucknow has been appointed Visiting Assistant Professor; Professor G. A. Hunt will be on leave at Princeton University during 1956–57; Dr. L. A. Rubel has been awarded a National Science Foundation Fellowship and will spend the year 1956–57 at the Institute for Advanced Study.

Emory University announces the following: Professor C. G. Latimer has retired as Chairman of the Department of Mathematics but will continue as Professor of Mathematics; Associate Professor Trevor Evans has been appointed Chairman of the Department; Mr. E. E. Grace of the University of North Carolina and Assistant Professor Henry Sharp of Georgia Institute of Technology have been appointed to assistant professorships; Mr. F. L. Hardy, formerly a graduate assistant at the University, has been appointed to an instructorship.

Illinois Institute of Technology reports: Mr. R. J. Mihalek and Mr. Barth Pollak, a fellow at Princeton University, have been appointed to instructorships; Associate Professor S. F. Bibb has retired.

Indiana University announces: Professor J. W. T. Youngs has been appointed Chairman of the Department of Mathematics and Director of the Graduate Institute for Mathematics and Mechanics; the Institute will become that portion of the Mathematics Department concerned with graduate affairs.

At Massachusetts Institute of Technology: Dr. Hartley Rogers, Jr., formerly a Visiting Lecturer at the Institute, and Assistant Professor I. M. Singer of the University of California at Los Angeles, have been appointed to assistant professorships; Dr. W. L. Baily of Princeton University, Dr. R. T. Prosser of Duke University, and Dr. E. M. Stein, University of Chicago, have been appointed C. L. E. Moore Instructors; Mr. K. J. Hoffman of the University of California at Los Angeles and Mr. Leonard Roberts, previously a research associate at the Institute, have been appointed to instructorships; Dr. M. A. Kervaire of Berne University, Switzerland, has been appointed Visiting Lecturer in Mathematics; Professor J. B. Diaz of the University of Maryland and Dr. S. M. Ulam, senior scientist at Los Alamos Scientific Laboratory, have been appointed to visiting professorships; Assistant Professor R. A. Clark of Case Institute of Technology has been appointed Visiting Assistant Professor; Dr. K. F. Roth, lecturer at the University of London, and Dr. J. T. Stuart of the National Physical Laboratory, Teddington, England, have been appointed to visiting lectureships; Dr. Marcel Berger of the Institute of Mathematics, University of Strasbourg, has been appointed Research Associate; Associate Professor R. V. Kadison of Columbia University is on partial leave and is also Research Associate at the Institute; Assistant Professor J. F. Nash, Jr. is on leave of absence at the Institute for Advanced Study; Associate Professor G. B. Thomas, Jr., is on leave of absence at Stanford University; Professors G. W. Whitehead and Norbert Wiener have returned from their leaves of absence.

Northwestern University reports the following: Dr. W. C. Fox of Massachusetts Institute of Technology has been appointed to an assistant professorship; Mr. W. L. Hoyt of the University of Chicago has been appointed to an instructorship; Assistant Professor J. A. Zilber of the University of Illinois has been appointed Lecturer.

Purdue University announces the following: Professor C. F. Kossack, previously director of the Statistical Laboratory, has been appointed Head of the Department of Mathematics; Professor V. L. Anderson is now Head of the Statistical Laboratory; Dr. Paul Brock of the ElectroData Corporation, Pasadena, California, has been appointed to an associate professorship; Assistant Professors V. L. Anderson, John Dyer-Bennet, Leonard Gillman, Meyer Jerison, and Henry Teicher have been promoted to associate professorships; Dr. A. H. Copeland, Jr., Dr. G. L. Krabbe, Mr. Clarence Lane, and Dr. C. J. Neugebauer have been promoted to assistant professorships; Mr. W. A. Stock who is at the Hammond Extension Center has been promoted to an assistant professorship; Professor Melvin Henriksen is on leave of absence during 1956–57 and is at the Institute for Advanced Study on an Alfred P. Sloan Foundation grant; Dr. A. H. Copeland received a Purdue University research grant for the summer of 1956.

United States Naval Academy reports: Associate Professors R. P. Bailey, T. J. Benac, L. H. Chambers, and J. P. Hoyt have been promoted to professorships;

Assistant Professor Joseph Milkman has been promoted to an associate professorship; Associate Professor R. W. Rector is on leave of absence for 1956–57 at the Ramo-Wooldridge Corporation, Los Angeles, California.

At the University of Nebraska: Dr. D. L. Guy of Washington University and Miss F. Eloise Pool have been appointed to instructorships; Associate Professor W. G. Leavitt has been promoted to a professorship; Dr. F. W. Anderson has been promoted to an assistant professorship; Professors F. W. Anderson and M. A. Basoco received Faculty Summer Research Fellowships; Professor H. B. Ribeiro is on leave of absence for six months at the University of Münster.

Miss Lucille J. Albers, previously a data processing mathematician for Land-Air, Dayton, Ohio, has a position as a mathematician at the Air Force Armament Center, Eglin Air Force Base, Florida.

Assistant Professor E. L. Arnoff of Case Institute of Technology has been promoted to an associate professorship. Professor Arnoff is also Assistant Director of the Operations Research Group.

Dr. Archie Blake, formerly an advisory engineer for Westinghouse Electric Corporation, Baltimore, Maryland, has a position as a systems staff mathematician at the Bendix Aviation Corporation, Detroit, Michigan.

Mr. R. M. Blumenthal of Cornell University has been appointed to an instructorship at the University of Washington.

Mr. A. P. Boblétt, formerly chief statistician at the U. S. Naval Ammunition Depot, Crane, Indiana, has accepted a position as Mathematical Consultant at the U. S. Naval Ordnance Laboratory, Corona, California.

Dr. S. R. Bodner, recently a senior mathematician at Republic Aviation Corporation, Farmingdale, New York, has a position as a senior scientist at the Avco Manufacturing Company, Stratford, Connecticut.

Mr. L. M. Bostick, previously a graduate assistant at Eastern New Mexico University, has a position as a nuclear engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Professor Louis Brand, head of the Department of Mathematics of the University of Cincinnati, has been appointed Whitney Visiting Professor at Trinity College, Hartford, Connecticut, for the academic year 1956–57.

Professor W. B. Carver of Cornell University was in charge of the program in mathematics for the Shell Merit Fellows at the University during the summer of 1956. This program was sponsored by the Shell Oil Company for high school science and mathematics teachers. Dr. H. D. Block of the university assisted Professor Carver in this program.

Dr. C. C. Chang of Cornell University has been appointed to an assistant professorship at the University of Southern California.

Acting Dean J. O. Chellevold of Wartburg College has been appointed Dean.

Assistant Professor S. D. Conte of Wayne University is on sabbatical leave and is with the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor E. A. Davis of the University of Nevada has been ap-

pointed to an assistant professorship at the University of Utah.

Assistant Professor R. B. Davis of the University of New Hampshire has been appointed to an associate professorship at Syracuse University.

Associate Professor R. D. Depew of State Teachers College, Florence, Alabama, has a position as a mathematician and instructor at Lincoln Laboratory, Lexington, Massachusetts, for the International Business Machines Corporation.

Assistant Professor H. W. Doss, Jr., of the University of Omaha has been appointed a research associate at the Engineering Research Institute, University of Michigan.

Mr. D. G. B. Edelen of Johns Hopkins University has accepted a position as an engineer for the Glenn L. Martin Company, Middle River, Maryland.

Assistant Professor M. P. Emerson of Harpur College has been appointed to an associate professorship at Southwest Missouri State College.

Assistant Professor J. M. Feld of Queens College has been promoted to an associate professorship.

Assistant Professor W. R. Ferrante of Lafayette College has been appointed to an assistant professorship in the Department of Mechanical Engineering at the University of Rhode Island.

Assistant Professor R. I. Fields of the Speed Scientific School, University of Louisville, has been granted a year's leave of absence by the University and is serving as an assistant professor for the year at Virginia Polytechnic Institute.

Associate Professor D. T. Finkbeiner, II, of Kenyon College has been promoted to a professorship.

Dr. H. K. Flesch of Cornell University has accepted a position with the Federal Telecommunication Laboratories, Nutley, New Jersey.

Associate Professor Harriet M. Griffin of Brooklyn College has been promoted to a professorship.

Professor D. W. Hall of the University of Maryland has been appointed to a professorship at Harpur College.

Associate Professor P. R. Halmos of the University of Chicago has been promoted to a professorship.

Associate Professor A. E. Halteman, previously of the University of Idaho, has a position with the Electronic Defense Laboratory, Mountain View, California.

Dr. Carl Hammer, formerly a senior research engineer at Franklin Institute, is Director of the UNIVAC European Computer Center, Germany.

Associate Professor E. H. C. Hildebrandt of Northwestern University has been awarded a Fulbright lectureship to teach mathematics at the Higher Teachers Training College in Baghdad, Iraq, during the academic year 1956–57.

Associate Professor P. G. Hodge, Jr., of Polytechnic Institute of Brooklyn has been promoted to the position of Professor of Applied Mechanics.

Mr. J. G. Horne, Jr., of Tulane University has been appointed to an assistant professorship at the University of Kentucky.

Dr. Rufus Isaacs of Lockheed Aircraft Corporation, Los Angeles, California, has accepted a position with the Hughes Aircraft Company, Culver City, California.

Mr. D. H. Jones, previously a teaching fellow at the University of Washington, is employed as a scientist at the Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Associate Professor C. W. Jordan, Jr., of Williams College has been promoted to a professorship.

Associate Professor Irving Kaplansky of the University of Chicago has been promoted to a professorship.

Assistant Professor L. A. Kokoris of Washington University has been appointed Visiting Lecturer at Yale University.

Mr. S. R. Lenihan, formerly a thermodynamics engineer for Consolidated-Vultee Aircraft Corporation, San Diego, California, has a position as a nuclear engineer for Ralph M. Parsons Company, Los Angeles, California.

Mr. A. M. Linn has a position as a mathematician with Pratt-Whitney Aircraft Corporation, East Hartford, Connecticut.

Dr. Mark Lotkin, previously a mathematician at Republic Aviation Corporation, Farmingdale, New York, is Chief of the Mathematical Section of the Advanced Development Division, Avco Manufacturing Company, Stratford, Connecticut.

Assistant Professor Nathaniel Macon of Alabama Polytechnic Institute is employed now as a numerical methods analyst with the General Electric Company, Cincinnati, Ohio.

Dr. W. E. Mientka of the University of Massachusetts has been appointed to an assistant professorship at the University of Nevada.

Dr. Clair E. Miller of the Ramo-Wooldridge Corporation has accepted a position as research mathematician with the California Research Corporation, Richmond, California.

Dr. M. E. Muller of Cornell University has accepted a position with International Business Machines Corporation, New York, New York.

Dr. Rufus Oldenburger, director of research for the Woodward Governor Company, Rockford, Illinois, has been appointed Professor of Engineering Sciences and Mechanical Engineering at Purdue University.

Mr. F. J. Palas, formerly a graduate student at the University of Oklahoma, has been appointed to an assistant professorship at Southern Methodist University.

Professor T. S. Peterson of Portland State College has been appointed Chairman of the Division of Science.

Assistant Professor Gustave Rabson of Antioch College has been promoted to an associate professorship.

Mr. Alfred Reichenthal, previously an engineer with the American Bosch Arma Corporation, New York, New York, has accepted a position as a research engineer with Hughes Aircraft Company, Los Angeles, California.

Mr. A. T. Rice, formerly a mathematician for the United States Air Force, Washington, D. C., is now a test calculations liaison specialist for the General Electric Company, Lynn, Massachusetts.

Associate Professor C. L. Rich of Stout State College is retiring from teaching this year.

Dr. P. R. Rider of the Wright Air Development Center, Wright-Patterson Air Force Base, Ohio, spent the summer on temporary duty with the Summer Research Group of the Holloman Air Development Center, New Mexico.

Dr. Lawrence Rosenfeld, previously a senior engineer at Raytheon Manufacturing Company, has a position as Head of Operations Research and Mathematical Services, Melpar, Inc., Cambridge, Massachusetts.

Assistant Professor W. C. Royster of Alabama Polytechnic Institute has been appointed to an assistant professorship at the University of Kentucky.

Mr. Jerome Sacks of Cornell University has been appointed to an instructorship at California Institute of Technology.

Dr. O. P. Sanders of Oklahoma Agricultural and Mechanical College has been appointed to an associate professorship at Southeastern State College, Oklahoma.

Dr. R. C. Seber of the State University of Iowa has been appointed to an assistant professorship at Western Michigan College of Education.

Assistant Professor A. L. Shields of Tulane University has been appointed to a visiting assistant professorship at the University of Michigan.

Associate Professor James Singer of Brooklyn College has been promoted to a professorship.

Associate Professor Jerome C. Smith of Lafayette College has been appointed Professor and Head of the Department of Mathematics of High Point College.

Mr. C. B. Solloway of the California Institute of Technology has been appointed to an assistant professorship at the University of Southern California.

Dr. W. C. Swift of Cornell University has accepted a position with Bell Telephone Laboratories, Murray Hill, New Jersey.

Mr. P. Y. Tani, previously a research engineer at North American Aviation, Los Angeles, California, has been appointed a research associate at Willow Run Laboratories, University of Michigan, Ypsilanti, Michigan.

Mr. R. P. Tapscott, a research assistant at Purdue University, has a position as a project engineer at the Allison Division of General Motors, Indianapolis, Indiana.

Mr. R. L. Uschold of Nazareth College, Rochester, New York, has been appointed to an instructorship at Canisius College.

Dr. M. A. Woodbury, previously at George Washington University, has been appointed Research Professor of Mathematics at New York University.

Dr. C. T. Yang, a member of the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Pennsylvania.

Dr. R. E. Zink has been appointed to an assistant professorship at Purdue University.

Professor Emeritus W. J. Berry of Polytechnic Institute of Brooklyn died on March 5, 1956. He was a charter member of the Association.

Professor Emeritus A. D. Butterfield of the University of Vermont died on August 22, 1955. He was a member of the Association for 31 years.

Professor G. R. Clements (retired) of the United States Naval Academy died on March 25, 1956. He was a member of the Association for 39 years and was a Governor of the Maryland-District of Columbia-Virginia Section from 1950–53.

Professor V. D. Gokhale, Chairman of the Department of Mathematics of Atlanta University, died on June 1, 1956.

Professor Emeritus G. W. Mullins of Columbia University died on March 11, 1956. He was a member of the Association for 36 years.

Captain A. R. Park, professor of mathematics and director of guidance and testing at Wentworth Military Academy, died on March 23, 1956. A memorial medal award (known as Capt. A. R. Park Memorial Medal for Proficiency in Mathematics) has been established and will be given to the junior college cadet excelling in mathematics.

Dr. M. M. Slotnick, a geophysical consultant with the Standard-Vacuum Oil Company, New York, New York, died May 7, 1956. He was a member of the Association for 19 years.

Professor Emeritus P. F. Smith of Yale University died June 3, 1956.

Associate Professor C. F. Strobel of North Carolina State College died April 21, 1956. He was a member of the Association for 20 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1956, by a mail vote of the membership of the Association in the Sections indicated:

Illinois	E. C. Kiefer, Millikin University
Iowa	Bernard Vinograde, Iowa State College
Louisiana-Mississippi	Z. L. Loflin, Southwestern Louisiana Institute
Maryland-District of Columbia-Virginia	O. J. Ramler, Catholic University of America

Michigan	B. M. Stewart, Michigan State University
Minnesota	G. K. Kalisch, University of Minnesota
Philadelphia	N. J. Fine, University of Pennsylvania
Southern California	P. H. Daus, University of California at Los Angeles
Texas	C. R. Sherer, Texas Christian University

Almost 35% of those eligible to vote in the above elections cast ballots for the election of a sectional governor. In the case of two sections, votes were received from more than 50% of the membership. In the Louisiana-Mississippi Section, the percentage of votes cast was 57% and in the Iowa Section 52%.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 127 persons have been elected to membership by the Board of Governors on applications duly certified.

S. L. ANDERSON, Student, Seattle University.

G. A. ANDREWES, M.S. (Massachusetts) Instr., Technical High School, Springfield, Mass.

BROTHER AUBIN, C.F.X., M.S. (St. John's U.) Chairman, Department of Mathematics, Mount St. Joseph High School, Baltimore, Md.

P. M. BAILYN, Student, Cooper Union.

J. G. BENNETT, B.S. (Trinity U.) Champaign, Illinois.

R. L. BERGGREN, M.A. (Minnesota) Computing Engr., Douglas Aircraft Co., Long Beach, Calif.

W. S. BISHOP, M.S. (Arkansas) Asst. Professor, Howard College.

R. E. BLEWSTER, JR., Student, Agricultural and Mechanical College of Texas.

CAROL A. A. BLOOMQUIST, Student, University of Minnesota.

N. R. BONAM, Student, Kansas State College.

LILLIAN K. BRADLEY, M.A. (Michigan) Chairman, Department of Mathematics, Texas College.

G. Y. BREITBARD, Student, California Institute of Technology.

W. J. BUCKINGHAM, Student, Stanford University.

R. E. BURGESS, B.S. (Purdue) Senior Math., Librascope, Burbank, Calif.

W. H. BURGIN, JR., Student, Dartmouth College.

W. R. CARNES, M.S. (Georgia I.T.) Instr., Georgia Institute of Technology.

BOMSHIK CHANG, M.A. (Seoul National U.) Teaching Asst., University of British Columbia.

R. K. CLARK, Student, Texas Christian University.

E. L. COHEN, Student, Harvard University.

R. T. CRAIG, Student, College of the Holy Cross.

P. L. CRAWLEY, Student, California Institute of Technology.

D. M. DAHM, Student, Rice Institute.

VASSILI DAIEV, Sea Cliff, New York.

E. A. DAVIS, Ph.D. (California) Asst. Professor, University of Utah.

D. A. DAWSON, Student, McGill University.

E. I. DEATON, B.A. (Hardin-Simmons) Teaching Asst., University of Texas.

R. S. DINSMORE, Student, Princeton University.

MRS. NANCY M. DISMUKE, M.S. (Brown) Math., Oak Ridge National Lab.

M. H. DIXON, Student, Siena College.

A. C. DOWNING, JR., M.S. (Michigan) Senior Math., Oak Ridge National Lab.

RUDOLPH DOWNS, Student, Rutgers University.

UNDERWOOD DUDLEY, Student, Carnegie Institute of Technology.

MANNING FEINLEIB, Student, Cornell University.

T. S. FERGUSON, Ph.D. (California) Lecturer, Statistics, University of California.

W. H. FLEMING, Ph.D. (Wisconsin) Asst. Professor, Purdue University.

D. A. FORD, Student, Occidental College.

J. W. FOTHERGILL, JR., B.A. (Sacramento S.C.) Math., University of California Radiation Lab.

LT. D. A. FRANKS, M.S. (Howard) Chief, Internal Guidance Section, Ordnance Guided Missiles School, Redstone Arsenal, Huntsville, Ala.

W. Y. GATELEY, S.M. (M.I.T.) Instr., Clarkson College of Technology.

L. D. GOLDBERG, Student, Yale University.

MRS. ELNA H. GRAHN, M.S. (Wisconsin) Visiting Instr., University of Idaho.

H. S. GREENFIELD, M.S. (Brigham Young) Res. Physicist, Dugway Proving Ground, Utah.

R. K. GRUENEWALD, M.A. in Ed. (Washington) Teacher, Eliot School, St. Louis, Mo.; Instr., Washington University.

MARTHA J. GUTHRIE, Student, Arizona State College.

IRWIN GUTTMAN, Ph.D. (Toronto) Asst. Professor, University of Alberta.

H. N. HADLEY, B.A. (Whitman) Head, Statistical Division, Inspection and Test Department, Naval Powder Factory, Indian Head, Md.

P. R. HARNESS, Student, Tennessee Polytechnic Institute.

R. W. HARRUFF, Student, Kent State University.

BERT HENRY, B.S. (Western Reserve) Director, Schillinger Center, Cleveland, Ohio.

PAUL HERZBERG, Student, Queen's University.

A. L. HESS, Ph.D. (Wisconsin) Asso. Professor, Montana State College.

F. H. HILDEBRAND, Student, Kent State University.

S. K. HOWRY, M.A. (California) Math., Remington Rand Univac, St. Paul, Minn.

R. C. HUBER, B.A. (Lehigh) Math. Engr., Renner Industrial Consultants, Philadelphia, Pa.

MRS. LOUISE F. HUTCHINSON, M.A. (Bryn Mawr) Grad. Student, Bryn Mawr College.

J. F. JAKOBSEN, A.M. (Missouri) Instr., University of Missouri.

N. M. KENDALL, Student, Baylor University.

P. L. KINGSTON, B.A. (Colgate) Grad. Student, University of Rochester.

P. E. KLEBE, JR., M.A. (Yale) Instr., Brooklyn Polytechnic Institute.

BARBARA F. KOROTKIN, Student, Temple University.

B. F. KRAMER, Student, City College of the City of New York.

H. F. KREIMER, JR., Student, Yale University.

THELMA E. KROPPER, Student, Skidmore College.

J. G. LEGHORN, M.S. (Wisconsin) Instr., University of Colorado.

D. A. LEWINSKI, M.S. (Manitoba) Member, Technical Staff, Bell Telephone Labs., New York, N. Y.

THERESA M. LOTITO, Student, LeMoyne College.

R. A. MALLGREN, Student, Wayne University.

K. O. MALMQUIST, JR., A.A. (Modesto Jr.C.) Math., University of California Radiation Lab.

WALLACE MANHEIMER, M.A. (Columbia) Chairman, Department of Mathematics, Franklin K. Lane High School, Flushing, N. Y.

ZACHARY MARTIN, Student, California Institute of Technology.

R. H. MCCOBB, Ed.D. (Temple) Asso. Professor, New Jersey State Teachers College, Glassboro.

B. J. MCDONALD, B.A. (Northland) Physicist, U. S. Navy Mine Defense Lab., Panama City, Fla.

R. A. MELTER, Student, Cornell University.

G. W. MELTON, JR., Student, Knox College.

FLORENCE M. MILLER, A.M. (Loyola) Math. Teacher, Wright Junior College.

H. G. MOORE, B.S. (Utah) Grad. Asst., University of Utah.

KENNETH MOOSMAN, Student, Utah State Agricultural College.

D. D. MORRISON, M.A. (Wayne) Res. Asso., Wayne University.

M. E. MULLER, Ph.D. (U.C.L.A.) Instr., Cornell University.

AMIN MUWAFI, B.S.E. (Florida) Grad. Asst., University of Florida.

D. E. NAGLE, Student, Arizona State College.

W. A. NASH, Ph.D. (Michigan) Asso. Professor, Engr. Mech., University of Florida.
D. A. NELSON, Student, Knox College.
RICHARD NUSINOW, B.S. (Chicago) Instr., Illinois Institute of Technology.
E. H. OBERST, Student, John Carroll University.
REV. D. J. O'BRIEN, S.J., M.A. (Bellarmine) Grad. Student, Fordham University.
W. G. OELLERICH, B.S. (Georgia) Teacher, University of Georgia, Augusta.
F. C. OGG, Ph.D. (Illinois) Chairman, Department of Mathematics, Bowling Green State University.
F. J. OSSIANDER, Student, University of Washington; Aeronautical Engr., Boeing Aircraft Co., Seattle, Wash.
H. A. PETERSON, M.A. (N.Y.U.) Math. Instr., Darien High School, Conn.
R. T. PFEIFER, Student, John Carroll University.
W. L. POPE, Student, Utah State Agricultural College.
T. A. PORSCHING, Student, Carnegie Institute of Technology.
C. W. PRICE, Consultant in Physics, Renner Industrial Consultants, Philadelphia, Pa.
L. B. RATCLIFF, M.S. (Florida S.U.) Ballistic Research Lab., Aberdeen Proving Ground, Md.
H. A. REGUSTERS, Engr. Writer, Renner Industrial Consultants, Philadelphia, Pa.
ARLINE S. REISS, Student, Brooklyn College.
MILTON ROSENBERG, Student, City College of the City of New York.
K. A. ROSS, Student, University of Utah.
F. M. ROURKE, Student, Siena College.
WALTER RUDIN, Ph.D. (Duke) Asso. Professor, University of Rochester.
M. T. SALHAB, Student, University of Texas.
D. J. SCHAEFER, Student, San Jose State College.
M. J. SCHRADER, B.S. in Ed. (Southern Ill.U.) Grad. Asst., Southern Illinois University.
W. L. SCHRADER, Student, Memphis State College.
A. H. SCOTT, A.B. (Middlebury) Grad. Asst., University of Vermont.
J. G. SEEBOLD, JR., Student, Stanford University.
S. L. SEGAL, Student, Wesleyan University.
JOAN M. SHAPIRO, Student, Wesleyan College.
R. L. SHAW, M.A. (Denver) Grad. Student, University of California.
D. J. SINE, B.S. (Frostburg S.T.C.) Grad. Student, West Virginia University.
J. L. SNELL, Ph.D. (Illinois) Asst. Professor, Dartmouth College.
A. P. STOCK, B.S. (Drexel I.T.) Electronic Engr., Renner Industrial Consultants, Philadelphia, Pa.
N. O. STOCKMAN, B.S. (Xavier U.) Aeronautical Res. Scientist, National Advisory Committee on Aeronautics, Cleveland, Ohio.
MRS. MARY R. STRANDTMANN, M.A. (Texas Tech. C.) Instr., Texas Technological College.
MRS. SALY R. STRUIK, Dr.Math. (Prague) Belmont, Massachusetts.
ARTHUR SULLIVAN, Student, Rutgers University.
C. C. THOMPSON, Student, Oklahoma Agricultural and Mechanical College.
E. R. VANCE, JR., Student, University of Redlands.
MARC VENNE, Student, University of Montreal.
D. W. WALL, Ph.D. (Michigan) Asst. Professor, University of North Carolina.
D. E. WALSTON, B.A. (A. & M.C. of Texas) Teaching Asst., University of Texas.
M. W. WEAVER, Ph.D. (Texas) Instr., University of Texas.
B. H. WESTFALL, B.A. (Kentucky) Instr., Berea College.
R. A. WIEDERANDERS, M.A. (Minnesota) Instr., Wartburg College.
C. T. YOUNG, M.A. (Vermont) Grad. Asst., University of Vermont.
HENRY ZEMEL, Student, McGill University.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 24, 1956, at Ann Arbor, Michigan, in conjunction with the meetings of the Michigan Academy of Science, Arts and Letters. Professor C. C. Richtmeyer of Central Michigan College of Education

presided at both morning and afternoon meetings and at the luncheon and business meeting.

A total of 90 persons attended the meeting including 55 members of the Association.

The nominating committee consisting of Professors J. S. Frame, chairman, C. H. Butler, and L. E. Mehlenbacher, proposed Professor R. S. Pate of Eastern Michigan College as Chairman and Professor Fred Beeler of Western Michigan College as Secretary-Treasurer. The slate was elected unanimously.

A resolution was adopted unanimously opposing the provision of the proposed revision of the certification code for the teachers of Michigan. A copy of the resolution is to be sent to each member of the State Board of Education.

The following papers were presented at the morning and afternoon meetings:

1. *Numerical solution methods in partial differential equations using high-speed digital computers*, by Professor J. W. Carr, III, University of Michigan.

A summary is given of the developing theory of numerical solutions of partial differential equations using high-speed computers. In particular, the differences between attacks or initial- and boundary-value problems are noted. The problems of convergence of the difference equation solution to the actual solution, and of the stability of the numerical solution under round-off are also noted. Finally, some estimates are made of time necessary for the solution of certain three-dimensional problems using the fastest machines available and the fastest machines now under design and construction.

2. *Some remarks on two-valued logic*, by Mr. Walter Hoffman, Wayne University.

Two Boolean functions are said to be of the same type, if one can be obtained from the other by permutations of variables, or by replacing variables x_i by $x_i' = 1 - x_i$. The logical significance of this definition is discussed.

3. *Functions of measure*, by Professor B. J. Eisenstadt, Wayne University, introduced by the Secretary.

In characterizing the space $\Lambda(\phi)$, introduced in "Some new functional spaces" by G. G. Lorentz, *Ann. of Math.* (2), 1950, the following question arises. When are certain functions on a Boolean algebra, which arise naturally in the study of the space $\Lambda(\phi)$, expressible as a function of a measure on the algebra? This paper answers the question with the following theorem.

THEOREM. *If $\Phi: B \to R$ is a function from a countably complete Boolean algebra to the reals such that*

(1) *Φ is positive and increasing*

(2) *Φ is atom free*

(3) *If $e_n \in B$ and $e_n \downarrow 0$, $\Phi(e_n) \to 0$*

(4) *If $e_1 \cap e_2 = 0 = e_3 \cap e_4$ and $\Phi(e_1) \geqq \Phi(e_3)$, $\Phi(e_2) \leqq \Phi(e_4)$ then $\Phi(e_1 \cup e_2) - \Phi(e_1) \leqq \Phi(e_3 \cup e_4) - \Phi(e_3)$,*

then there exists a positive atom free, totally finite measure μ on B and a continuous, concave, increasing function F, vanishing at zero, such that $\Phi = F(\mu)$; and conversely.

4. *On using the reciprocal function for a linear interpolation*, by Professor H. E. Stelson, Michigan State University.

This paper considers conditions for determining the better inverse interpolation in cases where it is optional whether a table for $f(x)$ or a table for $1/f(x)$ is used.

5. *Power series expansions for inverse functions*, by Professor J. S. Frame, Michigan State University.

Reverting a power series that expresses w as a function of z near $z=0$ consists in finding the coefficient b_k in a corresponding series that expresses z as a function of w near $w=0$. Letting $a(z)=\sum_1^\infty a_k z^k$ and $b(w)=\sum_1^\infty b_k w^k$, we generalize the problem by setting $(w/z)^t=1+a(z)$ and solving for $(z/w)^t=1+b(z)$, assuming that $w/z=1$ at the origin. The coefficient b_k is found as the residue at $w=0$ of $(z/w)^t w^{-k-1}$, which is transformed by integration by parts to be the residue at $z=0$ of $(t/(t+k))(w/z)^{-t-k}z^{-k-1}$. Thus we obtain

$$b_k=\frac{t}{t+k}\sum_{r=0}^{\infty}\binom{-(t+k)/s}{r}a_k^{(r)} \quad \text{where} \quad [a(z)]^r=\sum_{k=1}^{\infty}a_k^{(r)}z^k.$$

This formula is applied first to find convergent series for all the roots of the general trinomial equation, and then to obtain certain known expressions for the roots of Bessel functions.

6. *A one page table of fifteen-place logarithms*, by Mr. Abraham Nemeth, University of Detroit.

The method described here depends upon the fact that any number can be decomposed into factors which converge to 1. After a number has been so decomposed, the logarithms of the individual factors can be found from a short table, constructed as follows.

The first column is headed N, and entries appear from 1 to 9 inclusive. The next column contains the logarithms of these numbers to fifteen places. The next column contains the logarithms of the numbers from 1.1 to 1.9 inclusive. The next, the logarithms of the numbers from 1.01 to 1.09 inclusive. The table proceeds in this way, the last column containing the logarithms of numbers from 1.000000001 to 1.000000009 inclusive.

The process is also reversible and it is possible to find N when $\log N$ is known.

7. *Brook Taylor, his life, work, and theorem*, by Professor P. S. Jones, University of Michigan.

Brook Taylor lived from 1685–1731 but his mathematically productive period was largely compressed into the years 1708–1715. The first year marked his determination of centers of oscillation, published in 1714, and the latter year marked the appearance of the first editions of his two books *Linear Perspective and Methodus Incrementorum.*

"Firsts" found in the latter are: an interpolation formula (a refinement of Newton's) from which Taylor's theorem is derived, the formulas for integration by parts and interchange of dependent and independent variables in differentiation, and the recognition of singular solutions in some specific differential equations together with the application of his methods to the problems of the vibrating string, the free hanging and loaded string, capillary action, the density and diffraction of the atmosphere, centers of percussion.

Poor health and the emotional disturbances due to estrangement from his austere father at the time of his first marriage and to the deaths in childbirth of both his first and second wives probably account for his cessation of mathematical activity and his turning to religious and philosophical writing in his later years. Taylor's *Contemplatio Philosophica* was published by his grandson in 1793.

It still seems fair to give Taylor credit for priority in the publication of a general formula for expanding functions into infinite series, for an independent derivation however non rigorous, and for recognizing the generality and utility of his series in spite of Johannes Bernoulli's vigorously asserted claims to priority and the recent discovery in the papers of James Gregory of several series expansions using successive derivatives.

8. *The program of the Association*, by Professor G. B. Price, University of Kansas.

The objectives and responsibilities of the Association were discussed in this invited address. Some suggestions for improved methods of attaining these objectives were made.

9. *Projectively Euclidean Hermitian manifolds*, by Professor S. I. Goldberg, Wayne University.

Let M^n (complex dimension n) be a differentiable manifold of constant holomorphic curvature k. These spaces for positive, negative and zero values of k have recently been characterized (topologically) by J. Igusa. General measures of deviation from k have been defined by S. Bochner and it was found that for deviations within certain limits the Betti numbers remain unchanged. For example, for $k>0$, the projective curvature tensor gives a measure of the deviation from complex projective space. A subgroup of the group of projective transformations for Hermitian manifolds with an asymmetric connection is defined. This leads in a natural way to the definition of a manifold $\tilde{M}^n$ whose curvature tensor has the same formal description as that of M^n. The following results are established: (i) $\tilde{M}^n$ has constant holomorphic curvature, if and only if, k is a constant, (ii) an $\tilde{M}^n$ is conformal with some M^n, and (iii) if $\tilde{M}^n$ is compact it is torsion free.

10. *Conditional covariance and light*, by Professor G. Y. Rainich, University of Michigan.

Physicists use two models describing the phenomenon of light: one is the electromagnetic field, the other is a corpuscle. Four-dimensionally the first involves a matrix M whose elements are the components of the electric and magnetic vectors. In the general case M has two invariants but in the case of light (when the two vectors are at right angles and of equal length) both invariants vanish. As a result, the square roots of the diagonal elements of the square of M behave like the components of a four-vector of zero square which is the model used in the corpuscular theory. The situation is considered from the point of view of conditional covariance.

11. *Some geometric aspects of integration*, by Professor W. F. Eberlein, Wayne University.

Connections between properties of the ordinary Riemann integral and elementary separation properties of convex sets in Hilbert space are formulated. In particular, a geometric proof of the Osgood convergence theorem is obtained. The methods generalize to more abstract spaces and integrals.

12. *The ring of $n\times n$ matrices over an AW^*-algebra*, by Mr. S. K. Berberian, Michigan State University.

Let A be an AW^*-algebra of type II_1, A_n the ring of $n\times n$ matrices over A. It is conjectured that A_n is AW^*. We show: (a) A_n is a B_p^*-algebra; (b) if A (equivalently the center of A) is of denumerable type, then A_n is AW^* of denumerable type. The main technique is the regular ring attached to the continuous geometry of projections of A.

S. D. CONTE, *Secretary*

THE MARCH MEETING OF THE OKLAHOMA SECTION

The spring meeting of the Oklahoma Section of the Mathematical Association of America was held at the University of Tulsa, Tulsa, Oklahoma, on March 31, 1956. Professor Truman Wester, Chairman of the Section, presided. There were 57 persons in attendance, including 42 members of the Association.

The following papers were presented:

1. *A note on Borel's integral method of summation*, by Professor H. N. Carter, University of Tulsa.

Borel's definition for the sum of a series based on an integral is given by the equation

$$s = \int_0^\infty e^{-a}\left(u_0 + u_1 a + u_2\frac{a^2}{2!} + \cdots + u_n\frac{a^n}{n!} + \cdots\right)da.$$

Substitute $u(a)$ for

$$u_0 + u_1 a + u_2\frac{a^2}{2!} + \cdots + u_n\frac{a^n}{n!} + \cdots$$

and $s = \int_0^\infty e^{-a}\,u(a)da$ which is Borel's integral method for determining the sum of the series $\sum_{n=0}^\infty u_n$.

The sum of any series whose terms are integral powers of $(n+1)$ can be determined by Borel's integral method by deriving a general formula for $u(a)$ corresponding to the series $\sum_{n=0}^\infty (-1)^n(n+1)^k$, $k=0, 1, 2, \cdots$. For this series

$$u(a) = 1 - 2^k a + \frac{3^k a^2}{2!} - \frac{4^k a^3}{3!} + \cdots + (-1)^n(n+1)^k\frac{a^n}{n!} + \cdots.$$

A formula can be derived to represent the constant coefficients of the terms in the right member of this equation by using the identity

$$(x-1)^r = {}_rC_0x^r - {}_rC_1x^{r-1} + {}_rC_2x^{r-2} - \cdots - (-1)^v\,{}_rC_vx^{r-v} + \cdots + (-1)^r, \qquad r = 1, 2, 3, \cdots$$

Then $u(a) = \sum_{r=1}^\infty (-1)^{r-1}P_r a^{r-1}$ where

$$P_r = \sum_{v=0}^{v=r}(-1)^v\left[\frac{(r-v)^{k+1}}{(r-v)!v!}\right], \qquad v = 1, 2, 3, \cdots.$$

This formula for $u(a)$ enables one to readily determine the sum of the series $\sum_{n=0}^\infty (-1)^n(n+1)^k$, $k=0, 1, 2, 3, \cdots$ by Borel's integral method.

2. *Another method of steepest descent in linear programming*, by Professor R. B. Deal, Oklahoma Agricultural and Mechanical College.

3. *Elements of area in coordinate-transformation*, by Professors W. A. Rutledge, M. Schwartz and Simon Green, University of Tulsa.

In vector analysis the study of general surface coordinates usually involves the element of area expressed by $(\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v)dudv$ where $\mathbf{r}$ is a position vector and u, v are orthogonal surface coordinates. Generally the development omits the use of this form in relation to finding areas or expressing area elements for a change of variables. In this paper the usefulness of this form is discussed and illustrated with examples.

4. *An algorithm for simplifying games*, by Professor Harold Shniad, University of Arkansas.

A finite two-person game in normal form is given by a finite matrix. If the value v of the game is known to satisfy an inequality of the form $v_1 \leqq v \leqq v_2$, then it is shown how the original game may be replaced by another game such that each element of the new corresponding matrix satisfies the given inequality and such that the value of the new game is the same as that of the original one. The new matrix may be used to make sharper estimates of the value of the game and to simplify the finding of dominating strategies.

5. *The Cevian chain*, by Professor N. A. Court, University of Oklahoma.

If the Cevian triangle (T_1) of the point M for a triangle (T_0) be constructed, then the Cevian triangle (T_2) of M for (T_1), and again the Cevian triangle (T_3) of M for (T_2), *etc.*, a "Cevian chain"

of triangles is obtained such that the trilinear polars of M for all the triangles of the chain coincide, say, in a line m. In each triangle of the chain a conic may be inscribed so as to be circumscribed about the following triangle of the chain. The point M and the line m are pole and polar for each conic of the chain thus obtained. This paper is scheduled to appear in full in the quarterly *Scripta Mathematica*.

6. *Vectorial analytic geometry*, by Professors Simon Green, M. Schwartz and W. A. Rutledge, University of Tulsa, presented by Professor Green.

This paper is concerned with the use of the algebra of vectors in a first course in analytic geometry. Illustrations are given which compare the traditional methods of analytic geometry with vector methods.

7. *Convergence of a simple geometric process*, by Professor J. R. Foote, University of Oklahoma.

A geometric process is presented by examples which suggest that either convergence or divergence of a certain sequence of points can be obtained. An approximating function with an iterative procedure is easily written. Elementary methods at once yield the desired results in a single theorem, a portion of which follows. Let $F_i(x)=m_ix+b_i$, $i=1, 2$, and $M=-m_1/m_2\neq 1$, and $x_0=k/(m_1+m_2)$, where

$$F_1(x^i)+F_2(x^{i+1})=k+b_1+b_2 \text{ defines the iteration.}$$

If $x^1=x_0$, then $x^i=x_0$ for all i and independently of M. If $x^1\neq x_0$, as $i\to\infty$ then $x^i\to\infty$, or $x^i\to x_0$, according as $|M|>1$ or $|M|<1$ respectively.

8. *Some consequences of the Brouwer plane translation theorem*, by Professor O. H. Hamilton, Oklahoma Agricultural and Mechanical College.

The relation of the Brouwer plane translation theorem to some of the classical fixed point theorems is discussed. A proof that the 2-cell has the fixed point property is given using the plane ranslation theorem. Other more special fixed point theorems are discussed.

9. *Mathematical research in the petroleum industry*, by Dr. D. R. Shreve, Carter Oil Company, Tulsa, Oklahoma.

The applications of new mathematical techniques in the petroleum industry, made feasible by the development of high speed computers, was indicated. Mathematical controls were described, from use of statistically directed control of exploratory drilling following improved interpretation of geophysical data, with control of injection and production through predictions of production history of a reservoir, by calculated schedules through pipe lines and primary storage, through a refinery controlled by linear programming or other methods of operations research, with distribution to warehouses to meet predicted demands.

10. *Demonstration*, at Carter Research Laboratory, of the "IBM 650" computing machine with drum memory.

R. V. ANDREE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 16–17, 1956, at U. S. Navy Supply Corps School, Athens, Georgia. Professor R. H. Moorman, Chairman of the Section, and Professor D. F. Barrow, Vice-Chairman, presided over the general sessions; Professors S. T. Gormsen, C. G. Latimer, I. E. Perlin, C. G. Phipps, E. B. Shanks and W. L. Williams presided over subsections.

There were about 250 in attendance including 148 members of the Association.

The following officers were elected for the coming year: Chairman, Professor G. B. Huff, University of Georgia; Vice-Chairman, Professor C. G. Latimer, Emory University; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College.

The following program was presented:

1. *A probability problem arising in grading*, by Professor G. B. Huff, University of Georgia.

The student's knowledge of the chronological sequence of a series of events may be tested by listing the events in a random order and asking the student to order the events. A measure of the student's answer may be obtained by adding the absolute values of the differences between the student's answer and a key. This leads to the question: What is the expected total of the absolute values of the differences between a given permutation of the numbers from 1 to n and a permutation obtained by shuffling and drawing? It is shown that the answer is $(n^2-1)/3$.

2. *Rubber stamp problem*, by Mr. W. R. Carnes, Georgia Institute of Technology.

An infinite rubber stamp prints concentric circles whose radii are all irrational. What is the least number of times that the stamp must be placed on an infinite plane so that each point on the plane will be covered by a circle? An existence-type solution is given, showing that three times is sufficient if the centers are properly chosen. Three such points are demonstrated and a proof that every point is covered by a circle is given using the distance formula and properties of irrational numbers.

3. *Cayley-Hilbert spaces*, by Professor V. D. Gokhale, Atlanta University.

Cayley-Banach and Cayley-Hilbert spaces are generalizations of the classical Banach and Hilbert spaces. They are vector spaces over Cayley's 8-unit division algebra. After defining these spaces by a suitable set of postulates, a generalization of the Neumann-Jordan Theorem on the equivalence of Cayley-Banach and Cayley-Hilbert spaces is proved.

4. *Results of a questionnaire*, by Professor Tomlinson Fort, University of South Carolina.

Professor Fort reported on the returns of a questionnaire recently sent out by the Committee on Personnel and Education of the Association.

5. *A history of the Southeastern Section*, by Professor H. A. Robinson, Agnes Scott College.

Immediately after the formation of the Association, December, 1915, the late Dean R. P. Stephens attempted to organize the Section; however, organization did not occur until 1922. For the 31 annual meetings held, 34 guest mathematicians have been sponsored. In first eleven years there was an average of eight papers presented with 49 present including 18 Association members; since 1949, 39 papers, 193 present including 111 members.

6. *The use of analogy*, by Professor C. G. Phipps, University of Florida.

Reasoning by analogy permits us to transfer our training from one situation to an analogous one. Thus analogy yields alternate interpretations of deductive systems since two perfectly analogous situations have the same mathematical formulas. Algebra and geometry being analogous combine into analytic geometry.

7. *Mathematics and philosophy*, by Professor R. H. Moorman, Tennessee Polytechnic Institute.

The author traced the historical relationship of mathematics and philosophy in the Greek period, the middle ages and modern times. The type of analysis which Plato introduced into mathematics was introduced into philosophy by Descartes. The type of analysis which Descartes introduced into mathematics has been used successfully in philosophy only in the twentieth century in the field of symbolic logic. The concept of "universal mathematics" goes back as early as Pythagoras. Utilitarian sciences develop by the mathematization of philosophy. Certain modern pseudo-sciences do not have the validity they claim because they do not have functional relationships.

8. *Non-associative number theory*, by Professor Trevor Evans, Emory University.

On replacing the unary operation of successor in Peano's postulates for the natural numbers by a binary operation, a characterization is obtained of a non-associative arithmetic studied by Etherington. The number theory of this system is investigated and Fermat's Last Theorem proved in it. In a manner analogous to the embedding of the natural numbers in the ring of integers, this non-associative arithmetic is embedded in the left neoring recently studied by Bruck. The number theory of this left neoring is also studied in some detail.

9. *Rational univalent functions*, by Professor W. C. Royster, Alabama Polytechnic Institute. (By title.)

Let F be the class of functions $z+\sum_{n=2}^{\infty} a_n z^n$ which are univalent in the unit circle $E(|z|<1)$. Linis (this MONTHLY, vol. 62, 1955) gave a short proof of a theorem of Friedman (*Duke Math. Journal*, vol. 13, 1946) which states that if the a_n are rational integers then $f(z)$ is one of nine rational functions. The purpose of this note is to extend the method of Linis to quadratic fields with negative discriminant and to give criteria for selecting combinations of possible coefficients yielding rational functions which are regular and univalent in E.

10. *A note on the solution of algebraic equations*, by Professor Stephen Kulik, University of South Carolina.

A method is derived for computing by approximation any root of an algebraic equation, all the roots of which are real. It gives the lower and upper bounds of the root converging to it as a limit.

11. *Consistency of equations*, by Professor J. M. Thomas, Duke University. (By title.)

This paper points out the usefulness in teaching elementary differential equations of de la Vallée Poussin's condition (*Ann. Soc. Sci. Bruxelles*, vol. 64, 1950, pp. 74–75) for a differential whose coefficients are only assumed continuous, to be exact, and the possibility of using the idea to get a consistency (passivity) condition for a system of total differential equations under the same assumption.

12. *A note on lattice-ordered rings*, by Professor R. G. Blake, University of Florida.

Nearly ten years ago Garrett Birkhoff wrote "The theory of lattice-ordered rings . . . is not yet even in its infancy." The statement seems to be still true today. This paper suggests a set of axioms for a type of lattice-ordered ring which seems to promise some interesting results.

13. *The free lattice generated by a set of chains*, by Dr. H. L. Rolf, Vanderbilt University.

The free lattices $FL(2+2)$ and $FL(4+1)$ are shown to be infinite and their structures determined. It is shown that $FL(3+2)$ and $FL(5+1)$ each contains a sublattice isomorphic to $FL(1+1+1)$. Furthermore, $FL(1+1+1)$ contains a sublattice isomorphic to $FL(n+m)$.

14. *Wide open sets*, by Professor D. F. Barrow, University of Georgia.

Professor Barrow defines a wide open set of numbers as a set such that the sum of two numbers of the set is never a number of the set; and a complete wide open subset of a set S, as a wide open subset of S to which no other number of S could be adjoined without destroying its wide open property. (For example, the odd integers are a complete wide open subset of the integers.) A few simple theorems were presented.

15. *Matrices with elements in a Boolean ring*, by Professor A. T. Butson, University of Florida.

Let $\mathfrak{B}$ be a Boolean ring of at least two elements containing a unit 1. Form the set $\mathfrak{M}$ of matrices $A, B, \cdots$ of order n having entries $a_{ij}, b_{ij}, \cdots, i, j = 1, 2, \cdots, n$, which are members of $\mathfrak{B}$. For any matrix A of $\mathfrak{M}$ there exists a unimodular matrix U of $\mathfrak{M}$ such that $UA = H$ has the following properties: $h_{pq} = 0$ for $q > p$, $h_{pq}h_{qq} = 0$, and $h_{pq}h_{pp} = h_{pq}$ (note that if a diagonal element is 0, then the entire row consists of 0's). This form H is unique.

16. *Remarks on Boolean functions II*, by Dr. D. O. Ellis, Research Engineer, National Cash Register Electronic Division, Hawthorne, California.

It is shown that certain of the algebraic functions of two variables in a Boolean algebra give rise to Boolean algebras centered on given elements of the original algebra. The relations among the operations of these algebras are determined explicitly. A criterion is given for the possession by any Boolean function (of a finite number of variables) of a linear factor in one of its variables. The methods employed are elementary.

17. *Some results on a particular non-associative algebra*, by Professor Tom Baynham, Jr., University of Tennessee.

In considering a non-associative algebra A^* defined by G. C. Holt in his paper entitled "A non-associative algebra generated by square matrices," (*Tennessee Academy of Science Journal*, 27: 281–290, 1952), results are obtained concerning nilpotent elements, idempotent elements, ideals and the center of A^*.

18. *An orthotropic beam bent by its own weight*, by Professor C. B. Smith, University of Florida.

An infinitely long beam is considered to be in a state of plane stress under the action of its own weight and equal supporting reactions applied at a series of points situated at regular intervals along its lower edge. By considering various portions of the beam, it is possible to draw some conclusions concerning the bending of finite beams loaded by gravity when the ends are either simply supported or clamped.

19. *Application of shallow shell theory to the buckling of spherical shells*, by Professor W. A. Nash, University of Florida.

The linear differential equations governing the axi-symmetrical buckling of shallow spherical shells subject to uniform external pressure are developed for the case of small deformations. An approximate solution to these equations, based upon Galerkin's method, is presented for the case

of a simply supported spherical cap and a displacement pattern suitable for use with clamped edge spherical caps is indicated. The results obtained indicate the potentialities of shallow shell theory for use in approximating classical small deformation buckling loads of shells which are not shallow.

20. *A comparison of the Deming and Garwood fitting methods*, by Mrs. Nancy M. Dismuke, Oak Ridge National Laboratory.

The Deming and Garwood procedures for fitting data to a function non-linearly dependent on a set of parameters have been applied to several problems. Based on this experience a comparison is made of the convergence properties, the parameter variances obtained, and the labor involved in applying the two methods.

21. *Stress distribution in an orthotropic disk subjected to its own weight and supported by a concentrated force on its boundary*, by Professor J. B. Wilson, University of Florida.

An orthotropic disk subjected to its own weight and supported by a concentrated force applied at its boundary is considered as the limit of the case in which the supporting force is distributed over an arbitrarily small area on the boundary. Series solutions for the stresses and displacements are obtained, and the stress function for the orthotropic disk is shown to contain as a limiting case the stress function for an isotropic disk. The stress components are calculated at several points along the horizontal diameter of a plane-sawn disk of Sitka spruce, with the grain taken to be horizontal.

22. *Curvatures associated with a vector field*, by Professor J. D. Novak, University of South Carolina.

In this paper W. C. Graustein's associate curvature and T. K. Pan's normal curvature for a vector field in a surface are generalized for any vector field associated with the surface. These generalized curvatures are used in the study of curves on the surface.

23. *On picture writing*, by Dr. George Pólya, Visiting Lecturer, Mathematical Association of America. (By invitation.)

In several combinatorial problems we can arrive more intuitively at the expression, or defining equation, of the *generating function* that solves the problem if, instead of the letters of various alphabets, we use appropriate pictures as symbols for variables or indeterminates. This idea, which the author has used in research for more than twenty years but has not published before, is illustrated by three examples in the present lecture which is scheduled to appear in the MONTHLY.

24. *The elementary derivatives from a geometrical definition of e*, by Professor J. T. Moore, University of Florida. (By title.)

In the class of curves $y=a^x$ define e to be that value of a such that $y=a^x$ crosses the y-axis with a slope of 1. By definition, then, $D_x e^x=1$, *for* $x=0$. An application of the Δ-process then leads to $\lim_{\Delta x\to 0}(e^{\Delta x}-1)/\Delta x=1$ and $D_x e^x=e^x$ *for all* x*'s*. By writing $y=\ln x$ and $y=x^n$ in the respective forms $x=e^y$ and $\ln y=n \ln x$, the log and power function formulas are obtained immediately. The following advantages are claimed: (1) binomial theorem is not needed; (2) no complex identities or inductive methods are used; (3) definition of e is more understandable than the usual algebraic definition as a limit.

25. *Understanding mathematical induction*, by Professor R. W. Ball, Alabama Polytechnic Institute.

Mathematical induction can be taught successfully if it is related to reasoning processes the student has already used. The instructor should require, before the formal proof, a thorough dis-

cussion of the statement of the theorem, a specific example of the extension process, and a conclusion for some finite case. Only after the student is convinced that such reasoning does not depend on n should he attempt the general proof.

26. *A note on degenerate conics*, by Professor J. W. Lasley, Jr., University of North Carolina.

It is well known that a real conic degenerates into a pair of lines if the rank of its matrix is 2; into a repeated line if its rank is 1. If the lines are distinct, they are real if the 2×2 determinant in the upper left hand corner is minus; conjugate imaginaries, if this sign is plus. In the theory of real quadratic forms these criteria may be made in terms of the rank of the adjoint matrix and the sign of its trace. This paper reviews the theory of degenerate conics from this higher viewpoint.

27. *Partial fractions*, by Professor L. A. Dye, The Citadel.

The resolution of a proper rational fraction into a set of simpler fractions is demonstrated by evaluating the necessary parameters for the general case. Some examples of special methods for particular cases are given.

28. *Georgia Institute of Technology Computer Program—a progress report*, by Professor B. M. Drucker, Georgia Institute of Technology.

The School of Mathematics of Georgia Institute of Technology initiated a curriculum in digital computers and numerical analysis in the fall of 1954. The Rich Electronic Computer Center of the Experiment Station was activated in the fall of 1955. This report comprises comments as to future plans in light of the experience obtained to date and suggestions to other departments interested in setting up such a program.

29. *Locally one-to-one mappings on* S^1, by Professor M. K. Fort, Jr., University of Georgia.

Let f be a mapping on the unit circle S^1 into a one dimensional metric space Y. If f is locally one-to-one on S^1 (*i.e.*, each point of S^1 has a neighborhood on which f is one-to-one), then f is an essential mapping.

30. *A note on the matrix equation* $AX=Xf(A)$, by Professor R. H. Ackerson, University of Florida.

The matrix equation $AX=XB$ has been considered by W. V. Parker, with A in rational canonical form. Using Parker's method, a necessary and sufficient condition was determined for the equation $AX=Xf(A)$ to have all of its solutions expressible as polynomials in A, where A is non-derogatory, and $f(A)$ is a polynomial in A.

31. *On the simple group of order 5616*, by Professor F. A. Lewis, University of Alabama.

This paper gives some results of a study of $LF(3,3)$ in terms of generators corresponding to elementary operations used in reducing a determinant.

32. *On the rational equivalence of ternary quadratic forms*, by Mr. Amin Muwafi, University of Florida.

The first part is a method of obtaining a linear rational transformation, whose determinant has a given value $C\neq 0$, that takes a ternary quadratic form with rational coefficients into one with rational coefficients and without cross-products. Formulas are given by which the three columns of the matrix of the transformation are determined. In the second part, sufficient conditions are found in order that a given ternary quadratic form, of determinant $d\neq 0$, may be taken into an-

other given ternary quadratic form of the same determinant d by a linear rational transformation of determinant one.

33. *A note on the distribution of logical quantifiers*, by Professor R. S. Fouch, Florida State University.

Quine has proved a number of theorems, grouped into six chains, concerning the distribution of the universal and existential quantifiers over three truth-functions. However, there are 13 other truth-functions, of which 7 are non-trivial. Similar distributive theorems for these 7 functions are established by a simple decision method. A structure is determined for the theorems for each function and it is shown that there are only 3 different structures for the 10 functions. The interrelationships of these structures are also examined.

34. *Note on perturbations of systems having a periodic solution*, by Professor J. A. Nohel, Georgia Institute of Technology.

Let x, f be real vectors with n components and suppose t and μ are real. Consider the system (1) $x' = A(t)x + \mu f(t, x, \mu)$, $(' = d/dt)$, where $A(t)$ is a real periodic $n \times n$ matrix with period T and f has period 2π in t. Suppose that for $\mu = 0$ the linear system (2) $x' = A(t)x$ has at least one solution with period 2π. Sufficient conditions are given for the existence of periodic solutions of (1) for $|\mu|$ sufficiently small. This generalizes results of Coddington and Levinson (*Contributions to the Theory of Nonlinear Oscillation, vol. II*, Annals of Math. Studies no. 29, Princeton University Press) where $A(t)$ is a constant matrix.

35. *On the behavior of solutions of ordinary finite difference equations*, by Mr. A. C. Downing, Jr., Oak Ridge National Laboratory.

It is well known that when finite difference equations are substituted for differential equations, it does not follow *a priori* that the behavior of the solution of the difference equations will approximate the solution of the differential equations. This is demonstrated for the very simple ordinary differential equation $y'' - \lambda^2 y = 0$. When λ^2 is real, the character of the linearly independent solution depends upon whether $\lambda^2 > 0$, $\lambda^2 = 0$ or $\lambda^2 < 0$. The solutions of the corresponding finite difference equation (as usually formulated) fall into four categories. The anomalous character of the fourth category is readily recognized.

36. *Solution of a non-linear partial differential equation of the third order*, by Professor R. W. Cowan, University of Florida.

The type of third order equation to be solved is determined by the form selected for the intermediate integral. Then equations analogous to Monge's equations are derived and used to determine intermediate integrals. From the latter a general solution may sometimes be found. If this is not possible other types of solutions can be determined by specializing the arbitrary functions occurring in the intermediate integrals.

37. *Basic sets of polyharmonic polynomials*, by Professors E. P. Miles, Jr. and Ernest Williams, Alabama Polytechnic Institute.

The authors show that the four homogeneous polynomials of degree n, $n \geqq 4$,

$$B_\alpha^n(x, y) = \sum_{j=0}^{[(n-\alpha)/2]} (-1)^j (j+1)^{[\alpha/2]} \frac{n(n-\alpha)(n-\alpha-1)\cdots(n-\alpha-2j+1)}{(2j+\alpha)!} x^{n-\alpha-2j} y^{\alpha+2j} \quad (\alpha = 0,1,2,3)$$

constitute a basic set of solutions for the biharmonic equation, $\nabla^4 xy(u) = 0$; for $\alpha = 0, 1$, the polynomials are the real and imaginary parts of $(x+iy)^n$. These results are generalized to obtain a basic set of solutions for $\nabla^4_{x_1, x_2 \cdots x_k}(u) = 0$; this set contains the $C_{n+k-3,\ k-3}(2n/(k-2)+1)$ harmonic polynomials of the authors' basic set (*Proc. A.M.S.*, vol. 6, 1955) for $\nabla^2_{x_1, x_2, x_3, \cdots, x_k}(u) = 0$. A further generalization gives a basic set for $\nabla^{2k}_{x_1, x_2, x_3, \cdots, x_k}(u) = 0$.

38. *Periodic solutions of a non-linear wave equation*, by Professor F. A. Ficken, University of Tennessee.

Define $Lu=u_{tt}+2ku_t+au-u_{xx}$. We know that the problem $Lu=b(x, t)-\epsilon u^3$, $u(x, 0)=f(x)$, $u_t(x, 0)=g(x)(-\infty<x<\infty)$ has a unique solution $u(f, g; x, t)$ if $|\epsilon|$ is properly restricted. Here we give a new and simpler proof of a known further fact: if $b(x, t+p)=b(x, t)$, $k>0$, and $a>0$, then for certain f_1 and g_1, u is also p-periodic in t ($u(f_1, g_1; x, t+p)=u(f_1g_1; x, t)$). This solution is now seen to be, as in the linear one-dimensional case, a steady motion that is approached, as transients disappear, by any $u(f, g; x, t)$ meeting certain natural conditions. Special cases: $0\leqq x$ and $u(0, t)=0$; $0\leqq x\leqq\pi$ and $u(0, t)=0=u(\pi, t)$. All this work has been done in collaboration with B. A. Fleishman.

39. *From matrices to tiles*, by Dr. T. W. Hildebrandt, Oak Ridge National Laboratory.

Property (A) for matrices was defined by D. Young (*Trans. A.M.S.* vol. 76, 1954, pp. 92–111). For the matrix of the finite difference representation for a linear elliptic partial differential equation, this property has a simple geometrical interpretation, related to the problem of "tiling" [filling the plane (space) with congruent polygons (polyhedra)].

40. *The class of integral sets*, by Professor E. B. Shanks, Vanderbilt University.

The class of integral sets was defined in terms of the non-negative integers, using the binary system. It was shown that this system satisfies the axioms of set theory used by Gödel, provided the axiom of infinity postulates an infinite class instead of an infinite set. A relation that partially orders the system was discussed also.

41. *Fixed points for limit functions*, by Professor W. L. Strother, University of Miami.

That the pointwise limit of continuous functions is not necessarily continuous is well known. A new sort of limit is defined which agrees with the pointwise limit at all points of continuity of the latter and is weakly continuous. For certain spaces which have fixed point property for continuous functions it is shown that the limit of continuous functions has a fixed point.

42. *The effect of symmetrization on smoothness of functions*, by Professor W. S. Snyder, University of Tennessee.

The function $f(x, y)=f(r)=r^4(1+\cos\pi/r)$ has continuous first partial derivatives on $0\leqq x^2+y^2=r^2\leqq 1$, but its symmetrization with respect to the origin is not piecewise smooth. (For a definition of symmetrization, see Pólya, G. and Szegö, G., *Isoperimetric inequalities in mathematical physics*, Annals of Math. Studies, no. 27, pp. 151–152).

43. *A topological equivalence theorem*, by Professor R. A. Lytle, University of South Carolina.

Garrett Birkhoff shows in his paper, *The topology of transformation sets*, Annals of Mathematics, vol. 35, that the point open topology is equivalent to the uniform topology for the space of homeomorphisms of the unit interval onto itself. This paper generalizes this result as follows: Suppose A is a graph which is a continuum and H is the set of all homeomorphisms of A onto A. Then H with the overlap topology is equivalent to H with the uniform topology.

44. *A note on the Cauchy criterion*, by Mr. G. E. Duncan, Georgia Institute of Technology.

This paper demonstrates the sufficiency of the Cauchy criterion by a direct application of the nested set theorem.

45. *The new freshman mathematics*, by Professor J. D. Mancill, University of Alabama.

What is bringing about change in our first year college mathematics? Mathematics has fanned out in ever-increasing new directions with the effect that neatly compartmentalized subjects have become mixed beyond recognition. Importance of development is having its effect on curriculum in secondary schools and freshman college levels. We are confronted with the problem of putting elementary mathematics back in the center of the ring of "general education." It must be realized that to revitalize the position of elementary mathematics in the light of new developments involves change, or at least the willingness to change.

46. *Universal mathematics*, by Professor C. L. Seebeck, Jr., University of Alabama.

Universal mathematics course designed by the Committee on the Undergraduate Program of the Association was outlined and discussed. Report was favorable for the experimental course given at University of Alabama.

47. *General or unified mathematics*, by Professor F. W. Kokomoor, University of Florida.

Students having varied preparational backgrounds and taking mathematics only to obtain what "every college graduate ought to have" find integrated courses better than the traditional, which are mainly preparatory for something they will never reach. However, some of these students do become intensely interested and turn out to be excellent mathematicians. Integrated courses can also better serve majors in other fields by giving prerequisite material earlier and omitting unnecessary content. Majors in mathematics itself lose nothing by taking the integrated courses, but do just as well with the traditional, if they have no prerequisite deadlines to meet along the way.

48. *The postulational approach to freshman mathematics*, by Professor Herman Meyer, University of Miami.

The postulational approach to freshman mathematics taught at the University of Miami for the past seven years is described. Some examples of the methods, proofs, and procedures used are set out, along with a subjective analysis of the strength and weakness of the program.

49. *Some remarks on the location of the roots of an algebraic equation*, by Professor A. T. Brauer, University of North Carolina.

It is well known that every theorem on characteristic roots of a matrix contains a theorem on algebraic equations as a special case. It is shown that the following known theorems can be obtained easily by using the columns and the rows of the same matrix. (See O. Perron, *Algebra*, 2nd ed., Berlin, 1933, pp. 31–33.) Let τ be the absolute greatest root of the equation $x^n+a_1x^{n-1}+\cdots+a^n=0$, and $m, m_1, m_2, \cdots, m_{n-1}$ arbitrary positive numbers. Then,

(I) $|\tau| \leqq \max(|a_1| + m_1^{-1}, |a_2| m_1 + m_2^{-1}, \cdots, |a_{n-1}| m_1 m_2 \cdots m_{n-2} + m_{n-1}^{-1}, |a_n| m_1 m_2 \cdots m_{n-1})$;

(II) $|\tau| \leqq \max(m^{-1}, |a_1| + |a_2| m \cdots + |a_n| m^{n-1})$.

Moreover, improvements of these results are obtained.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-sixth regular meeting of the Southern California Section of the Mathematical Association of America was held at Pomona College, Claremont, California, on March 17, 1956. Professor S. E. Urner, Chairman of the Section, presided. The attendance was 107 including 69 members of the Association.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor H. F. Bohnenblust, California Institute of Technology; Vice-Chairman, Professor P. B. Johnson, Occidental College; Secretary-Treasurer, Professor P. H. Daus, University of California at Los Angeles; Program Committee: Professor A. R. Harvey (Chairman), San Diego State College; Professor V. C. Harris, San Diego State College; Professor R. A. Dean, California Institute of Technology; Mr. R. E. Horton, Los Angeles City College.

The following program was presented:

1. *Differentiation in vector spaces*, by Dr. Walter Noll, University of Southern California, introduced by Professor D. V. Steed.

Originally, differentials were thought of as "infinitely small" increments. Since no precise meaning can be given to the "infinitely small," they are now regarded as linear relations between arbitrary finite increments $dx=\xi$ and corresponding increments $dy=\eta=f'(x)\xi$. Such a relation can be written also in the form

$$\eta = \nabla f(x)[\xi] = \frac{d}{d\epsilon} f(x+\epsilon\xi)\Big|_{\epsilon=0}.$$

This definition can be extended in a straightforward manner to scalar or vector fields simply by replacing both x and ξ by vectors. The "differential" thus obtained is a linear form or a linear transformation. By referring the underlying space to arbitrary curvilinear coordinates, the differential of a vector field can be shown to be identical to what is usually called the "covariant derivative".

2. *Rearrangements of infinite series*, by Professor V. L. Klee, University of Washington.

The speaker reviewed some of the principal results concerning rearrangement of infinite series of vectors in Euclidean n-space, especially the basic Levy-Steinitz theorem and a closely related result due to Jarnik, Behrend, and Wald. A proof of the latter was presented, and an outline of a proof of the former. Some results of Hadwiger were also discussed.

3. *College mathematics for non-science students*, a panel discussion by Professor May M. Beenken, Immaculate Heart College; Professor P. B. Johnson, Occidental College; Mr. R. B. Herrera, Los Angeles City College; Dr. C. J. A. Halberg Jr., University of California, Riverside.

This was a panel discussion by members of a special committee of a subcommittee on mathematics appointed by the California Committee for the Study of Education. The chairman, Proffessor May M. Beenken, explained the history and purpose of the committee. Professor P. B. Johnson discussed current practices in the field of mathematics for general education as gleaned from a questionnaire study made by the committee. Mr. R. B. Herrera spoke on textbooks and literature

in this field of mathematics and distributed an annotated bibliography compiled by the committee. Dr. C. J. A. Halberg, Jr., gave the committee's recommendations.

4. *The role of the mathematician in industry*, by Mr. Ross Miller, Northrop Aircraft Corporation, introduced by Professor Clifford Bell.

The profit motive is the underlying force in industry. A corporation makes money by pleasing its customers, by satisfying their wants through better products made more rapidly, more reliably, more cheaply, etc. Frequently it creates in its customers new desires which industry is in a position to satisfy. And we note in this case the opportunity for basic research.

The mathematician's goals are those of job satisfaction and interest, a feeling of challenge and responsibility. Pleasant working conditions with adequate financial rewards are also important considerations.

The goals of industry and the goals of the mathematician can be compatible and can find fulfillment in a merger of interests. The mathematician's contribution is basic and becomes ever increasing in its importance. Industry, on the other hand, stands prepared to offer the mathematician the personal rewards which are so important to the individual.

5. *New developments in numerical analysis*, by Dr. G. E. Forsythe, Numerical Analysis Research, University of California at Los Angeles.

Automatic digital computing machines, virtually unknown ten years ago, continue breathtaking developments in speed, capacity, quantity manufactured, and demands for numerical analysis. Some 20 university departments use such machines regularly as a research tool. Academic mathematics seems to contact machines in four ways. (1) Huge mathematical experiments are carried out on such problems as Fermat's last theorem. (2) Mathematics is a principal tool in analyzing computing methods and in devising new ones. (3) Computers and their use raise many new logical and mathematical problems undreamed of before computers ran. (4) Such machines are a boon to teachers in advertising and glamorizing mathematics, and they can be understood by all who know elementary arithmetic.

6. *The theory of games and the game of poker*, by Dr. Rodrigo Restrepo, California Institute of Technology, introduced by Professor H. F. Bohnenblust.

In the last few years considerable effort has been devoted to the construction of economic, tactical or gambling models that can be analyzed by means of the theory of games. Using a simplified version of the game of poker, the author constructs a class of games in which each player must decide how much to bet. The problem is solved completely, and the results show that the familiar technique of "bluffing" is a necessary element of any optimal strategy.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The thirtieth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Geneva College, Beaver Falls, Pennsylvania, on April 28, 1956. Dean L. T. Moston, Chairman of the Section, presided during both the morning and afternoon sessions.

There were 59 persons present, including 41 members of the Association.

The Secretary reported that in accordance with the wishes of the members involved and with the approval of the Executive Committee of the Section the Board of Governors of the Association had voted to transfer Cabell County, West Virginia, from the Allegheny Mountain Section to the Ohio Section.

The following papers were presented:

1. *Semantics in information theory*, by Professor Evan Johnson, Jr., Pennsylvania State University.

It was pointed out that Warren Weaver defined the semantic problem of communication theory as that concerned with the precision with which transmitted signals convey the desired meaning. Meaning is assigned by the individual to either a single communication signal or to a set of signals, each of which is a continuous function of time. A first problem of semantics, therefore, is the definition of the set of all functions of time which are distinguishable by the individual, and to which meaning can potentially be assigned.

2. *A generalization of the game of Nim*, by Professor Donald C. Benson, Carnegie Institute of Technology.

A class of two-person games was considered which includes "Nim" and "Kayles" as special cases. (Definitions of "Nim" and "Kayles" can be found in Ball's *Mathematical Recreations*.)A criterion was given to determine the winning and losing positions.

3. *Games of evasion*, by Professor L. E. Dubins, Carnegie Institute of Technology.

The speaker discussed an interesting class of games introduced by R. P. Isaacs. He sketched some solutions and mentioned some unsolved problems.

4. *Geology as a source of problem material in elementary mathematics*, by Professor E. A. Sturley, Allegheny College.

Many of the students in undergraduate courses do not have mathematics or physics as primary interests. These courses should include more problem material from other areas to help these students relate the mathematics course to their own fields. Some geological applications of mathematical ideas were presented as illustrations.

5. *Coordination with the secondary school and possible secondary curricular changes*, by Professor J. H. Neelley, Carnegie Institute of Technology.

This paper suggests curtailing the geometry courses, specially solid, and filling in with more algebra and analytic trigonometry. Personal contacts, luncheons, meetings of college and high school mathematics teachers are also advised.

6. *Computers in business and science*, by Mr. L. P. Rosenberry, I. B. M. Corporation, Detroit, Michigan.

The history of digital computer usage in commercial and engineering applications was reviewed. Several examples of present and past problems successfully solved were presented. Future plans for use of computers by several companies were indicated—linear programming, inventory control, marketing, plant design, process control, scientific calculations. Some general prognostications were presented.

I. D. PETERS, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-third regular meeting of the Iowa Section of the Mathematical Association of America was held at Grinnell College, Grinnell, Iowa, on April 20–21, 1956.

The attendance was 55, including 29 members of the Association.

The following officers were elected: Chairman, Professor F. W. Lott, Iowa State Teachers College; Vice -Chairman, Professor A. H. Blue, Cornell College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following motion carried unanimously:

WHEREAS, the cooperation of the high-school and college teachers of mathematics in Iowa is deemed to be of vital concern to the members of the Iowa Section of the Mathematical Association of America.

NOW THEREFORE, BE IT RESOLVED that the Iowa Section of the Mathematical Association of America hold, each year, a meeting in addition to the annual Spring meeting to be devoted to the pedagogic interests of collegiate mathematics and held in conjunction with a meeting attended by large numbers of Iowa high-school teachers of mathematics.

The following papers were presented:

1. *Pascal's arithmetical triangle*, by Professor R. B. McClenon, Grinnell College.

Professor McClenon presented Pascal's construction of the arithmetical triangle, and several theorems which Pascal proved by mathematical induction. It was pointed out that they, as well as the well-known applications which Pascal made, could be proved easily by using modern notation.

2. *An application of the functional equation* $f(x+y)=f(x)f(y)$ *in elementary differential equations*, by Professor L. E. Pursell, Grinnell College, and Professor R. F. Reeves, Ohio State University.

It is well known that if $y(x)$ satisfies the differential equations $dy/dx=ky$, then $r(x)=y(x)/y(0)=\exp(kx)$ satisfies the functional equation $r(x_1+x_2)=r(x_1)r(x_2)$; and, conversely if $r(x)$ is differentiable at $x=0$ and satisfies this functional equation, then for any choice of $y(0)$ the function $y(x)=y(0)r(x)$ satisfies the differential equation $dy/dx=[r'(0)]y$. In some classroom applications in which this differential equation is involved, it is more convincing to the students to demonstrate first that the functional equation is satisfied. The window problem as given in section 3.3 of *Differential Equations* by Agnew is solved as an example.

3. *Common elements*, by Professor F. A. Brandner, Iowa State College.

A rigorous and complete treatment of the subject, "Common Elements", is not to be found in any text-book of statistics, and almost no research has been done in that direction. Several definitions and their equivalence are discussed. The elementary idea is extended. Properties of common elements equations and their possible uses are developed.

4. *Charts for zeros of cross product Bessel functions*, by Professor Don Kirkham, Iowa State College.

In an earlier report (this MONTHLY, vol. 59, 1952, p. 501) some methods for obtaining the zeros of $J_n(x_p)Y_m(kx_p)-J_m(kx_p)Y_n(x_p)$ and of some similar cross products having the inner pair of terms, or all four terms differentiated were presented. Now, utilizing these methods, a series of large graphs (charts) of x *vs.* kx, from which the zeros of the cross products, with and without the primes, for $n=m=0, 1, 2, 3$, $p=1, 2, \cdots 9$, $0\leqq k\leqq\infty$, $k=1$ excluded, can be read to about two

decimal accuracy, has been prepared. The zeros of $J_n(x)$ and of $J_n'(x)$ lie on the graphs and are seen to be closely enough disposed to enable one to draw, to about two decimal accuracy, except for x or kx near zero, graphs of x *vs.* kx from a set of zeros of $J_n(x)$ and/or of $J_n'(x)$ alone. Zeros of $Y_n(x)$ and of $Y_n'(x)$ also lie on the graphs and are identified. The charts are of further particular interest in that they provide approximate values of x_p for large k, values which are not tabulated in the literature; show clearly that a zero of the cross products exists at $k=\infty$ and $k=0$; and show that a zero at $k=\infty$ or $k=0$ is a zero of either $J_n(x)$ or of $J_n'(x)$.

5. *History of the first forty years of the Iowa Section of the Mathematical Association of America*, by Professor Fred Robertson, Iowa State College.

The author discussed the formation of the section on April 28, 1916, in Des Moines. The names of the charter members were given. The minutes of the first meetings were reviewed. The fact that the section has met jointly with the Iowa Academy of Science since its founding and met with the State Teachers Association for the first five years of its existence was emphasized in the discussion of time, and place of its meetings. Changes over the years were shown and hopes for future progress expressed.

6. *Remarks on commuting automorphisms of rings*, by Professor M. F. Smiley, State University of Iowa.

Nathan Divinsky (*Trans. Royal Soc. Canada*, Sec. III, (3), vol. 49, 1955, pp. 19–22) discusses automorphisms T of an associative ring A which satisfy $x\cdot x^T=x^T\cdot x$ for every x in A. The present note frees this discussion from the chain condition employed by Divinsky by using results of the Jacobson structure theory of rings.

7. *A characterization of a certain type of distribution*, by Professor R. V. Hogg, State University of Iowa.

If $x_1, x_2, \cdots, x_n$ be n ordered, initially independent observations of a random variable having a frequency of the continuous type on the interval $(0, b)$, then a necessary and sufficient condition that x_j and x_i/x_j, $i<j$, be stochastically independent is that the frequency be of the form cx^{c-1}/b^c, $0<x<b$, zero elsewhere, where $c>0$.

8. *A certain limiting distribution*, by Professor A. T. Craig, State University of Iowa.

Let $F(x)$ denote the c.d.f. of a random variable x and $G_n(\bar{x})$ that of the arithmetic mean $\bar{x}$ of a random sample of n values of x. It is well known that if $E(x)=\mu$ is finite, the sequence of distributions $G_n(\bar{x})$ converges to the distribution

$$G(\bar{x}) = 0, \qquad \bar{x} < \mu,$$
$$= 1, \qquad \bar{x} \geqq \mu,$$

at every continuity point of $G(\bar{x})$. If $E(x)$ does not exist, little is known concerning the limit of the sequence $G_n(\bar{x})$. For the class of distributions having

$$F(x) = \int_{-\infty}^{1} \frac{cy^{2[p-1]}}{(1+y^2)^p}\,dy,$$

p a positive integer, it is proved that the sequence of distributions $G_n(\bar{x})$ converges to the distribution

$$G(\bar{x}) = \frac{1}{\pi}\int_{-\infty}^{\bar{x}} \frac{K}{(K^2+t^2)}\,dt.$$

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section of the Mathematical Association of America was held on April 28, 1956 at the University of Kentucky, Lexington, Kentucky. Professor V. F. Cowling, Chairman of the Section, presided at both the morning and afternoon sessions.

There were 52 persons in attendance, including 41 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor G. G. Roberts, Berea College; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor J. C. Eaves, University of Kentucky.

By invitation of the Committee, Professor Fred Ficken of the University of Tennessee delivered an hour address at the afternoon session on "Isolated Singularities of Continuous Vector Fields in the Plane." An abstract of this address follows.

This paper was an expository account of the elementary classical theory. The behavior near the origin of streamlines of fields $(ax+by,\ cx+dy)$, $(ad-bc \neq 0)$ was treated in detail by reduction to familiar canonical forms (nodes, pass, focus, and center). After brief remarks on perturbation of these fields, the speaker introduced for any continuous field the class Γ of simple closed sectionally smooth curves not passing through a singularity of the field, and discussed the isotopy classes of Γ. Finally, the standard results on the Poincaré index of $\gamma \in \Gamma$ with respect to the field were presented.

The following papers were presented:

1. *A criterion for the irreducibility of a polynomial*, by Professor A. W. Goodman, University of Kentucky.

This was an expository talk based on a result due to Perron, *Jour. fur die Reine und Ang. Math.* vol. 132, 1907, pp. 288–307.

2. *The comparability of cardinal numbers*, by Mr. K. E. Stoll, University of Kentucky.

The usual proof of the important theorem on the comparability of cardinal numbers depends on the elementary theory of ordinal numbers and the well ordering theorem. In this paper a proof of the comparability theorem was given without introducing ordinal numbers, and without direct use of the well ordering theorem. The proof did require the axiom of choice.

3. *Gibbs phenomenon for Taylor means*, by Mr. Joseph Cornelison, University of Kentucky.

The preservation of Gibbs Phenomenon under various linear transformations was discussed, including recent work by O. Szasz [*Acta Scientiarum Mathematicarum*, vol. 12, 1950, pp. 107–111]. New results on "Gibbs Phenomenon for Taylor Means" were stated and the method of proof outlined.

4. *The transcendence of e*, by Mr. Richard Sprague, University of Kentucky.

The speaker presented a proof of the transcendence of e as found in Edmund Landau, *Vorles-*

ungen uber Zahlentheorie, vol. 3, 1927, Leipzig, p. 90+, pointing out the relatively elementary nature of the necessary ideas.

5. *Some remarks on polynomials*, by Mr. Garnett Stephens, University of Kentucky, introduced by Professor V. F. Cowling.

Descartes' rule provides an upper bound on the number, N, of real zeros of a polynomial, $P(x)$, with real coefficients. Using a generalization of a lemma of Fejer [*C.R. Acad. Sci. Paris*, vol. 158, 1914, pp. 1328–1331], a criterion for obtaining a lower bound on N is obtained based on the Tchebychef polynomial representation of $P(x)$. The method is similar to that used by Turan [*Bull. Amer. Math. Soc.*, vol. 55, 1949, pp. 797–800] with Laguerre polynomials.

6. *An approach to determinants via matrices*, by Professor Frank Levin, University of Kentucky, introduced by Professor A. W. Goodman.

The relationship between multiplication of determinants and of the matrices associated with them provides the well-known rules regarding row and column operations on determinants. This relationship can be made geometrically intuitive by a consideration of the effects of various linear transformations of the volume of a parallelepiped. Further, the linear transformations yield the rule for matrix multiplication. The paper gives an elementary method of proof of the various geometrical propositions involved.

7. *Recent trends in freshman mathematics programs*, by Professor W. L. Moore, University of Louisville.

The speaker reported on his visits to the University of Chicago and Case Institute of Technology during a semester's leave of absence. A symposium on the freshman program by Professor Price of the University of Kansas was held at the Institute. Two programs suggested to Dr. Moore are to be tried at the University of Louisville in the fall semester. One of these, suggested by the Chicago staff, will give an introduction to the principles of mathematics and the other a course in calculus not requiring analytic geometry as a prerequisite.

A. W. GOODMAN, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The spring meeting of the Missouri Section of the Mathematical Association of America was held jointly with the Missouri Council of Teachers of Mathematics at Fontbonne College, St. Louis, Missouri, on April 21, 1956. Professor H. D. Brunk, Vice-Chairman of the Section, and Miss Frances Story, Chairman of the Missouri Council of Teachers of Mathematics, presided at two parallel morning sections. Professor Francis Regan, Chairman of the Section, presided at the afternoon session. There were 86 persons in attendance, including 42 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor R. J. Michel, Southeast Missouri State College; Vice-Chairman, Professor W. R. Utz, University of Missouri; Local Secretary-Treasurer, Professor C. H. Dalton, Southeast Missouri State College. Professor Margaret F. Willerding, Harris Teachers College, was re-elected Association Secretary for a period of another five years.

The following papers were presented:

1. *A report of a conference between high school and college teachers of mathe-*

matics and science, by Professor C. A. Johnson, University of Missouri, School of Mines and Metallurgy at Rolla.

A joint panel discussion by college professors and high school teachers concerned three problems: (1) How can we induce more students to complete high school? (2) How can we revise or revitalize the present high school curriculum to make it suitable to progress in scientific areas such as engineering? (3) How can we secure the personnel to carry out the teaching of science and mathematics on the high school and college level, especially in view of the salary inducements provided by industry?

2. *College arithmetic for prospective teachers*, by Professors Marie A. Moore and Jesse Osborn, Harris Teachers College, presented by Professor Moore.

The results of Arithmetic Proficiency Tests given at Harris Teachers College have proven that many of the students are poorly equipped in arithmetic content. The need of more effort to strengthen these students in the content before they attempt courses in methods was emphasized. Permanent progress in teaching methods must begin with and must not lose contact with the experiences and judgments of successful teachers of all times. Such progress has been hindered by promoters of catch-words urging swings from one extreme position to another.

3. *The Boolean geometry of the integers*, by Professor J. L. Zemmer, University of Missouri.

An abstract set S is called a Boolean space if there is a function $d(x, y)$ defined on the set of pairs of elements of S with values in a Boolean algebra B and satisfying the usual requirements of a distance function, namely, (1) $d(x, y)=d(y, x)$; (2) $d(x, y)=0$ if and only if $x=y$; (3) $d(x, y) \subseteq d(x, z) \cup d(z, y)$. In this paper a distance function is defined on the ring of integers with values in a suitable Boolean algebra. Some of the geometric properties of this Boolean space are discussed. The main result is a description of the group of isometries.

4. *Analytical functionals and symbolic calculus*, by Professor Maria Castellani, University of Kansas City.

The speaker outlined the theory of Fantappie's indicatrices and their basic concepts and applications to the theory of analytical functionals and symbolic calculus. A special consideration was given to the use of these indicatrices in methods of solving partial differential equations.

5. *Recreational mathematics for use in the elementary classroom*, by Mrs. Ruth H. Nies, Ladue School System, introduced by the Secretary.

Recreational arithmetic makes use of entertaining materials which stimulate interest, create favorable attitudes, organize and review fundamental processes, and strengthen basic concepts. Classroom experiences have shown that recreational methods, properly used, result in improved achievement as evidenced on standardized tests and in practical application of problem solving. Students at all grade levels may benefit from a recreational approach, which is worthwhile for the average and remedial groups as well as for those in an enrichment program. The values of recreational arithmetic in modern teaching should not be overlooked.

6. *What position for recreation in high school classrooms?* by Mrs. Mattie B. Ryland, Higginsville High School, introduced by the Secretary.

Adaptation of teaching methods to meet needs of the high school mathematics student may well include increase in the use of and variety of recreational mathematics. Material used must prove enjoyable to the student lest he consider the term a misnomer. Use provokes questions. Shall all be required to participate? Shall the purpose be motivation, a method used with expected regularity, a guard against loss of interest, or an insert between units? Careful selection of material

and consideration of these questions by the instructor will determine position in the individual classroom.

7. *Glimpses of mathematical recreations on the college campus*, by Professor Margaret F. Willerding, Harris Teachers College.

The speaker discussed two phases of recreational mathematics on the college campus. The first phase was an outline of an in-service course for teachers called "Recreational Mathematics for Use in the Elementary and Secondary Classroom". The second phase was the use of recreational mathematics as a motivating factor in undergraduate courses of mathematics. Recreational mathematics was particularly stressed in teacher training courses.

8. *Mathematics in the liberal arts curriculum*, by Professor A. E. Ross, Notre Dame University. (By invitation.)

In the present lecture we propose what we feel to be a well rounded and effective program of training in mathematics for non-mathematical specialists. The proposed program begins with the study of the role of language as a means of communicating ideas, of the role of technical terminology and of the very basic type of mathematical experience leading to the study of the algebra of classes and the Boolean algebra. The program continues with the study of number theory, projective geometry, real numbers, vectors, analytic geometry, and complex numbers.

Some basic ideas of calculus are introduced at the conclusion of the course. Without minimizing the importance of the acquisition of manipulative skills of mathematics, one must realize that mathematics achieves the standing of a liberal arts subject only if the teaching reaches beyond manipulation into the wealth of mathematical ideas and if it does not neglect to bring out the dynamic or creative side of the subject.

MARGARET F. WILLERDING, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-second annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on April 21, 1956. Professor L. K. Jackson of the University of Nebraska presided.

Thirty-eight persons attended the meeting. Of these, twenty-two were members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor J. M. Earl, University of Omaha; Vice-Chairman, Professor L. K. Jackson, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

The following papers were presented:

1. *Infinite exponentials*, by Mr. D. W. Miller, University of Nebraska.

By analogy with infinite series and infinite products, we define infinite exponentials in the following manner. Suppose given a countably infinite sequence $a_1, a_2, a_3, \cdots$ of non-negative real numbers. For real numbers a and b, with a, b not both zero, we denote a^b by $[a, b]$. Now define the sequence $X_1, X_2, X_3, \cdots$ by

$$X_n = [a_1, [a_2, [a_3, \cdots, [a_{n-1}, a_n]] \cdots], \qquad n = 1, 2, 3, \cdots .$$

Then $[a_1, [a_2, [a_3, [\cdots$ is called an infinite exponential written $X_{i=1}^{\infty} a_i$. If $\lim X_n = E < \infty$, we say that the exponential converges to E, otherwise we say that the exponential diverges. Certain special classes of infinite exponentials are considered, and questions of convergence and divergence are discussed.

2. *About the arithmetic and geometric mean*, by Professor Arne Magnus, University of Nebraska, introduced by the Secretary.

Various arithmetic and geometric proofs of the inequality between the arithmetic and geometric means and related inequalities were presented.

3. *A locus on the sphere*, by Professor O. C. Collins, University of Nebraska, introduced by the Secretary.

In a circle, a chord TPT', perpendicular to the radius through O, subtends a central angle $2K$. A sphere touches the plane of the circle at O. S and S' are those points on the sphere which are nearest to T and T'. In the plane of the circle lies an ellipse, which has for its major axis that diameter of the circle which passes through O, and for the value of its eccentricity the sine of one half of the arc SS' of the sphere. A point P, on the sphere, is determined so that the foot of a perpendicular let fall from it onto the plane of the circle lies on the ellipse. It is proved that in the spherical triangle SPS' the angle at P is equal to K.

4. *New mathematics courses for freshmen and sophomores*, by Professor G. B. Price, University of Kansas. (By invitation).

5. *Lattice of divisibility and multiplication of the natural numbers*, by Professor H. B. Ribeiro, University of Nebraska.

6. *Functions defined by integrals*, by Professor W. G. Leavitt, University of Nebraska.

The method of defining a function by means of an integral is illustrated in a simple way by considering such a definition for $\log x$. Many of the properties of the logarithm can then be derived by using elementary properties of integrals.

7. *Quasi-convex subsets of finite projective geometries*, by Dr. E. J. Schweppe, University of Nebraska.

A set Q of points in a projective space is called quasi-convex if at least one of the points C, D is in Q whenever C and D are separated harmonically by points A and B in Q. It is easily shown that the pointwise complement of a quasi-convex set is also quasi-convex. Using this result it is shown that, in a projective geometry which contains only a finite number of points, all quasi-convex subsets are almost trivial and those which do exist are then described explicitly.

8. *The decline and fall of the general solution empire*, by Mr. H. W. Becker, Radio Engineering Institute, Omaha, Nebraska.

Since only a few of the simplest problems of Diophantine analysis can have a general solution in polynomials, a more realistic program must be set up. A problem is solved through n if all solutions $T(n)$ in integers equal to or less than n are tabulated. Let the number of solutions covered by a particular algebraic formula, and all such formulas known, be $A'(n)$ and $A(n)$ respectively. Then (adapting an idea from Dickson's *History* II, xx) the virility of the particular formula is $V'(n) = A'(n)/T(n)$, or $V(n) = A(n)/T(n)$. In most problems of plane and solid arithmogeometry, the virility of algebra is a small percent.

EDWIN HALFAR, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-fifth annual meeting of the Illinois Section of the Mathematical Association of America was held at Eastern Illinois State College, Charleston, Illinois, on May 11 and 12, 1956. Dean Hugh Beveridge, Chairman of the Sec-

tion, presided at all sessions.

There were 63 persons in attendance, including 43 members of the Association.

At the business meeting on Friday afternoon the following officers were elected to serve for the coming year: Chairman, Professor F. E. Hohn, University of Illinois; Vice-Chairman, Professor C. T. McCormick, Illinois State Normal University; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. Professor Arnold Wendt reported on the work of the Committee on Contests and Awards stating that about 1000 more high school students participated this year than a year ago. Professor L. A. Ringenberg reported that the Committee on the Strengthening of the Teaching of Mathematics in conjunction with a committee representing the Illinois Council of Teachers of Mathematics were working cooperatively with the State Office of Public Instruction in arranging for correspondence work in higher mathematics in the smaller schools. Among other things, they were making it easier to publish articles on the teaching of mathematics in elementary and secondary schools.

The following program was presented:

1. *Three problems*, by Professor A. W. McGaughey, Bradley University.

This paper presented three problems leading to the same two pairs of algebraic equations in two unknowns. Four pairs of solutions were obtained, one pair being positive integers, another pair being rational but non-integers and the other two pairs being complex numbers. It pointed out that the positive integers were the only solution for one problem, the rational pair could be used also for the second problem. It then showed how the complex numbers could be interpreted as real solutions for the third problem.

2. *Preface to undergraduate research*, by Miss Rose Lariviere, University of Illinois, Navy Pier.

Some important features of the training required by the immature research worker were discussed, and group projects for elementary college classes, intended to increase confidence and to improve mathematical style, were described.

3. *Differentials*, by Professor M. E. Munroe, University of Illinois, introduced by the Secretary.

A valuable part of calculus, the algebra of differentials, is usually presented as suggestive only and scrupulously avoided in rigorous treatments. Development of calculus is strengthened by introducing the theory of differentiable manifolds in a suitably specialized, concrete form. Even this theory is enhanced by using the differential in the definition of the integral. Replace the usual differences $x_i - x_{i-1}$ by values of the operator dx on segments of the tangent line. When this is properly done, a rigorous proof of the substitution formula becomes trivial. A similar treatment of multiple integrals makes obvious the Jacobian formula for change of variable.

4. *Scientists and the secondary schools*, by Dr. J. R. Mayor, Director, Science Teaching Improvement Program, American Association for Advancement of Science.

Professional scientific organizations are now more deeply concerned about science and mathematics education in the secondary schools than at any other time in this century. Scientists recognize that they have a responsibility for the improvement of secondary-school teaching which has

not been adequately met. Throughout the nation there is evidence of new interest in teacher education and of a desire on the part of college and university staff members in academic departments to cooperate with secondary-school teachers and administrators in improving secondary-school programs. If this interest and spirit of cooperation can be properly coordinated and fostered, desirable modification of some goals and trends in secondary education that have been questioned by scientists could result.

5. *Gunderson-Hoffman semi-groups*, by Mr. Gayne Gunderson and Mr. D. K. Hoffman, presented by Professor Gertrude Hendrix, Eastern Illinois State College.

Gunderson and Hoffman have found that the "set of eight numbers" reported by Porges (this MONTHLY, vol. 52, 1945, pp. 379–382) exists not because of peculiar properties of those eight natural numbers, but because of the base-ten notation in which they are symbolized. Using Porges' operator on numbers written to other bases reveals a large family of semi-groups, probably infinite. In their paper, *Groupoids determined by summing squares of digits of a natural number written to an arbitrary base*, theorems are proved establishing that exhaustive results for each base can be obtained by considering only numbers less than the square of the base. Two theorems predicting elements of two semi-groups for each odd base also are proved.

Further investigation takes two directions: (1) search for further relations by which elements of semi-groups for a given base can be predicted; (2) consideration of semi-groups which may appear when higher powers of the digits are summed.

6. *Mathematics clubs in Illinois*, by Professor D. R. Bey, Illinois State Normal University.

The present status of mathematics clubs is important because it reflects educational philosophy. At the time of the state-wide study during the school year 1954–1955 about one secondary school in ten had an active club in mathematics; indications were that the number of mathematics clubs was decreasing. In schools where clubs were in existence, teachers placed a higher evaluation on their contributions to the mathematics program than did the teachers in schools without clubs. There is a need for a state association of mathematics clubs in order to establish a unified policy, strengthen the programs of individual clubs, and encourage the formation of new clubs.

7. *A mathematical introduction to logic circuits*, by Professor F. E. Hohn, University of Illinois.

This paper outlines those aspects of propositional logic which can be represented by electronic circuits and shows how logical diagrams for these circuits may be developed.

A. W. MCGAUGHEY, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The May meeting of the Minnesota Section was held in conjunction with the annual meeting of the Minnesota Academy of Science at Augsburg College in Minneapolis on May 5, 1956. There were 64 persons attending the meeting including 54 members of the Association.

Professor George Soberg of Augsburg College presided at the morning session and Professor E. J. Camp of Macalester College, the retiring Chairman of the Section, presided at the afternoon session.

At the business meeting, the following officers were elected to serve during the coming year: Chairman: Professor Walter Fleming, Mankato State Teachers College; Secretary-Treasurer: Professor F. C. Smith, College of St. Thomas;

Executive Committee: Professor E. J. Camp of Macalester College, Professor W. B. Fulks, University of Minnesota, and Professor F. L. Wolf, Carleton College.

By invitation of the Executive Committee, Professor W. L. Hart of the University of Minnesota delivered an address entitled "Certain Iterative Gradient Methods for the Solution of Systems of Equations." The abstract of this address follows:

With $x=(x_1, x_2, \cdots, x_n)$ in real n-space, $f(x)$ real valued, and $x^{(0)}$ a given point, the author first described the well-known vector correction $\Delta x^{(0)}$ in the path of steepest descent to obtain $x^{(1)}=x^{(0)}+\Delta x^{(0)}$ as a next approximation to a solution of $f(x)=0$. Then, the paper discussed two methods for obtaining a sequence $\{x^{(m)}\}$ converging to a solution of a system $f(x)=0$, $f=(f_1, \cdots, f_k)$, where gradient corrections are involved, as follows. (a) The sequential projection method (due to S. Kaczmarz for linear systems and to C. B. Tompkins for general systems). (b) The composite Newton-Raphson gradient method (due to W. L. Hart and T. S. Motzkin) which reduces, for linear systems, to a special case of a method for the linear case due to M. R. Hestenes and M. L. Stein. The method of proof for convergence of $\{x^{(m)}\}$ in (b) was outlined.

The following short papers were presented:

1. *A measurable mapping from one circle into another*, by Professor Hidehiko Yamabe, University of Minnesota.

Let C denote the circle group, *i.e.*, the real numbers mod 1. We can define a mapping of C into itself associated with an integer m which maps x into mx. Then, for any two given circles, say C_1 and C_2, does there exist a measurable mapping f so that $f(mx)=n(fx)$ holds for almost all x on C_1? For the case $m \geqq n$, there exists such a mapping f with meas $(f(x), x \in C_1)=1$. The author proposes a conjecture that when $m<n$, meas $(f(x), x \in C_1)=0$ if such a mapping f exists.

2. *An application of mapping*, by Mrs. Hildegarde Howden, St. Olaf College.

Proposed: To show that the laws of the 12-tone technique of musical composition form a group by satisfying the following conditions: (1) The identity case 0 belongs to the set T of transformations; (2) If R (In, RIn) belongs to T, then its inverse $R^{-1}(In^{-1}, (RIn)^{-1})$ does; (3) If R and In (or R and RIn, In and RIn) belong to T, then the composite RIn does. In general, composition of mappings is not commutative; however, the transformations in the 12-tone technique of musical composition are commutative, *i.e.*, $RIn=InR$.

3. *Some elementary aspects of linear programming*, by Mr. J. S. Hill, Actuary, Minnesota Mutual Life Insurance Company, St. Paul, Minnesota.

The most popular text on the subject is by Charnes, Cooper and Henderson, and it was prepared from materials developed in a research seminar at the Carnegie Institute of Technology. The "simplex method" of computation is credited to G. B. Dantzig of the Rand Corporation. The subject has sparked the interest of economists, engineers, statisticians, mathematicians, and persons associated with business and industry.

The theory rests on the proposition that within an n-dimensional space there can be constructed a convex polyhedral cone whose boundaries are determined by the limitations which are basic to the problem. It is then possible to proceed from point to point in a manner such that the distance from the origin (*i.e.*, the value of a linear functional) is never decreased. Thus by a methodical process, a maximum value of the functional is obtained. The process consists of setting up a matrix or "simplex tableau" and using specific rules and formulae to transform the matrix by successive steps. At each step a simple test is applied to see if the maximum has been reached.

The types of problems to which the technique has been applied are machine productivities and capacities, optimum storage, shipment and distribution of goods, minimizing set-up times in

machine shop operations, and optimal labor allocation. Military combat and logistics problems, personnel selection, procurement, transportation, and Leontif-type problems are also actively being explored with this technique. The author hopes to experiment in certain life insurance management problems. As in all areas of applied mathematics, the highest order of skill is required in setting up the problem, developing reliable functions for the variables and in making a valid interpretation of the result.

4. *A panel discussion on the state committee for evaluation of secondary mathematics*, by Professor D. A. Johnson, University of Minnesota, Mr. Clarence Olander, St. Louis Park High School, and Professor F. L. Wolf, Carleton College.

The panel discussed the efforts of a state committee to develop methods and procedures for studying and improving the mathematics program in the secondary schools of Minnesota. The panel presented the suggested methods which the committee has developed to date and asked the audience for comments and criticisms to guide the committee in its future work.

F. C. SMITH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Spring, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS

KENTUCKY, Berea College, Berea, April, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College of William and Mary, Williamsburg, Virginia, December 1, 1956.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, March 23, 1957.

MINNESOTA, Concordia College, Moorhead, October 6, 1956.

MISSOURI, Southeast Missouri State College, Cape Girardeau, Spring, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 27, 1957.

NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.

NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.

OHIO

OKLAHOMA, Oklahoma City University, October 26, 1956.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, Muhlenberg College, Allentown, Pa., November 24, 1956.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, Spring, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May, 1957.

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OCTOBER 1956

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ELEMENTARY INTEGRALS

D. H. POTTS, Navy Electronics Laboratory, San Diego

1. Introduction. This paper is concerned with the following problem: What are the conditions that an elementary function possess an elementary indefinite integral? This question was first successfully treated by J. Liouville in a series of papers which appeared between 1833 and 1840. The work of Liouville and subsequent investigators is presented in Ritt's monograph [1]. Although the problem is one which arises very naturally in the first course in calculus, its solution, as presented in Ritt [1], requires mathematical techniques of a considerably more advanced level. It is the purpose of this paper to show that a more elementary treatment is possible if one uses a more restrictive definition of elementary function.

2. Elementary functions. In the calculus an elementary function is one which can be constructed out of the variable x by repeated use of: (a) algebraic operations, (b) the taking of exponentials, (c) the taking of logarithms. To make "algebraic operation" more explicit we recall the notion of algebraic function. One says that y is an algebraic function of x if it is defined by an irreducible relation

$$a_0y^n + a_1y^{n-1} + \cdots + a_n = 0$$

where $a_0, a_1, \cdots, a_n$ are polynomials in x, and a_0 is not identically zero.

It is immediately clear from this definition why one must use more advanced techniques of analysis in treating the problem under consideration. For the algebraic functions are, in general, multiple-valued, and one is naturally led to consideration of branch points, Riemann surfaces, and related topics in theory of functions of a complex variable.

It is also clear that one may avoid this complication if one is willing to restrict the nature of "elementary function" as regards "algebraic operation", if one requires that y be elementary in x only if one has an *explicit* relationship $y=f(x)$ in which the operator f is constructed from the three basic types, *viz.* algebraic, exponential, logarithmic. In this case one means by an algebraic operation one which involves only the extracting of root in addition to the usual rational operations. However, one may even dispense with extraction of root, since such an operation can be replaced by a combination of rational, exponential, and logarithmic operations, *i.e.*,

$$x^{1/n} = e^{(1/n)\log x}.$$

Henceforth, we shall consider "elementary function" as being understood in this narrower sense.

3. The construction of an elementary function. In order to state more explicitly the way in which an elementary function is constructable we shall employ Ostrowski's method of field extensions (see Ritt, p. 42). This device offers us a two-fold advantage: (i) it is simpler than the structure given by Ritt ([1], p. 10–11); (ii) one can carry out the theory for a "generalized" concept of elementary function with no additional difficulty.

At the core of this method is the concept of a differential field, *i.e.*, a class of functions which form an algebraic field and is closed with respect to the operation of differentiation. To be more specific, we shall say that a class $\mathfrak{F}$ of real valued functions of a real variable, each of which is piecewise differentiable on a common finite interval I, is a *differential field* provided that

(i) $\mathfrak{F}$ contains all the constant functions

(ii) $\alpha, \beta \in \mathfrak{F} \Rightarrow \alpha+\beta, \alpha\cdot\beta \in \mathfrak{F}$

(iii) $\alpha \in \mathfrak{F} \Rightarrow \alpha' \in \mathfrak{F}$

(iv) $\alpha \in \mathfrak{F} \Rightarrow 1/\alpha \in \mathfrak{F}$ unless $\alpha \equiv 0$ on I.

We note that (iii), (iv) imply that the zeros of any non-constant member of $\mathfrak{F}$ are isolated.

We now introduce the concept of *field extension*. If $\mathfrak{F}$ is a differential field and θ is a piecewise differentiable function not in $\mathfrak{F}$, then the class of rational functions of θ with coefficients in $\mathfrak{F}$ is a field. If it is also a differential field, then we call this field a *simple extension* of $\mathfrak{F}$ and denote it $\mathfrak{F}(\theta)$.

We generalize the last. $\mathfrak{F}(\theta_1, \cdots, \theta_n)$ is a simple extension of $\mathfrak{F}(\theta_1, \cdots, \theta_{n-1})$. We call $\mathfrak{F}(\theta_1, \cdots, \theta_n)$ an extension of $\mathfrak{F}$.

It is clear now how one may describe a structure for a given elementary function. One starts from the field of rational functions of x and makes the necessary extensions in each case by adjoining the appropriate exponential or logarithm. For example, suppose that

$$\alpha(x) = e^x + \log (x + e^x).$$

If $\mathfrak{F}$ is the field of rational functions of x then

$$\alpha \in \mathfrak{F}(e^x, \log (x + e^x)).$$

4. Theorems on differential fields. In the following we shall assume that $\mathfrak{F}$ is a given differential field. Identities will be understood to be identities on the interval I on which the functions of the field are defined.

THEOREM 1.

$$\left.\begin{array}{ll} \text{(i)} & \theta \notin \mathfrak{F} \\ \text{(ii)} & \theta' \in \mathfrak{F} \\ \text{(iii)} & \beta_0, \cdots, \beta_n \in \mathfrak{F} \\ \text{(iv)} & \beta_0\theta^n + \beta_1\theta^{n-1} + \cdots + \beta_n \equiv 0 \end{array}\right] \Rightarrow \beta_0 \equiv \beta_1 \equiv \cdots \beta_n \equiv 0.$$

Proof. If false, then we can assume $\beta_0 \not\equiv 0$ and that n is minimal. We can then rewrite the equation of (iv) in the form

$$\theta^n + \gamma_1\theta^{n-1} + \cdots + \gamma_n = 0$$

where

$$\gamma_i = \frac{\beta_i}{\beta_0} \in \mathfrak{F}.$$

Differentiating, we get

$$(n\theta' + \gamma_1')\theta^{n-1} + [\gamma_1(n-1)\theta' + \gamma_2']\theta^{n-2} + \cdots + \gamma_n' = 0.$$

All coefficients must vanish; otherwise n is not minimal. Thus $n\theta' = -\gamma_1'$. Therefore, $n\theta = c - \gamma_1$. But this contradicts (i).

THEOREM 2.

(i) $\alpha \in \mathfrak{F}$

(ii) $e^{p\alpha} \notin \mathfrak{F}$, $p = 1, 2, \cdots, n$

(iii) $\beta_0, \cdots, \beta_m \in \mathfrak{F}$, *where* $m < n$

(iv) $\beta_0\theta^m + \beta_1\theta^{m-1} + \cdots + \beta_m \equiv 0$, *where* $\theta = e^{\alpha}$

$\Rightarrow \beta_0 \equiv \beta_1 \equiv \cdots \equiv \beta_m \equiv 0.$

Proof. Suppose $\beta_0 \not\equiv 0$. Then, letting $\gamma_i = \beta_i/\beta_0 \in \mathfrak{F}$, we have

$$\theta^m + \gamma_1\theta^{m-1} + \cdots + \gamma_m = 0. \tag{A}$$

Differentiating, we get

$$m\alpha'\theta^m + [\gamma_1' + (m-1)\alpha'\gamma_1]\theta^{m-1} + \cdots + \gamma_m' = 0. \tag{B}$$

For any fixed value of x, the m values of θ determined by (A) also satisfy (B). Thus the left-hand sides of these equations must be equivalent as polynomials in θ. Hence $m\alpha'\gamma_1 = \gamma_1' + (m-1)\alpha'\gamma_1$, $m\alpha'\gamma_2 = \gamma_2' + (m-2)\alpha'\gamma_2$, $\cdots$, $m\alpha'\gamma_m = \gamma_m'$. If $\beta_k \not\equiv 0$, then $k\alpha' = \gamma_k'/\gamma_k$. And so $k\alpha = c + \log|\gamma_k|$. Thus $e^{k\alpha} = a\gamma_k \in \mathfrak{F}$. This contradicts (ii).

THEOREM 3.

(i) $\alpha, e^{n\alpha} \in \mathfrak{F}$

(ii) $e^{p\alpha} \notin \mathfrak{F}$, $p = 1, 2, \cdots, n-1$

(iii) $\beta_0, \cdots, \beta_{n-1} \in \mathfrak{F}$

(iv) $\theta = e^{\alpha}$

$\Rightarrow$ *There exist* $\gamma_0, \cdots, \gamma_{n-1}$, *not all zero,* $\in \mathfrak{F}$ *such that*

$$\left[\sum_{k=0}^{n-1} \beta_k\theta^k\right]\left[\sum_{k=0}^{n-1} \gamma_k\theta^k\right] \in \mathfrak{F}.$$

Proof.

$$\left[\sum_{k=0}^{n-1}\beta_k\theta^k\right]\left[\sum_{k=0}^{n-1}\gamma_k\theta^k\right] = \sum_{k=0}^{2n-2}\theta^k\sum_{j=0}^{k}\beta_{k-j}\gamma_j$$

$$= \sum_{k=0}^{n-1}\theta^k\sum_{j=0}^{k}\beta_{k-j}\gamma_j + \sum_{k=n}^{2n-2}\theta^k\sum_{j=0}^{k}\beta_{k-j}\gamma_j$$

$$= \sum_{k=0}^{n-1}\theta^k\left[\sum_{j=0}^{k}\beta_{k-j}\gamma_j + \theta^n\sum_{j=k+1}^{n-1}\beta_{k+n-j}\gamma_j\right].$$

If this product is to be in $\mathfrak{F}$, then, by Theorem 2,

$$\sum_{j=0}^{k}\beta_{k-j}\gamma_j + \theta^n\sum_{j=k+1}^{n-1}\beta_{k+n-j}\gamma_j \equiv 0$$

for $k=1, 2, \cdots, n-1$.

Now suppose that β_m is the last member of $\beta_0, \cdots, \beta_{n-1}$ which is not identical to zero. Each γ_j except γ_{n-m} occurs in at least one equation with the coefficient β_m. For if $k-j=m$, then $j=k-m$ has values 0, 1, 2, $\cdots$, $n-1+m$, while if $k+n-j=m$, then $j=k+n-m$ has values $n-m+1$, $n-m+2$, $\cdots$, $n-1$. The coefficients of γ_{n-m} are obtained from $j=n-m$. Thus $k-j=k+m-n$ and $k+n-j=k-m$. And so the following β's multiply γ_{n-m}: $\beta_0, \beta_1, \cdots, \beta_{n-1-m}$, $\beta_{m-1}, \beta_{m-2}, \cdots, \beta_0$. We can assume one of these is not zero, for if all were, then the result of the theorem would obviously hold.

We thus have a system of $n-1$ equations in n unknowns $\gamma_0, \cdots, \gamma_{n-1}$. If the rank of the matrix of this system is $n-r$, we may transpose r of the γ's leaving a non-vanishing coefficient determinant on the left. We can easily choose the γ's on the right so that not all right-hand terms are zero. We then solve for the remaining γ's.

THEOREM 4.

(i) $\alpha, e^{n\alpha} \in \mathfrak{F}$

(ii) $e^{p\alpha} \notin \mathfrak{F}$, $p = 1, 2, \cdots, n-1$

(iii) $\beta_0, \beta_1, \cdots, \beta_m \in \mathfrak{F}$

(iv) $P(\theta) = \sum_{k=0}^{m}\beta_k\theta^k \equiv 0$, *where* $\theta = e^{\alpha}$

(v) $\omega^n = 1$

$\Rightarrow P(\omega\theta) \equiv 0.$

Proof. Let $[m/n]=r$. Then

$$P(\theta) = \sum_{k=0}^{m}\beta_k\theta^k = \sum_{k=0}^{n-1}\beta_k\theta^k + \sum_{k=n}^{2n-1}\beta_k\theta^k + \cdots + \sum_{k=rn}^{m}\beta_k\theta^k$$

$$= \sum_{k=0}^{n-1}[\beta_k + \theta^n\beta_{k+n} + \cdots + \theta^{rn}\beta_{k+rn}]\theta^k \equiv 0.$$

By Theorem 2, $\beta_k+\theta^n\beta_{k+n}+\cdots+\theta^{rn}\beta_{k+rn}\equiv 0$, $k=1, 2, \cdots, n-1$. Since these last equations are unchanged by the substitution of $\omega\theta$ for θ, it follows that $P(\omega\theta)\equiv 0$.

THEOREM 5.

(i) $\alpha \in \mathfrak{F}$

(ii) $\theta = e^{\alpha} \notin \mathfrak{F}$

(iii) $\beta \in \mathfrak{F}(\theta)$

(iv) $\beta \notin \mathfrak{F}$

$\Rightarrow \beta' \notin \mathfrak{F}$.

Proof. Case 1. If $e^{p\alpha}\notin\mathfrak{F}$, $p=1, 2, \cdots$, then it follows from Theorem 2 that β can be written uniquely in the form

$$\beta = P(\theta, x) + \frac{Q(\theta, x)}{R(\theta, x)}$$

where P, Q, R are polynomials in θ with coefficients in $\mathfrak{F}$, the degree of Q being less than that of R and $(Q, R)=1$, *i.e.*, no common factor in $\mathfrak{F}(\theta)$.

Differentiating, we have

$$\beta' = P_x + P_\theta\theta\alpha' + \frac{RQ_x - QR_x + \theta\alpha'(RQ_\theta - QR_\theta)}{R^2}.$$

If $\beta'\in\mathfrak{F}$, it follows from Theorem 2 that $P_x+P_\theta\theta\alpha'\in\mathfrak{F}$. Thus, if

$$P = \sum_{k=0}^{n} \rho_k\theta^k,$$

then

$$\sum_{k=0}^{n} [\rho_k' + k\rho_k\alpha']\theta^k \in \mathfrak{F}.$$

Using Theorem 2 we have $\rho_n'+n\rho_n\alpha'=0$. And so $n\alpha' = -\rho_n'/\rho_n$, giving $n\alpha = c-\log|\rho_n|$. Thus $e^{n\alpha}\in\mathfrak{F}$; therefore $n=0$.

We now have

$$R^2(\beta' - P_x) = RQ_x - QR_x + \theta\alpha'(RQ_\theta - QR_\theta).$$

As polynomials in θ, the degree on the left is greater than that on the right. In view of Theorem 2, this is impossible.

Case 2. If $e^{p\alpha}\notin\mathfrak{F}$, $p=1, 2, \cdots, n-1$, but $e^{n\alpha}\in\mathfrak{F}$, then, by Theorem 3, every quotient of polynomials in θ of degree less than n can be written as a product of such polynomials and hence eventually as a polynomial in θ of degree less than n. Thus we may write

$$\beta = \beta_k\theta^k + \beta_{k-1}\theta^{k-1} + \cdots + \beta_0 \text{ where } k < n.$$

Differentiating, we get

$$\beta' = (\beta_k' + k\alpha'\beta_k)\theta^k + \cdots + \beta_0'.$$

If $\beta' \in \mathfrak{F}$, then, by Theorem 2, $\beta_k' + k\alpha'\beta_k = 0$. This leads to $e^{k\alpha} = a/\beta_k \in \mathfrak{F}$, which contradicts our assumption.

THEOREM 6.

(i) $\alpha \in \mathfrak{F}$
(ii) $\theta = \log |\alpha| \notin \mathfrak{F}$
(iii) $\beta \notin \mathfrak{F}$ but $\beta \in \mathfrak{F}(\theta)$
(iv) $\beta' \in \mathfrak{F}$

$\Rightarrow$ *There exists* $\gamma \in \mathfrak{F}$ *and a constant* c *such that* $\beta = \gamma + c \log |\alpha|$.

Proof. From Theorem 1 we see that β has a unique representation in the form $\beta = P + Q/R$ where P, Q, R are polynomials in θ with coefficients in $\mathfrak{F}$, $(Q, R) = 1$, and the degree of Q is less than that of R. Differentiating, we get

$$\beta' = P_x + \frac{\alpha'}{\alpha} P_\theta + \frac{RQ_x - QR_x + \dfrac{\alpha'}{\alpha}(RQ_\theta - QR_\theta)}{R^2} \in \mathfrak{F}.$$

It follows from Theorem 1 that $P_x + (\alpha'/\alpha)P_\theta \in \mathfrak{F}$. Thus, if $P = \rho_n\theta^n + \cdots + \rho_0$, then

$$\rho_n'\theta^n + \left(\rho_{n-1}' + n\rho_n \frac{\alpha'}{\alpha}\right)\theta^{n-1} + \cdots + \rho_0' \in \mathfrak{F}.$$

And so

$$\rho_n' = \rho_{n-1}' + n\rho_n \frac{\alpha'}{\alpha} = \rho_{n-2}' + (n-1)\rho_{n-1}\frac{\alpha'}{\alpha} = \cdots = \rho_1' + 2\rho_2\frac{\alpha'}{\alpha} = 0.$$

Since $\rho_n' = 0$, ρ_n is a constant. But then $\rho_{n-1} = c - n\rho_n \log \alpha$. Therefore, $\rho_n = 0$. And so ρ_{n-1} is constant. Continuing in this way, we see that $\rho_n = \rho_{n-1} = \cdots = \rho_2 = 0$ and ρ_1 is a constant. Thus

$$\beta' = \rho_1 \frac{\alpha'}{\alpha} + \rho_0' + \frac{RQ_x - QR_x + \dfrac{\alpha'}{\alpha}(RQ_\theta - QR_\theta)}{R^2}.$$

Rearranging, we get

$$\left(\beta' - \rho_1\frac{\alpha'}{\alpha} - \rho_0'\right)R^2 = RQ_x - QR_x + \frac{\alpha'}{\alpha}(RQ_\theta - QR_\theta)$$

which, by Theorem 1, is possible only if the degree of R is zero. Thus $\beta = \rho_0 + \rho_1 \log |\alpha|$, where $\rho_0 \in \mathfrak{F}$ and ρ_1 is constant.

THEOREM 7.

(i) $\alpha \in \mathfrak{F}$

(ii) $\int \alpha \in \mathfrak{F}(\theta_1, \cdots, \theta_n)$

(iii) *each θ_i is either an exponential or a logarithm*

$\Rightarrow$ *There exist $\beta_0, \cdots, \beta_n \in \mathfrak{F}$ and constants $c_1, \cdots, c_n$ such that*

$$\int \alpha = \beta_0 + c_1 \log \beta_1 + \cdots + c_n \log \beta_n.$$

Proof. Suppose $n=1$. Then the result follows from Theorems 5, 6. Assume the theorem is true for $n=1, 2, \cdots, s$. We consider the case $n=s+1$. Let $\mathfrak{G}=\mathfrak{F}(\theta_1)$. Then $\alpha\in\mathfrak{G}$ and $\int\alpha\in\mathfrak{G}(\theta_2, \cdots, \theta_{s+1})$. Thus

$$\int \alpha = \beta_0 + c_1 \log \beta_1 + \cdots + c_s \log \beta_s$$

where $\beta_0, \cdots, \beta_s\in\mathfrak{G}$ and $c_1, \cdots, c_s$ are constants. Let $\theta=\theta_1$. Then we have

$$\int \alpha = \beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x)$$

and

$$\alpha = \beta_{0,\theta}\theta' + \beta_{0,x} + \sum \frac{c_i}{\beta_i} [\beta_{i,\theta}\theta' + \beta_{i,x}].$$

Case 1. Suppose $\theta=e^\gamma$, where $\gamma\in\mathfrak{F}$ and $e^{p\gamma}\notin\mathfrak{F}$, $p=1, 2, \cdots$. Then $\theta'=\theta\gamma'$ and we have

$$\alpha = \beta_{0,\theta}\theta\gamma' + \beta_{0,x} + \sum \frac{c_i}{\beta_i} [\beta_{i,\theta}\theta\gamma' + \beta_{i,x}].$$

This must be an identity in θ by Theorem 2, since each β_i is a rational function of θ. If we replace θ by $\mu\theta$ we get

$$\alpha = \beta_{0,\theta}(\mu\theta, x)\mu\theta\gamma' + \beta_{0,x}(\theta\mu, x) + \sum \frac{c_i}{\beta_i(\mu\theta, x)} [\beta_{i,\theta}(\mu\theta, x)\mu\theta\gamma' + \beta_{i,x}(\mu\theta, x)]$$

which is the derivative of

$$\beta_0(\mu\theta, x) + \sum c_i \log \beta_i(\mu\theta, x).$$

Thus

$$\beta_0(\mu\theta, x) + \sum c_i \log \beta_i(\mu\theta, x) = \beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x) + f(\mu).$$

Differentiating with respect to μ and setting $\mu=1$ we have

$$\beta_{0,\theta}(\theta, x)\theta + \sum \frac{c_i}{\beta_i(\theta, x)} \beta_{i,\theta}(\theta, x)\theta = f'(1).$$

Therefore

$$\beta_{0,\theta}(\theta, x) + \sum \frac{c_i}{\beta_i(\theta, x)} \beta_{i,\theta}(\theta, x) = \frac{f'(1)}{\theta}.$$

This is an identity in θ. Thus

$$\beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x) = f'(1)[\gamma - \log \theta_0] + \beta_0(\theta_0, x) + \sum c_i \log \beta_i(\theta_0, x).$$

And so we can express $\int\alpha$ in the desired form.

Case 2. Suppose $\theta=\log|\gamma|$. Then

$$\alpha = \beta_{0,\theta} \frac{\gamma'}{\gamma} + \beta_{0,\theta} + \sum \frac{c_i}{\beta_i}\left[\beta_{i,\theta} \frac{\gamma}{\gamma} + \beta_{i,x}\right].$$

By Theorem 1 this must be an identity in θ. We replace θ by $\theta+\mu$. Thus

$$\alpha = \beta_{0,\theta}(\theta+\mu, x) \frac{\gamma'}{\gamma} + \beta_{0,x}(\theta+\mu, x) + \sum \frac{c_i}{\beta_i}\left[\beta_{i,\theta}(\theta+\mu, x) \frac{\gamma'}{\gamma} + \beta_{i,x}(\theta+\mu, x)\right],$$

which is the derivative of

$$\beta_0(\theta+\mu, x) + \sum c_i \log \beta_i(\theta+\mu, x).$$

And so

$$\beta_0(\theta+\mu, x) + \sum c_i \log \beta_i(\theta+\mu, x) = \beta_0(\theta, u) + \sum c_i \log \beta_i(\theta, x) + f(\mu).$$

Differentiating with respect to μ and setting $\mu=0$ we get

$$\beta_{0,\theta}(\theta, x) + \sum \frac{c_i}{\beta_i(\theta, x)} \beta_{i,\theta}(\theta, x) = f'(0).$$

This is an identity in θ. Thus

$$\int \alpha = \beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x) = f'(0)(\theta - \theta_0) + \beta_0(\theta_0, x) + \sum c_i \log \beta_i(\theta_0, x).$$

And so $\int\alpha$ is of the desired form.

Case 3. Suppose $\theta=e^\gamma$ where $\gamma\in\mathfrak{F}$ and $e^{p\gamma}\notin\mathfrak{F}$, $p=1, 2, \cdots, n-1$, but $e^{n\gamma}\in\mathfrak{F}$. We get

$$\alpha = \beta_{0,\theta}\theta\gamma' + \beta_{0,x} + \sum \frac{c_i}{\beta_i}\left[\beta_{i,\theta}\theta\gamma' + \beta_{i,x}\right]$$

where, in view of Theorem 3, each β_i is a polynomial in θ of degree less than n.

By Theorem 4 we can replace θ by $\omega^k\theta$, $k=0, 1, \cdots, n-1$, where $\omega^n=1$, in

the last. We thus have

$$\int \alpha = \beta_0(\omega^k\theta, x) + \sum c_i \log \beta_i(\omega^k\theta, x), \qquad k = 0, 1, \cdots, n-1.$$

And so

$$\int \alpha = \frac{1}{n}\sum_{k=0}^{n-1}\beta_0(\omega^k\theta, x) + \frac{1}{n}\sum c_i \log\left[\prod_{k=0}^{n}\beta_i(\omega^k\theta, x)\right].$$

But the expressions $\sum\beta_0(\omega^k\theta, x)$, $\prod\beta_i(\omega^k\theta, x)$ are symmetric polynomials in the quantities $\theta, \omega\theta, \cdots, \omega^{n-1}\theta$. Since these are the roots of $\theta^n - \delta = 0$, where $\delta \in \mathfrak{F}$, it follows that the expressions are rational in δ, and hence are in $\mathfrak{F}$. This gives $\int\alpha$ the desired form.

5. Applications. Theorem 7 of the last section is our version of the Liouville-Ostrowski Theorem ([1], p. 42). Much of the subsequent development in Ritt's monograph may now be carried through with suitable modifications. For example, we have the following version of a result due to Liouville ([1], p. 47):

THEOREM 8.

(i) $\alpha, \beta \in \mathfrak{F}$

(ii) $e^{n\alpha} \notin \mathfrak{F}$, $n = 1, 2, \cdots$

(iii) $\gamma = \int e^{\alpha}\beta$ *is elementary with respect to* $\mathfrak{F}$, *i.e.,* γ *is contained in an extension of* $\mathfrak{F}$

$\Rightarrow$ *There exists* $\delta \in \mathfrak{F}$ *such that* $\gamma = e^{\alpha}\delta + constant.$

Proof. Let $e^{\alpha} = \theta$. Then $e^{\alpha}\beta = \theta\beta \in \mathfrak{F}(\theta)$. Thus, by Theorem 7,

$$\gamma = \lambda_0(\theta, x) + \sum_{i=0}^{k} c_i \log \lambda_i(\theta, x)$$

where $\lambda_i \in \mathfrak{F}(\theta)$, $i = 0, 1, 2, \cdots, k$. Differentiating, we have

$$\theta\beta = \lambda_{0,\theta}\theta\alpha' + \lambda_{0,x} + \sum \frac{c_i}{\lambda_i}[\lambda_{i,\theta}\theta\alpha' + \lambda_{i,x}].$$

By Theorem 2 this must be an identity in θ. We replace θ by $\mu\theta$ and integrate. Then

$$\mu\gamma = \lambda_0(\mu\theta, x) + \sum c_i \log \lambda_i(\mu\theta, x) + f(\mu).$$

Differentiating with respect to μ and setting $\mu = 1$ gives

$$\gamma = \theta\lambda_{0,\theta}(\theta, x) + \sum \frac{c_i\theta\lambda_{i,\theta}(\theta, x)}{\lambda_i} + f'(1).$$

Let $\omega(\theta, x) = \lambda_0(\theta, x) + \sum c_i \log \lambda_i(\theta, x)$. We have

$$\omega(\theta, x) = \theta\omega_{,\theta}(\theta, x) + f'(1).$$

Hence

$$\gamma = \omega(\theta, x) = \frac{\theta}{\theta_0}\left[\omega(\theta_0, x) - f'(1)\right] + f'(1).$$

And so γ has the desired form.

Consider, for example, the case of $\gamma = \int e^{x^2} dx$. Theorem 8 applies where $\mathfrak{F}$ is the field of rational functions of x. If γ is to be elementary, then $\gamma = e^{x^2}\delta + \text{constant}$. But $\gamma' = e^{x^2} = e^{x^2}\delta' + 2xe^{x^2}\delta$. Hence $\delta' + 2x\delta = 1$. Now δ is rational in x and thus can be written in the form $\delta = P + Q/R$, where P, Q, R are polynomials in x, $(Q, R) = 1$, Q of degree less than R. We have

$$P' + \frac{RQ' - QR'}{R^2} + 2xP + \frac{2xQ}{R} - 1 \equiv 0.$$

This is not possible for any polynomials P, Q, R (since coefficients must vanish). And so $\int e^{x^2} dx$ is not elementary.

6. Trigonometric integrals. The restriction to real functions leaves out the integrands involving trigonometric functions, these being included in the complex case in their exponential form. But it is clear that, in order to treat such an integrand, one may replace $\sin x$ by $(1/a)\sinh ax$ and examine the resulting integrand. If its elementary integral is obtained, we replace a by $\sqrt{-1}$ to obtain an integral for the original expression.

Reference

[1] J. F. Ritt, Integration in Finite Terms, Columbia University Press, 1948.

INTEGRAL APPROXIMATE SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

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1. Statement of the problem. Consider

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad (i = 1, \cdots, m). \tag{1}$$

This is a system of m linear equations in n unknowns, where m and n are arbitrary positive integers. The coefficients, a_{ij}, are real. The case of complex coefficients can be disposed of by doubling the number of equations.

As several kinds of solutions will be mentioned, the following terms are used:

e-solution is an exact solution;

a-solution is an approximate solution;

i-solution is an integral solution;

l-solution is a least square solution.

There would also be the combinations: *ia*-solution, meaning an approximate solution where the unknowns have integral values; *lia*-solution, meaning an *ia*-solution which compared with other *ia*-solutions has the sum of the squares of its deviations a minimum.

Many different situations may arise. For example, in 3-space, the equations of the system may represent planes which form a closed polyhedron, a prism, they may all pass through a point or a straight line, or they may all be parallel.

Following are two interesting systems, each of one equation in two unknowns:

System A. $$y = \sqrt{2}x - 0.5.$$

System B. $$y = 1.1x - \sqrt{2}.$$

System A has, among others, the two *ia*-solutions $(-1, -1)$, and $(-35, -49)$. The former is more convenient to use, since the values of x and y are small integers. The latter has much the smaller error. Further *ia*-solutions with still larger values of x and y, but with still smaller errors, exist. Similar situations in higher dimensions are easily imagined by the reader. It will be shown that each of the two *ia*-solutions cited can be obtained as an *lia*-solution of a so-called Complete System.

System B has the *ia*-solutions $x = 4 + 10q$, $y = 3 + 11q$, for any integral value of q. These all have the same error, and all other *ia*-solutions have larger error, no matter how large values of x and y may be considered.

The method to be proposed for finding *ia*-solutions will be the same for any system of linear equations. It is not necessary to determine in advance which kind of a system you have. After you have carried out the solution you will know whether the system is like System A or System B, or unlike both of them.

The practical computer will object that he is not interested in *ia*-solutions. He may want the values of x_j carried out to 4 decimal places. Then he should multiply the constant terms of the system by 10^4, and proceed to find *ia*-solutions, which later can be divided by 10^4. In this way, he can be sure that his results are correct to 4 decimal places.

The procedure requires the adjoining of certain auxiliary equations, forming the so-called Complete System. The *lia*-solution of the latter depends on finding the least value of a positive quadratic form for integral values of the variables. The method of handling the quadratic form, devised by Minkowski, will be briefly outlined.

2. System of congruences. The method will also solve systems of congruences. Such a system can be converted into a system of linear equations, with one new unknown for each congruence. For example, the system of congruences

$$\begin{aligned} ax + by &\equiv c, \pmod{p} \\ dx + ey &\equiv f, \pmod{q} \end{aligned} \tag{2}$$

can be converted to the system of equations

$$(3) \qquad \begin{aligned} ax + by + up &= c, \\ dx + ey + vq &= f, \end{aligned}$$

where u and v must have integral values. The i-solutions of the System of Equations will give the i-solutions of the System of Congruences.

3. Notation. System (1) is called a non-homogeneous system. The corresponding homogeneous system is

$$(4) \qquad \sum_{j=1}^{n} a_{ij}x_j = 0, \qquad (i = 1, \cdots, m).$$

Systems (1) and (4) may each be written as a single vector equation, by introducing the vectors:

$$\begin{aligned} A_j &= (a_{1j}, a_{2j}, \cdots, a_{mj}), \qquad (j = 1, \cdots, n), \\ B &= (b_1, b_2, \cdots, b_m), \\ 0 &= (0, 0, \cdots, 0). \end{aligned}$$

Systems (1) and (4), then, become

$$(1') \qquad A_1x_1 + A_2x_2 + \cdots + A_nx_n = B.$$

$$(4') \qquad A_1x_1 + A_2x_2 + \cdots + A_nx_n = 0.$$

4. The complete system. In order to find an e-solution of System (4), m quantities must be made to vanish, namely the left members of the m equations of the system. For an a-solution, it is enough to have them close to 0.

It is also desirable that the variables, x_j, shall be in some way limited in size, as in the case of System A, where severe restrictions on the variables might lead to the ia-solution $(-1, -1)$. Less severe restrictions might lead to a more accurate solution, with larger absolute values of x and y, namely $(-35, -49)$. So there are n further quantities which we would like to have small, namely the values of x_j.

All of the $m+n$ quantities which are to be small are now listed, each with a weight, w_k, with k running from 1 to $m+n$, as follows:

$$(5) \qquad \begin{aligned} w_1(x_1 \qquad\qquad\qquad\qquad) &= v_1 \\ w_2(\qquad x_2 \qquad\qquad\qquad) &= v_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ w_n(\qquad\qquad\qquad\qquad x_n) &= v_n \\ w_{n+1}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) &= v_{n+1} \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ w_{n+m}(a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n) &= v_{n+m}. \end{aligned}$$

This is called the Complete System.

The values of the weights, w_k, are at our disposal. By decreasing the size of the first n weights, larger values of the variables are possible. In this way, you may choose when formulating the problem whether you prefer an *ia*-solution with small values of the variables and large errors or an *ia*-solution with large values of the variables and smaller errors. You can decide which of the two solutions of System A is better, $(-1, -1)$, or $(-35, -49)$. It is useless to demand the most accurate *ia*-solution, regardless of the size of the variables.

As in the case of Systems (1) and (4), the Complete System, (5) can also be written as a single vector equation:

$$V_1x_1 + V_2x_2 + \cdots + V_nx_n = V, \tag{5'}$$

where the vectors V_j and V are the columns of the extended matrix of System (5). The components of those vectors involve the weights. V is called the deviation vector. The first n components of V are called the solution part, and the last m components are called the error part of V.

DEFINITION. *A lia-solution of a system* (5) *for a given set of weights,* w_k, *consists of a set of integral values of* x_j *for which the deviation vector,* V, *has minimum length.*

As there are at most a finite number of sets of integral values of x_j for which $|V|$ is less than a particular value, one or more of these must minimize $|V|$.

The *lia*-solution of System (5) as defined has the useful property that no other set of integral values of x_j can exist for which the solution part of the deviation vector is shorter without having a longer error part. It will give you the most accurate *ia*-solution of System (4) considering the size of the variables, x_j.

5. The solution-error space, S. The vectors $V_1, \cdots, V_n$ are the basis of a space S, called the solution-error space. We shall be concerned with the integral points of S, namely those points obtained by taking integral linear combinations of vectors V_j, such as

$$P = a_1V_1 + a_2V_2 + \cdots + a_nV_n,$$

where $a_1, \cdots, a_n$ are integers. The square length of P is

$$P^2 = P \cdot P = (a_1V_1 + \cdots + a_nV_n)^2, \tag{6}$$

a positive quadratic form in the variables $a_1, \cdots, a_n$.

6. Minkowski on quadratic forms. The problem of finding the minimum value of a positive quadratic form for integral values of the variables has been treated in detail and solved by Minkowski.*

A digest of Minkowski's paper, sufficient for the practical computer, will

* H. Minkowski, Diskontinuitätsberich für arithmetische Äquivalenz, Crelle Journal 129, pp. 220–274, 1905, reprinted in collected works (Gesammelte Abhandlungen) vol. 2, pp. 51–105.

be given. Proofs are omitted. Notations are simplified. Minkowski's paper contains a proof that the proposed procedure will always lead to the smallest value of the quadratic form for integral values of the variables. The example which we work out will suggest the steps of the proof.

7. The method. The shortest non-null integral vector of S is found by a reduction process. We start with $V_1, V_2, \cdots, V_n$, an integral basis of S. At each reduction, one vector, V_c, is culled, after a suitable replacement V_r has been found. To be suitable, V_r must equal V_c plus an integral linear combination of the other vectors of the set, and V_r^2 must be less than V_c^2. When no vector can be culled because of the non-existence of a suitable replacement, the reduction is at an end, and the minimal integral basis of S has been obtained.

It will be shown by examples that in many cases replacements can be found by inspection. Suggestions will be given to aid or improve the inspection process. Nevertheless, when n is 5 or more, the inspection method may fail to uncover replacements which are present. At this point a more systematic procedure will be followed, and this (Minkowski has proved) certainly will find replacements if any exist. It would be uneconomical to use this last procedure when replacements can be found by inspection.

8. Replacements by inspection. In the early stages of a solution, replacements are frequently obvious. It may be noted that V_1+V_2 or V_1-V_2 is shorter than V_1. This happens when the large components of V_1 have the same signs as the corresponding components of V_2, or when they all have opposite signs. Any such obvious relation should be tested.

9. Aided inspection. After culling all the vectors whose replacements can be found by casual inspection, a slightly more systematic procedure must be followed.

Let V be the matrix whose columns are the vectors V_j. Then form the matrix product

$$M = V' \cdot V \tag{7}$$

of V with its transpose, V'.

Any vector of the form

$$V_r = k_1V_1 + k_2V_2 + \cdots + k_nV_n \tag{8}$$

is a suitable replacement for V_c, provided:

1. $k_1, k_2, \cdots, k_m$ are all integers.
2. k_c, the coefficient of the cull, is 1.
3. $V_r^2 < V_c^2$.

With $k_c=1$, V_c can be expressed as an integral linear combination of V_r and the other vectors, so that after the replacement, these will still be an integral linear basis of S.

Condition 3 is somewhat laborious to check in a particular case. V_r^2 is seen to be the value of the quadratic form,

$$\sum M_{ij}k_ik_j.$$

When the values of k_j are limited to 0 and $+1$, the value of this quadratic form is merely the sum of the elements of a certain principal minor of M, namely the minor with rows and columns corresponding to the values of i and j for which k_i and k_j are $+1$. The case when some of the values of k_j are -1 can be handled by changing the corresponding signs in the minor. The illustrations will show how this is done.

It is convenient to speak of a replacement of order 2 or 3 when that replacement is a linear combination of 2 or 3 vectors of the basis. All replacements of order 2 can be found by the following theorem.

THEOREM 1. *$V_r = V_i \pm V_j$ is a suitable replacement for V_i if and only if $\mp 2M_{ij} > M_{jj}$.*

To use Theorem 1 for finding a second order replacement, find M_{ij}, an off-diagonal element of matrix M which in absolute value exceeds one half of M_{jj}, one of the corresponding diagonal elements. If M_{ij} is negative, then $V_r = V_i + V_j$. If M_{ij} is positive, then $V_r = V_i - V_j$ is a replacement for V_i.

Conversely, if no such off-diagonal element exists, then there is no second order replacement for any vector of the set. Every second order replacement can be found by forming the matrix M, and scanning its off-diagonal elements, one by one.

THEOREM 2. *If no suitable second order replacement exists, and $V_r = V_1 + V_2 + V_3$ is a suitable replacement for V_1, then*

(a) *all the off-diagonal elements of the corresponding 3-row principal minor of M must be negative, and*

(b) *their sum in absolute value must exceed $M_{22} + M_{33}$.*

To find a replacement for any vector, V_c, with the aid of any two other vectors of the set, call the vectors V_1, V_2, and V_3. Consider only the 3-row principal minor of M whose elements are $V_1 \cdot V_1$, $V_1 \cdot V_2$, $\cdots$, $V_3 \cdot V_3$. See if the signs of all the off-diagonal elements can be made negative, by changing if necessary the signs of V_2 and V_3. If not, then the replacement is impossible. If the said off-diagonal elements all are negative, then see if their sum satisfies condition (b). This procedure will give every third order replacement.

To discover, if it exists, a replacement for V_1 involving only V_1, V_2, V_3, and V_4, use the following theorem:

THEOREM 3. *If there exists no suitable replacement of the form*

$$V_r = V_1 + k_2V_2 + k_3V_3 + k_4V_4, \tag{9}$$

for V_1, where the values of k_j are limited to -1, $+1$, and 0, then there are none with any integral values of k_j.

In seeking suitable replacements of orders 2, 3, and 4, all 2, 3, and 4 row

principal minors of M must be examined. Theorems 1, 2, and 3 eliminate most of these quickly, so that relatively few remain to be examined quantitatively.

A similar procedure can be used for replacements of order higher than 4, except that in some very rare cases, coefficients, k_j, greater than 1 will have to be used. For this reason, it is very difficult to be sure that all possible replacements have been found, since a thorough search involves such a large number of trials.

10. The final roundup of replacements. After making all the replacements that have been found by inspection, the vectors of the resulting basis of S are arranged in descending order of magnitude, and renamed $\overline{V}_1, \overline{V}_2, \cdots, \overline{V}_n$. To be systematic, we first attempt to find replacements for $\overline{V}_1$; that failing, we try for replacements of $\overline{V}_2$, and so forth. So we must try to find

$$\overline{V}_r = k_1\overline{V}_1 + k_2\overline{V}_2 + \cdots + k_n\overline{V}_n$$

where $k_1 = 1$, all the other k's are integers, and $\overline{V}_r \cdot \overline{V}_r < \overline{V}_1 \cdot \overline{V}_1$. We demand of the method that either it deliver the required vector, or it assure us that no such vector exists.

If no replacement for $\overline{V}_1$ exists, then a similar procedure can be followed to find replacements for $\overline{V}_2$, and so forth. In the end, we shall either have a replacement for one of the vectors, or will know that no replacement exists for any.

From the definition of $\overline{V}_r$, it is seen that $V_r \cdot V_r$ is a quadratic form in k_j. It can be written as a sum of squares, each with a positive coefficient:

$$\begin{aligned} V_r^2 = {} & Y_1(k_n + y_{1,n-1}k_{n-1} + y_{1,n-2}k_{n-2} + \cdots + y_{11}k_1)^2 \\ & + Y_2(k_{n-1} + y_{2,n-2}V_{n-2} + y_{2,n-3}k_{n-3} + \cdots + y_{21}k_1)^2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + Y_{n-1}(k_2 + y_{n-1,1}k_1)^2 \\ & + Y_n(k_1)^2 < V_1^2. \end{aligned} \tag{10}$$

The coefficients, Y_j, are all positive. The value of V_1^2, which is the right member of inequality (10) is a known number, and so are the coefficients, namely the Y's and the y's.

For the purpose of finding a necessary condition on k_r, any of the squared terms of inequality (10) may be dropped. For a sufficient condition, all the terms must be retained.

So, first, retain only the final squared term. This puts limitations on the value of k_1. If $Y_n > V^2$, it requires k_1 to be 0, and no replacement for V_1 exists. Suppose that $k_1 = 1$ is not yet ruled out. Then consider another term of inequality (10), namely the next to last squared term. This limits the value of k_2. Proceeding in this way, every possible integral solution of equation (9) can be found.

To make matters clearer, we give two numerical illustrations. The first is

too simple to justify such a ponderous method, but the reader looking at two unknowns is asked to imagine there are 25 or more.

11. Example 1. We shall now solve System A. The homogeneous system is first formed, replacing the constant term by 0. Then the complete system is

$$\begin{aligned} y &= 0 \\ x &= 0 \\ 100(y - 1.41421x) &= 0. \end{aligned}$$

Note the choice of weight, $w=100$, for the last equation. This means, very roughly stated, that we wish the error in the equation, $y-\sqrt{2}x=0$ to be about one hundredth the size of the values of x and y. A choice of $w=10{,}000$ would give a more accurate solution, with larger values of x and y.

Also, in a real problem where there are 25 or so variables, any easily obtained simplification would be justified. So, we could give the first equation, $y=0$, a weight 0. The weight on x would keep both x and y within bounds, since the ratio of x and y would be determined by the last equation.

The vectors forming the basis of the space S are the columns of the complete system, namely

$$\begin{aligned} V_1 &= 1, \quad 0, \quad 100. \\ V_2 &= 0, \quad 1, \quad -141.421. \end{aligned}$$

An obvious replacement is

$$V_3 = V_1 + V_2 = 1, \quad 1, \quad -41.421.$$

V_3 replaces V_2, the longer of V_1 and V_2. The present basis of S is (V_1, V_3). An obvious replacement for V_1 is

$$V_4 = V_1 + V_3 = 2, \quad 1, \quad 58.579.$$

Similarly,

$$\begin{aligned} V_5 &= V_3 + V_4 = 3, \quad 2, \quad 17.158. \\ V_6 &= V_3 + 2V_5 = 7, \quad 5, \quad -7.105. \end{aligned}$$

Let us suppose that no further replacements are obvious. The present basis of S is V_5 and V_6, which we rename $\overline{V}_1$ and $\overline{V}_2$. Then the matrix is

$$V = \begin{Vmatrix} 3, & 2, & 17.158 \\ 7, & 5, & -7.105 \end{Vmatrix}.$$

For M, we have

$$M = V \cdot V' = \begin{Vmatrix} 307.397 & -90.908 \\ -90.908 & 124.481 \end{Vmatrix}.$$

Application of Theorem 1 of Section 9 would immediately direct our attention to the off-diagonal element of M, namely -90.908, which in absolute value exceeds one half of 124.481. This would give us the replacement, V_5+V_6 for the longer of the two vectors, namely V_5. However, we shall overlook this, in order to see how the method of Section 10 operates.

We wish to find, if possible,

$$V_r = \overline{V}_1 + k\overline{V}_2,$$

where k is an integer, and $V_r^2 < \overline{V}_1^2 = 307.397$.

$$\begin{aligned}\overline{V}_r^2 &= (\overline{V}_1 + k\overline{V}_2)^2,\\ &= \overline{V}_1\cdot\overline{V}_1 + 2k\overline{V}_1\cdot\overline{V}_2 + k^2\overline{V}_2\cdot\overline{V}_2,\\ &= 307.397 + 2k(-90.908) + k^2(124.481),\\ &= 124.481(k - 0.73029)^2 + 241.008 < 307.397.\end{aligned}$$

The two members of the inequality are seen to be equal for $k=0$, so it is easily seen to be satisfied for $k=1$.

Following is the suggested arrangement of the computation:

Name	How Obtained	Components			Replaced by
V_1	Given	1	0	100.	V_4
V_2	Given	0	1	−141.421	V_3
V_3	V_1+V_2	1	1	−41.421	V_6
V_4	V_1+V_3	2	1	+58.579	V_6
V_5	V_4+V_3	3	2	+17.158	V_7
V_6	V_3+2V_5	7	5	−7.105	
V_7	V_5+V_6	10	7	+10.053	

Each vector in the final basis of S, namely V_6 and V_7, gives an *ia*-solution of the homogeneous system. V_6, the shortest vector in this basis, gives the *lia*-solution. It is easily seen from the first two components of V_6 that $V_6 = 7V_1+5V_2$, so the said *lia*-solution is $y=7$, $x=5$.

To solve the non-homogeneous complete system, let R be the vector made up of the right hand members of the equations. In the present example, $R=0$, 0, 50, and the system is

$$\begin{aligned} y &= 0\\ x &= 0\end{aligned}$$

$$100y - 141.421x = 50.$$

For the *lia*-solution, we must find the shortest vector, V_s, of the form

$$V_s = -R + k_1V_1 + k_2V_2$$

where k_1 and k_2 are integers. In fact, k_1 and k_2 will be the values of y and x.

As a practical matter, any vector, V_i, from the table of culls can be added to $-R$, since it will be a linear combination of V_1 and V_2. The values of k_1 and k_2 can be read immediately from V_s as its first two components.

In the example, it is noted that $-R+V_4$ has the components 2, 1, 8.579 and so is much shorter than V_7 and V_8. Hence, by inspection, we have the result that

$$V_s = -R + V_4 = -R + 2V_1 + V_2,$$

and the *lia*-solution that $y=2$, $x=1$.

To be absolutely certain that the vector (2, 1, 8.579) is the shortest vector of the prescribed form, we write

$$V_s = k(2,\ 1,\ 8.579) + k_6V_6 + k_7V_7$$

and consider the minimum possible value of $V_s \cdot V_s$ for $k=1$ and integral values of k_6 and k_7. The result is a quadratic form in k, k_6, and k_7. It is written as a sum of squares, as in the solution of the homogeneous case. The result $k_6=k_7=0$ would confirm the result of our inspection. Any other value for k_6 and k_7 would give a better result, one that was overlooked by inspection. The details of the computation are as follows:

$$\begin{aligned} V_s \cdot V_s = {} & 124.481k_6k_6 + 33.573k_6k_7 - 41.954k_6k \\ & + 33.573k_7k_6 + 250.063k_7k_7 + 113.245k_7k \\ & - 41.954kk_6 + 113.245kk_7 + 78.599kk \\ = {} & 11.332k^2 + 241.008(k_7 + 0.51683k)^2 \\ & + 124.481(k_6 + 0.26970k_7 - 0.33703k)^2. \end{aligned}$$

For this to be less than 78.599, with $k=1$, it is easy to see from the squared form that $k_1=-1$ and $k_6=+1$. Therefore,

$$V_s = -R + V_4 + V_6 - V_7 = (-1,\ -1,\ -8.579).$$

Therefore, the *lia*-solution of the non-homogeneous system is $y=-1$, $x=-1$.

Following is the suggested arrangement of the computation:

Name	How Obtained	Components		
$-R$	Given	0	0	−50.
V_8	$-R+V_4$	2	1	+8.579.
V_9	$V_8+V_6-V_7$	−1	−1	−8.579.

The solution part of V_9 is the vector with the first two components, which gives the values of y and x. The error part, namely −8.579, gives w times the error, so the latter is −0.08579.

For a solution with a smaller error, larger values of x and y will have to be allowed. For this, change the weight from $w=100$ to $w=10{,}000$, by shifting the decimal point two places in the fifth column of the table of culls, and revise the computation accordingly. This will give the result $y=-49$, $x=-35$, with an error of about .0025. With still larger values of w, solutions with still smaller errors, but larger absolute values of x and y, can be obtained.

For this reason, when attacking a new system, the weights can be assigned arbitrarily, for a first run. Then you can change the weights, as may seem desirable in the light of experience.

12. Example 2. For the second example, we want more than one equation in two variables. The important thing to notice is that the procedure is just the same as in the simpler example.

Solve the system of congruences:

$$\begin{aligned} x_1 \cos 30^\circ + x_2 \sec 40^\circ &= \csc 55^\circ, \pmod 1, \\ x_1 \sin 30^\circ + x_2 \tan 40^\circ &= \cot 55^\circ, \pmod 1. \end{aligned} \tag{11}$$

The corresponding system of equations is:

$$\begin{aligned} x_1 \cos 30^\circ + x_2 \sec 40^\circ + x_3 &= \csc 55^\circ, \\ x_1 \sin 30^\circ + x_2 \tan 40^\circ + x_4 &= \cot 55^\circ. \end{aligned} \tag{12}$$

The homogeneous system is:

$$\begin{aligned} x_1 \cos 30^\circ + x_2 \sec 40^\circ + x_3 &= 0, \\ x_1 \sin 30^\circ + x_2 \tan 40^\circ + x_4 &= 0. \end{aligned} \tag{13}$$

The complete system is:

$$\begin{aligned} 1(x_1 \qquad\qquad\qquad\qquad\quad\ ) &= v_1 \\ 1(\quad x_2 \qquad\qquad\qquad\quad\ ) &= v_2 \\ 0(\qquad\qquad + x_3 \qquad\ ) &= v_3 \\ 0(\qquad\qquad\qquad + x_4) &= v_4 \\ W(x_1 \cos 30^\circ + x_2 \sec 40^\circ + x_3 \quad) &= v_5 \\ W(x_1 \sin 30^\circ + x_2 \tan 40^\circ + x_4) &= v_6. \end{aligned} \tag{14}$$

Choice of weights 0 for the third and fourth equations is for the purpose of reducing work. Such a possibility was suggested in Example 1.

'Arbitrary choice of weight $W=200$ enables one to start the solution. Later W can be adjusted up or down, according as a solution is wanted with a smaller error or with smaller integral values of $x_1, \cdots, x_4$. If you are more concerned in keeping the error small in equation 11a than in equation 11b, the former can be given a larger weight. While the weights are important, their precise values are not.

The vectors forming the basis of space S are, as read from the columns of

the matrix of coefficients of system (14):

$$V_1 = (1,\ 0,\ 0,\ 0,\ W\cos 30°,\ W\sin 30°),$$
$$V_2 = (0,\ 1,\ 0,\ 0,\ W\sec 40°,\ W\tan 40°),$$
$$V_3 = (0,\ 0,\ 0,\ 0,\ W,\ 0\ ),$$
$$V_4 = (0,\ 0,\ 0,\ 0,\ 0,\ W\ ).$$

In the table of culls, each row concerns one vector. The first column gives the number of the vector, the second gives the numbers of the vectors from which it was obtained. The next 6 columns give the six components of each vector. The value shown for each component must still be multiplied by the weight shown at the top of the column.

The next column shows the norm, or squared length of the vector. This takes into account the weights. For example, in the case of V_1, the components are 1, 0, 0, 0, $100\sqrt{3}$, 100. The sum of the squares of these numbers is 40,001. These norms are not necessary for the procedure.

The final column shows which vector finally replaces the vector of that row. V_1 is replaced by V_6, and V_2 by V_5, and so forth. At any point in the formation of this table, one can tell at a glance which vectors have been culled, and which form the current basis of S. The vectors of the final basis of S are seen to be V_{12}, V_{16}, V_{18}, and V_{20}.

TABLE OF CULLS FOR EXAMPLE 2

j	How Obtained	Weights						Norm	Replaced by
		1	1	0	0	200	200		
1		1	0	0	0	0.866025	0.500000	40,001	6
2		0	1	0	0	1.305407	0.839100	96,328	5
3		0	0	1	0	1.000000	0.000000	40,000	7
4		0	0	0	1	0.000000	1.000000	40,000	8
5	2−1	−1	1	0	0	0.439382	0.339100	12,324	11
6	1−2(5)	3	−2	0	0	−0.012739	−0.178200	1,290	14
7	3−5	1	−1	1	0	0.560618	−0.339100	17,173	10
8	4−5	1	−1	0	1	−0.439382	0.660900	25,196	9
9	8+7	2	−2	1	1	0.121236	0.321800	4,738	13
10	7−2(6)	−5	+3	1	0	0.586096	0.017300	13,786	12
11	5−9	−3	3	−1	−1	0.318146	0.017300	891	15
12	10−2(11)	1	−3	3	2	−0.050196	−0.017300	123	
13	9+2(6)	8	−6	1	1	0.095758	−0.034600	515	16
14	6−12	2	1	−3	−2	0.037457	−0.160900	1,097	17
15	11+5(12)	2	−12	14	9	0.067166	−0.069200	520	18
16	13+12	9	−9	4	3	0.045562	−0.051900	353	
17	14−16	−7	10	−7	−5	−0.008105	−0.109000	628	19
18	15−16	−7	−3	10	6	0.021604	−0.017300	89	
19	17−18	0	13	−17	−11	−0.029709	−0.091700	541	20
20	19−12	−1	16	−20	−13	0.020487	−0.074400	495	

All the replacements used in forming the table were found by inspection. Each involved at most the cull and one other vector. This is usual, unless the number of components is large, in which case a few third and fourth order replacements might be needed. These can be found by Theorems 1, 2, and 3 of Section 9. In very rare cases, one replacement might be overlooked, without the procedure of Section 10. The manner of finding the next to the last replacement, V_{19}, by this method will be shown in detail.

Just before finding V_{19}, the current basis of space S is the set (V_{17}, V_{16}, V_{12}, and V_{18}). These have been arranged, for convenience, in order of descending magnitude, and now will be renamed $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, and $\overline{V}_4$. We seek a replacement for the longest, namely $\overline{V}_1$. So, write

$$V_r = k_1\overline{V}_1 + k_2\overline{V}_2 + k_3\overline{V}_3 + k_4\overline{V}_4. \tag{15}$$

To find a suitable replacement for $\overline{V}_1$, we must find a set of integral values for $k_1, \cdots, k_4$, with $k_1 = 1$, and $V_r \cdot V_r < \overline{V}_1 \cdot \overline{V}_1$. But $V_r \cdot V_r$ is seen to be a quadratic form in k_j with matrix $M = V \cdot V'$, where V is the matrix of the vectors $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, $\overline{V}_4$, which computes out in the present case to be

$$M = \begin{Vmatrix} 626.8676 & 58.5128 & 54.7015 & 87.4240 \\ 58.5128 & 352.7802 & -19.5664 & 39.2877 \\ 54.7015 & -19.5664 & 122.7571 & -29.4058 \\ 87.4240 & 39.2877 & -29.4058 & 88.6409 \end{Vmatrix}. \tag{16}$$

Application of Theorem 1 of Section 9 would direct attention to the off-diagonal element 87.4240 of M whose value exceeds half the corresponding diagonal element, 88.6409. Thus, replacement V_{19} would be found. However, we wish to show how the method of Section 10 operates.

So the quadratic form is expressed as a sum of squares:

$$\begin{aligned} \sum M_{ij}k_ik_j = {} & 447.5242k_1^2 + 334.9894(k_2 + 0.304787k_1)^2 \\ & + 113.0032(k_3 - 0.057814k_2 + 0.740719k_1)^2 \\ & + 88.6409(k_4 - 0.33174k_3 + 0.44322k_2 + 0.98627k_1)^2. \end{aligned} \tag{17}$$

We wish to choose k_1, k_2, k_3, k_4 such that

$$\sum M_{ij}k_ik_j < 626.8676. \tag{18}$$

If the coefficient of k_1^2 had turned out to exceed 627, the norm of $\overline{V}_1$, then a mere glance at that term would show that the choice $k_1 = 1$ would contradict inequality (18), and no replacement for $\overline{V}_1$ would be possible. However, the possibility $k_1 = 1$ is not so ruled out in this example.

Now consider the later squared terms (with positive coefficients) one at a time, and in each case choose the new k_j for that term so as to make the new term as small as possible. We try, of course, to keep the sum of the terms less than 626.8676. We know in advance by equation (15) that the values $k_1 = 1$,

$k_2=k_3=k_4=0$ will not satisfy the inequality, for it will make the two members equal. So we find the solution, $k_1=1$, $k_2=0$, $k_3=-1$, $k_4=-1$. These values put into equation (15) would give V_{20}. There is also the solution $k_1=1$, $k_2=k_3=0$, $k_4=-1$, which gives V_{19}. When there are several solutions, it may be a little difficult to make a complete list of all possible ones. When there are no possible solutions, the arithmetic of inequalities will make that fact clear.

The *lia*-solution of the complete homogeneous system can be read from the components of the shortest of the four unculled vectors, namely $V_{18}=-7$, -3, 10, 6, $\cdots$, without weights. The *lia*-solution is $x_1=-7$, $x_2=-3$, $x_3=10$, $x_4=6$. Substitution of these values into the complete homogeneous system gives the successive errors, -7, -3, $10\cdot0$, $6\cdot0$, $(0.021604)(200)$, $(-0.017300)(200)$, the components of V_{18}.

13. Example 3. To save space, only the high points of Example 3 are given. The basis of the space S this time has 5 vectors, so that instead of equation (15), we have

$$V_r = k_1V_1 + k_2V_2 + k_3V_3 + k_4V_4 + k_5V_5, \tag{15'}$$

and for (16), we have, for purposes of illustration,

$$M = \begin{Vmatrix} 26 & -5 & -5 & -5 & -3 \\ -5 & 26 & -5 & -5 & -3 \\ -5 & -5 & 26 & -5 & -3 \\ -5 & -5 & -5 & 26 & -3 \\ -3 & -3 & -3 & -3 & 7 \end{Vmatrix}. \tag{16'}$$

The interesting thing about this example is that Theorems 1, 2, and 3 suggest no replacement for V_1. In fact, trial of the values $+1$, -1, and 0 in every possible combination for the values of $k_1, \cdots, k_5$ in equation (15') fails to give any replacement. If it becomes necessary to try also the values $k_j=+2$ and -2, and if n is 10 or more, then it becomes prohibitive to make an exhaustive search for replacements. So it becomes interesting to see how the method of Section 10 handles the situation. So, write

$$\begin{aligned} \sum M_{ij}k_ik_j = {} & 14.95k_1^2 + 20.43(k_2 - 0.52k_1)^2 \\ & + 23.12(k_3 - 0.34k_2 - 0.34k_1)^2 \\ & + 24.91(k_4 - 0.25k_3 - 0.25k_2 - 2.25k_1)^2 \\ & + 7(k_5 - 0.43k_4 - 0.43k_3 - 0.43k_2 - 0.43k_1)^2. \end{aligned} \tag{17'}$$

For a replacement, we must have

$$\sum M_{ij}k_ik_j < 26. \tag{18'}$$

Taking only the first squared term, we have

$$14.95k_1^2 < 26,$$

so we start with $k_1=1$. In fact for any hope of a replacement for V_1, we must have $k_1=+1$. Next take two squared terms,

$$14.95 + 20.43(k_2 - 0.52)^2 < 26,$$

$k_2=1$ or 0, preferably 1. (If that fails, we can return to the value $k_2=0$.)

Proceeding in this manner, it is found that $k_3=k_4=1$, but $k_5=2$. So we have the desired replacement, $k_r=k_1+k_2+k_3+k_4+2k_5$.

In a more complicated example, with a larger value of n, before you could be sure that no replacements were possible, you may find it necessary to try several different hopeful starts, each of which would prove impossible by the time you considered all the squared terms. But the number of these starts would be finite, and much smaller than the number of trials necessary under the method of inspection of Section 9.

14. The solution of system (13). System (13) is the homogeneous system of Example 2 from which the complete system was formed. If the values $x_1=-7$, $x_2=-3$, $x_3=10$, $x_4=6$, obtained at the end of Section 12, are substituted into System (13), instead of the zeros demanded the results obtained are 0.021604 and -0.017300, namely the last two components of V_{18}. No *ia*-solution with smaller values of x_j (measured by the sum of the squares) can have a smaller error.

15. Solution of system (12). To solve the given non-homogeneous system, (12), put it in the form:

$$(12') \qquad \begin{aligned} -x_0 \csc 55^\circ + x_1 \cos 30^\circ + x_2 \csc 40^\circ + x_3 &= 0, \\ -x_0 \cot 55^\circ + x_1 \sin 30^\circ + x_2 \tan 40^\circ + x_4 &= 0. \end{aligned}$$

For a solution, we must have $x_0=1$. The sum of two solutions would give x_0 the value 2, so the procedure of Section 12 cannot be followed.

Instead, we find a particular *ia*-solution of (12'), namely $x_0=1$, $x_1=x_2=x_3=x_4=0$ (V_{21} below), and to it add a linear combination of *ia*-solutions of (13), in which $x_0=0$. In the result, x_0 would always have the desired value of 1. As

TABULAR SOLUTION OF SYSTEM (12)

i	How Obtained	Weights						
		1	1	1	0	0	200	200
21		1	0	0	0	0	−1.220775	−0.700208
22	21+ 2	1	0	1	0	0	+0.084632	+0.138892
23	22+12	1	1	−2	3	2	+0.034436	+0.121592
24	23+17	1	−6	8	−4	−3	+0.026331	+0.012592
25	24+12	1	−5	5	−1	−1	−0.023865	−0.004708
26	25+18	1	−12	2	9	5	−0.002261	−0.022008

in the tabular solution of system (13), weights are assigned, and the shorter the vector the better the corresponding *ia*-solution. The computation may be put into tabular form.

The shortest vector obtained by inspection is V_{25}. This can be combined with the various vectors in the basis of S in an attempt to find a shorter vector. Among others, this gives V_{26}, which is longer than V_{25}, though its error part is shorter.

To be absolutely sure that you have the *lia*-solution of the non-homogeneous complete system, the matrix can be expressed as a sum of squares, as was illustrated in the case of Example 1.

From V_{25}, the answer to Example 2 is read as $x_1 = -5$, $x_2 = +5$. The values of x_3 and x_4 are each -1. These are the multiples of the modulus which are disregarded when solving a congruence. Substitution of these values of x_1 and x_2 in System (11) leads to errors -0.023865 and -0.005208. V_{26} would give smaller errors, but would use larger values of x_1 and x_2.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

NOTE ON A QUARTIC CONGRUENCE

L. CARLITZ, Duke University

Known results on the Galois group of the equation

$$x^4 + mx^2 + n = 0 \tag{1}$$

as well as some general theorems about the factorization of a rational prime in an algebraic number field yield information about the congruence

$$x^4 + mx^2 + n \equiv 0 \pmod{p}, \tag{2}$$

where p is an odd prime and m, n are rational integers. It is however easy to obtain such results concerning (2) in a very elementary way as we shall now indicate.

Since p is odd we may assume $m \equiv -2a \pmod{p}$. It is convenient to consider separately two cases: (i) n a square, (ii) n a non-square. Thus in case (i) we consider the congruence

$$x^4 - 2ax^2 + b^2 \equiv 0 \pmod{p} \tag{3}$$

and assume first that $(a^2-b^2)b^2 \not\equiv 0$. Then none of the numbers

$$2(a - b), \qquad 2(a + b), \qquad a^2 - b^2 \tag{4}$$

is divisible by p; moreover since their product is a square it follows that either all of them are squares or exactly one is a square. We shall prove the following

THEOREM 1. *If* $(a^2-b^2)b^2 \not\equiv 0$ *and each of the numbers* (4) *is a square* (mod p), *then the polynomial* $f(x)=x^4-2ax^2+b^2$ *is congruent to the product of four linear factors, while if only one of the numbers* (4) *is a square then* $f(x)$ *is congruent to the product of two irreducible quadratics.*

To prove the first part of the theorem it suffices to note that if

$$2(a-b) \equiv 4r^2, \qquad 2(a+b) \equiv 4s^2, \qquad a^2-b^2 \equiv 4r^2s^2,$$

then

$$x^4 - 2ax^2 + b^2 \equiv (x+r+s)(x+r-s)(x-r+s)(x-r-s).$$

To prove the second part suppose for example that $2(a-b) \equiv c^2$. Then we may verify that

$$f(x) \equiv (x^2 + cx - b)(x^2 - cx - b). \tag{5}$$

Since $c^2+4b \equiv 2(a+b)$, which by hypothesis is not square, it follows that each of the quadratics in (5) is irreducible (mod p).

The excluded sub-case $(a^2-b^2)b^2 \equiv 0$ is of little interest and obviously yields

$$f(x) \equiv (x^2 - a)^2 \qquad (a^2 \equiv b^2),$$
$$f(x) \equiv x^2(x^2 - 2a) \qquad (b^2 \equiv 0),$$

so that we now have repeated factors.

Turning next to case (ii) we consider the congruence

$$x^4 - 2ax^2 + n \equiv 0 \pmod{p} \tag{6}$$

with n a non-square. It follows that (6) cannot have four roots. We have in this case

THEOREM 2. *Assume* n *not a square. Then the polynomial* $g(x)=x^4-2ax^2+n$ *is irreducible* (mod p) *when* a^2-n *is a non-square. When* a^2-n *is a non-zero square,* $g(x)$ *is congruent to the product of an irreducible quadratic and two distinct linear factors. (The possibilily* $a^2-n \equiv 0$ *is ruled out by the hypothesis concerning* n.)

To prove this theorem suppose first that (6) has a root r. Then

$$g(x) \equiv (x^2 - r^2)(x^2 - n/r^2) \tag{7}$$

and clearly the second factor is irreducible. Moreover since (7) implies $2a \equiv r^2+n/r^2$, we get

$$4(a^2 - n) \equiv \left(r - \frac{n}{r}\right)^2.$$

Conversely when a^2-n is a square we get a factorization

$$g(x) \equiv (x^2 - c)(x^2 - d). \tag{4}$$

Since $cd \equiv n$, one of c, d is a square, the other a non-square and (8) reduces to (7). This takes care of the second part of the theorem. Suppose now that a^2-n is not a square. Clearly $g(x)$ cannot have a linear factor. Also neither the factorization (8) nor

$$g(x) \equiv (x^2 - cx + d)(x^2 + cx + d) \tag{9}$$

can occur, for (8) implies $a^2-n \equiv (c-d)^2$ while (9) implies $n \equiv d^2$.

To give a few numerical examples we cite

$x^4 + 3x^2 + 2 \equiv x^4 - 14x^2 + 2 \equiv (x - 4)(x + 4)(x - 7)(x + 7) \pmod{17}$,

$x^4 - 6x^2 + 4 \equiv (x + 4)(x - 4)(x + 20)(x - 20) \pmod{41}$,

$x^4 + 5x^2 + 1 \equiv x^4 - 6x^2 + 1 \equiv (x^2 + 2x - 1)(x^2 - 2x - 1) \pmod{11}$,

$x^4 - 2x^2 + 2$ irreducible $\pmod{11}$,

$x^4 + 6 \equiv (x - 2)(x + 2)(x^2 - 7) \pmod{11}$,

$x^4 - 2x^2 + 2 \equiv (x - 3)(x + 3)(x^2 - 6) \pmod{13}$,

$x^4 + 13x^2 + 6 \equiv (x - 3)(x + 3)(x^2 + 5) \pmod{17}$,

$x^4 - 4x^2 + 6$ irreducible $\pmod{37}$,

$x^4 - 3x^2 - 1 \equiv (x^2 - 8)^2 \pmod{13}$.

NOTE ON AN ELEMENTARY PROPERTY OF TRIANGLES

GUNNAR STEENSHOLT, University of Oslo

Let P be an arbitrary point within a triangle ABC. Let its distances from A, B and C be R_1, R_2, R_3 ,and let its distances from the sides BC, CA and AB (of lengths respectively a, b, c) be r_1, r_2, r_3. Then we have the theorem of Erdös-Mordell:

$$R_1 + R_2 + R_3 \geqq 2(r_1 + r_2 + r_3).$$

A simple and elementary proof of this theorem is not known (see Toth [1]). However, an elementary theorem of some intrinsic interest, and formally related to the Erdös-Mordell theorem, can be derived very simply as shown below. To the best of the author's knowledge this result does not seem to have been recorded in the literature.

It is plain that for the areas of the triangles involved:

$$ABC - ABP = BCP + CAP,$$

with two other similar equations. If now h_1, h_2, h_3 denote the heights from the vertices A, B, C respectively, and if T is the area of the triangle ABC, then

$$(h_1 - r_1)a + (h_2 - r_2)b + (h_3 - r_3)c = 4T.$$

But now clearly

$$2T = ar_1 + br_2 + cr_3,$$
$$R_1 \geqq h_1 - r_1,$$
$$R_2 \geqq h_2 - r_2,$$
$$R_3 \geqq h_3 - r_3.$$

Therefore

$$aR_1 + bR_2 + cR_3 \geqq 2(ar_1 + br_2 + cr_3),$$

which is the desired result.

If we now introduce the "weighted" means R and r of R_i and r_i respectively $(i=1, 2, 3)$ by putting

$$R = (aR_1 + bR_2 + cR_3)/(a + b + c)$$
$$r = (ar_1 + br_2 + cr_3)/(a + b + c)$$

we obtain

$$R \geqq 2r,$$

which may be regarded as a kind of analogue to the Erdös-Mordell theorem.

If $a=b=c$, we obviously get

$$R_1 + R_2 + R_3 \geqq 2(r_1 + r_2 + r_3).$$

This proves the original Erdös-Mordell theorem for equilateral triangles.

Reference

1. L. Fejes Toth, Lagerungen in der Ebene, auf der Kugel und im Raum, Springer Verlag, Berlin, 1953. See in particular p. 12 and p. 28.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON THE SEQUENCE FOR EULER'S CONSTANT

S. K. LAKSHMANA RAO, Indian Institute of Science

The convergence of the sequence $a_n = 1 + \frac{1}{2} + \cdots + 1/n - \log n$ as $n \to \infty$ is proved below by expressing a_n as an infinite integral. This procedure has the advantage of leading at once to the integral representation for $\lim a_n = \gamma$ (Euler's Constant) and we may also examine the behaviour of $a_n - \gamma$ for large n.

Noting that $1/r = \int_0^\infty e^{-rx}dx$ $(r>0)$ and $\log n = \int_0^\infty (e^{-x} - e^{-nx})dx/x$ (Frullani's

integral*), we write

$$a_n = \int_0^\infty (e^{-x} + e^{-2x} + \cdots + e^{-nx})dx - \int_0^\infty \frac{e^{-x} - e^{-nx}}{x}\,dx$$
$$= \int_0^\infty \left(\frac{e^{-x} - e^{-(n+1)x}}{1 - e^{-x}} - \frac{e^{-x} - e^{-nx}}{x}\right)dx$$

which can now be split up into a sum of two convergent infinite integrals in the form

$$a_n = \int_0^\infty \left(\frac{e^{-x}}{1 - e^{-x}} - \frac{e^{-x}}{x}\right)dx + \int_0^\infty e^{-nx}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)dx.$$

The following inequality

$$\frac{1}{2} - \frac{x}{8} < \frac{1}{x} - \frac{1}{e^x - 1} < \frac{1}{2} \qquad (x > 0) \tag{1}$$

can be easily proved by showing that it depends on the elementary inequality $\tanh x < x (x > 0)$. Therefore

$$\int_0^\infty e^{-nx}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)dx < \int_0^\infty \frac{e^{-nx}}{2}\,dx = \frac{1}{2n}.$$

This proves that a_n converges to a limit γ given by

$$\gamma = \int_0^\infty \left(\frac{e^{-x}}{1 - e^{-x}} - \frac{e^{-x}}{x}\right)dx.$$

Further,

$$a_n - \gamma = \int_0^\infty e^{-nx}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)dx$$

and by the inequality (1), we have

$$\int_0^\infty e^{-nx}\left(\frac{1}{2} - \frac{x}{8}\right)dx < a_n - \gamma < \int_0^\infty e^{-nx}\cdot\frac{1}{2}\,dx$$

or

$$\frac{1}{2n} - \frac{1}{8n^2} < a_n - \gamma < \frac{1}{2n}.$$

Therefore $a_n - \gamma$ behaves like $1/2n$ and $n(a_n - \gamma) = n(1 + \frac{1}{2} + \cdots + 1/n - \log n - \gamma)$ converges to the limit $\frac{1}{2}$.

* Referee's comment: "Frullani's integral" is a consequence of integration from 1 to n of the previous integral.

A NEW INTERPRETATION OF THE FIELD POSTULATES†

DONALD MILLER, Brookhaven National Laboratory

In this note we shall consider an arithmetical operation $*$ defined as $a * b = a^{\log b}$. The operations $*$ and ordinary multiplication satisfy the field axioms and have certain other interesting properties.

Consider the standard postulates for a field in terms of the quantities $\mathbf{a} \oplus \mathbf{b}$, $\mathbf{a} \otimes \mathbf{b}$, $-\mathbf{a}$, $\mathbf{a}^{-1}$, $\mathbf{0}$, and $\mathbf{1}$. It is easy to verify that the postulates are also satisfied by the following interpretation. Let the positive real numbers be the set of elements, let ab be the interpretation for $\mathbf{a} \oplus \mathbf{b}$, $a * b$ be that for $\mathbf{a} \otimes \mathbf{b}$, the base of the logarithms z (a positive real number, not equal to 1) be the interpretation for $\mathbf{1}$, 1 for $\mathbf{0}$, a^{-1} for $-\mathbf{a}$, and $\bar{a} = z^{1/\log a}$ for $\mathbf{a}^{-1}$.

Suppose one chooses to take ab in its usual role of a multiplication and then takes $a * b$ as a new type of addition. Then an interesting and characteristic property of Boolean algebras is found to hold; namely, the distributivity of addition over multiplication. This property,

$$a * (bc) = (a * b)(a * c)$$

is useful in showing that there is nothing especially unique in numerical systems about the standard distributive law

$$a(b + c) = ab + ac.$$

If $*$ is taken as an addition operation in the interpretation $(K, \cdot, *)$, then there are certain propositions, trivial in the $(K, \cdot, +)$ interpretation, which are no longer trivial since $z \neq 0$. Some examples are

$$a * z^n = a^n, \qquad a * \bar{a}^n = z^n,$$
$$a^{-1} = a * z^{-1}, \qquad az = \bar{a} * az (\neq z).$$

† Research carried out under the auspices of the U. S. Atomic Energy Commission.

THE VOLUME OF A TETRAHEDRON AS A DETERMINANT

S. B. TOWNES, University of Hawaii

In Classroom Notes in the April 1955 MONTHLY, G. M. Petersen gave a simple derivation of the area of a triangle as a determinant in the coordinates of its vertices. It is easy to obtain the volume of a tetrahedron by analogous procedure. In a space of three dimensions a plane area is equal to the square root of the sum of the squares of its projections upon the coordinate planes. It follows that the area of the triangle with the vertices (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is, (writing only the first lines of the determinants),

$$\tfrac{1}{2}\sqrt{\left|\begin{matrix} x_2 & y_2 & 1 \end{matrix}\right|^2 + \left|\begin{matrix} y_2 & z_2 & 1 \end{matrix}\right|^2 + \left|\begin{matrix} z_2 & x_2 & 1 \end{matrix}\right|^2}.$$

The equation of the plane through these same points is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Consider the tetrahedron with the vertices (x_i, y_i, z_i), $i=1, 2, 3, 4$. Using the normal form of the preceding equation, one gets the altitude through (x_1, y_1, z_1) in the form

$$\pm \frac{1}{\sqrt{\begin{vmatrix} x_2 & y_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_2 & z_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \end{vmatrix}^2}} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

From this it readily follows that the volume of the tetrahedron is

$$\pm \frac{1}{3!} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

By mathematical induction the content of a simplex in n dimensions can be found by the same method.

THE EVALUATION OF TWO INFINITE INTEGRALS

C. D. Olds, San Jose State College

The "differential equation" approach to the evaluation of definite integrals has great student appeal, and the advanced calculus teacher should have a few examples tucked away in his notes. For example, consider the integral

$$u \equiv \int_0^\infty \frac{a \cos mx}{a^2 + x^2}\, dx, \tag{1}$$

where, in order to make the discussion as brief as possible, we shall assume that $a>0$ and that $m>0$. Using double integration by parts, we see at once that

$$u = -\frac{1}{m^2} \int_0^\infty \cos mx \frac{d^2}{dx^2}\left(\frac{a}{a^2 + x^2}\right) dx. \tag{2}$$

But

$$\frac{d^2}{dx^2}\left(\frac{a}{a^2 + x^2}\right) = -\frac{d^2}{da^2}\left(\frac{a}{a^2 + x^2}\right),$$

and hence (2) shows that u satisfies the differential equation

$$\frac{d^2u}{da^2} = m^2u,$$

whose solution is

$$u = Ae^{ma} + Be^{-ma}, \tag{3}$$

where A and B are functions of m and not of a. However, if in (1) we let $x=ay$ we find that

$$u = \int_0^\infty \frac{\cos may}{1+y^2}\,dy, \tag{4}$$

which shows that u is a function of ma; hence the constants A and B in (3) are independent of m. Letting, then, $m\to\infty$ in (3), and using the fact that (4) converges, we deduce that $A=0$. Likewise letting $m\to 0$ we conclude that $B=\pi/2$. Thus

$$\int_0^\infty \frac{a\cos mx}{a^2+x^2}\,dx = \frac{1}{2}\pi e^{-ma}.$$

Now if

$$v \equiv \int_0^\infty \frac{a\sin mx}{x(a^2+x^2)}\,dx,$$

then

$$\frac{dv}{dm} = u = \frac{1}{2}\pi e^{-ma}$$

so that $v=\pi(1-e^{-ma})/2a$. Hence

$$\int_0^\infty \frac{a^2\sin mx}{x(a^2+x^2)}\,dx = \frac{1}{2}\pi(1-e^{-ma}). \tag{5}$$

Differentiating (5) twice with respect to m, we find that

$$\int_0^\infty \frac{x\sin mx}{a^2+x^2}\,dx = \frac{1}{2}\pi e^{-ma}. \tag{6}$$

Adding (5) and (6) we obtain the evaluation

$$\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{1}{2}\pi.$$

It is clear, of course, that all the operations involved can be established on the basis of uniformity.

A CONTINUATION PROBLEM

C. P. GADSDEN, Tulane University

If one solution y_1 of $y''+f_1y'+f_2y=0$ is known, a second (linearly independent) solution is given by any solution of the reduced equation

$$y_1y' - y_1'y = \exp\int_{x_0}^{x} -f_1dx = w(x). \tag{1}$$

Thus $y_2=y_1\int_{x_2}^{x}(w/y_1^2)dx$. Clearly a problem arises when y_1 has zeros. Suppose $y_1(x_1)=0$ (hence $y_1'(x_1)\neq 0$, since we assume of course $y_1\not\equiv 0$). Choose x_2 and x_3 so that x_1 is the only zero of y_1 in the interval $x_2<x_3$, and consider a third solution $y_3=y_1\int_{x_3}^{x}(w/y_1^2)dx$. From (1) we have, by continuity at x_1, $y_2(x_1)=y_3(x_1)=-w(x_1)/y_1'(x_1)$. To compare the derivatives of y_2 and y_3 at x_1 we integrate the easily verified relation $d(y'/w)=(-f_2y/w)dx$, first between x_1 and x_2 for $y=y_2$, and then between x_1 and x_3 for $y=y_3$; thus $y_2'(x_1)=w(x_1)\left[1/y_1(x_2)-\int_{x_2}^{x_1}(f_2y_2/w)dx\right]$ and $y_3'(x_1)=w(x_1)\left[1/y_1(x_3)-\int_{x_3}^{x_1}(f_2y_3/w)dx\right]$, so that generally $y_2'(x_1)\neq y_3'(x_1)$, and y_3 is in general not the continuation of y_2 across x_1. The continuation is readily obtained, however, in the form $y_2=Ky_3+Cy_1$, since y_3 and y_1 form a fundamental set of solutions. The constants are easily evaluated as $K=y_2(x_1)/y_3(x_1)=1$ and $C=\left[y_2'(x_1)-y_3'(x_1)\right]/y_1'(x_1)$, which can be computed from the formulas obtained above. Continuation of y_2 across further zeros of y_1 is carried out interval by interval using the same method and basic relations.

Reference

Witold Hurewicz, Ordinary differential equations in the real domain, Brown University, 1943, pp. 45–47.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1231. *Proposed by A. W. Walker, University of Toronto*

Show that $x^3+y^3+z^3-3xyz=a^3$ is a surface of revolution.

E 1232. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Arbitrary parallel lines drawn through the vertices A, B, C of a triangle

intersect the circumcircle in A', B', C'. Show that A'', B'', C'', the symmetrics of these points with respect to the midpoints of BC, CA, AB, respectively, lie on a line perpendicular to the parallel lines and passing through a fixed point of the triangle.

E 1233. *Proposed by Joseph Andrushkiw, Seton Hall University*

Denote the sides and inradius of a triangle by a_0, b_0, c_0, and r_0. The points of contact form a new triangle whose sides and inradius are a_1, b_1, c_1, and r_1. Repeating the process one obtains the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{r_n\}$. Show that as $n \to \infty$

$$\lim r_n/a_n = \lim r_n/b_n = \lim r_n/c_n = \sqrt{3}/6.$$

E 1234. *Proposed by Leo Moser and J. R. Pounder, University of Alberta*

(a) Find 3 distinct positive integers, relatively prime in pairs, such that the sum of any two is divisible by the third. Prove that these integers are unique.

(b) Find 3 distinct positive integers such that the product of any two leaves a remainder of 1 on division by the third. Prove that the numbers are unique.

E 1235. *Proposed by D. S. Greenstein, University of Pennsylvania*

Let $f_0(x)$ be bounded and integrable over $a \leqq x \leqq b$, and let

$$f_n(x) = \int_a^x f_{n-1}(t)dt, \qquad n = 1, 2, 3, \cdots, \ a \leqq x \leqq b.$$

Evaluate

$$\sum_{n=1}^{\infty} f_n(x).$$

SOLUTIONS

Maximum Cross Section of Unit Cube

E 1201 [1956, 120]. *Proposed by C. S. Ogilvy, Hamilton College*

What is the area of the maximum cross section of the unit cube?

Solution by A. R. Hyde, West Hartford, Conn. Equatorial sections of the circumscribed sphere cut the cube in the following possible configurations:

(a) section includes 4 vertices of the cube (2 diametrically opposed pairs) and separates the others, 2 on each side of the plane; configuration a rectangle $\sqrt{2}$ by 1 with area $\sqrt{2}$;

(b) section includes 2 vertices (diametrically opposed) and separates the others, 3 on each side of the plane; configuration a parallelogram ranging in area from a rhombus with diagonals $\sqrt{3}$ and $\sqrt{2}$ and area $\sqrt{3/2}$ (plane bisecting 2 edges of the cube) to the rectangle of (a) as a limiting case;

(c) section separates vertices into two coplanar groups of 4; configuration

a parallelogram ranging from a unit square (plane parallel to a surface of the cube) to the rectangle of (a) as a limiting case;

(d) section separates vertices into two noncoplanar groups of 4; configuration a hexagon with pairs of equal opposite sides and equal opposite angles, and whose limits are the rhombus of (b) and the rectangle of (a).

It can be shown that planes parallel to any of these equatorial planes cut off successively smaller areas (or, at most, no greater areas) as the latitude increases. In cases (a), (b), and (c) the equatorial section is the same as if made on a unit-square prism of indefinite length. Parallel sections of this prism are congruent, whereas in cases (a) and (b) the areas decrease with increasing latitude because of the bounding surface, and in case (c) they are constant until a vertex is reached, then decrease. The hexagon of (d) may be considered as the parallelogramic section of the indefinitely long prism diminished by congruent triangles at two opposite vertices. For parallel planes these triangles remain similar as the base of one moves toward the vertex, the base of the other away from the vertex, at the same rate. Thus the sum of the triangles increases and the hexagon correspondingly decreases until a vertex is reached and the sectional area becomes pentagonal; thereafter, case (d) is like the others.

Hence the greatest section is that of one of the equatorial planes, namely that of case (a), the $\sqrt{2}$ by 1 rectangle with area $\sqrt{2}$.

Also solved by Leon Bankoff, M. S. Klamkin, and Azriel Rosenfeld.

Klamkin gave an analytical solution and remarked that it would be more difficult to determine the maximum cross section of the higher ordered regular polyhedra. The other two solutions were intuitive answers.

Editorial Note. Since the maximum section of a convex body possessing a center of symmetry must pass through the center of symmetry, one needs to consider only those sections that pass through the center of the cube.

Property of a General Simplex

E 1202 [1956, 120]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let O be an arbitrary point on an arbitrary line λ passing through the centroid G of a tetrahedron $ABCD$. If λ cuts the planes BCD, CDA, DAB, ABC in A', B', C', D', show that

$$A'O/A'G + B'O/B'G + C'O/C'G + D'O/D'G = 4.$$

Solution by Azriel Rosenfeld, Columbia University. Clearly

$$A'O/A'G = OBCD/GBCD = 4\,OBCD/ABCD,$$

with similar expressions for $B'O/B'G$, $C'O/C'G$, and $D'O/D'G$. Therefore

$$A'O/A'G + B'O/B'G + C'O/C'G + D'O/D'G$$
$$= 4(OBCD + OCDA + ODAB + OABC)/ABCD = 4.$$

Also solved by Hüseyin Demir, A. R. Hyde, P. W. M. John, M. S. Klamkin, Josef Langr, Jože Ulčar, Chih-yi Wang, and the proposer.

Many of the solutions were analytical. N. A. Court pointed out that the problem is stated and solved in *National Mathematics Magazine*, vol. 15, no. 6, March 1941, p. 275, art. 9b. It is stated there that the analogous proposition for (affine) two dimensional space is also true. Since the corresponding proposition for affine one dimensional space is easily established, one might conjecture that the proposition holds for the general simplex in affine n-dimensional space. Such is the case; Ulčar furnished an analytical proof, and Rosenfeld's synthetic proof is easily generalized.

Klamkin pointed out that the given problem is a special case of the following problem found in Altshiller-Court, *Modern Pure Geometry*, p. 25, Ex. 38: "The areas of the faces of a given polyhedron are inversely as the perpendiculars from a point O, and OO' meets these planes in P_1, P_2, P_3, $\cdots$, P_n, respectively; prove that

$$O'P_1/OP_1 + O'P_2/OP_2 + \cdots + O'P_n/OP_n = n."$$

Taking $n=4$ and replacing O' by O and O by G, we obtain the desired result.

An Extension of Poretsky's Law

E 1203 [1956, 120]. *Proposed by S. I. Birnbaum, Polytechnic Institute of Brooklyn*

Prove that for given sets X, Y, T we have $X=Y=0$ if and only if

$$T = (X \cap T') \cup (Y \cap T') \cup (X' \cap T \cap Y').$$

I. *Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J.*

Necessity: If X is non-empty, it contains an element a. If a is in T, it is not in $X \cap T'$, $Y \cap T'$, $X' \cap T \cap Y'$, nor in their union T. Assuming a is in T' also leads to a contradiction. Hence $X=0$. Similarly, $Y=0$.

Sufficiency: Substituting $X=Y=0$ in the given expression yields $T=T$.

II. *Solution by Lawrence Glasser, University of Wisconsin.*

Necessity: $(X \cap T') \subset T$, whence $X=0$; $(Y \cap T') \subset T$, whence $Y=0$.

Sufficiency: $(0 \cap T') \cup (0 \cap T') \cup (1 \cap T \cap 1) = T$.

III. *Solution by R. M. Conkling, New Mexico College of Agriculture and Mechanic Arts.* Factoring T' from the first two summands, we have

$$T = [T' \cap (X \cup Y)] \cup [T \cap (X' \cup Y')].$$

The right side is the symmetric difference of T and $X \cup Y$, *i.e.*, the set of elements in either T or $X \cup Y$, but not in both. This symmetric difference is T if and only if $X \cup Y = 0$, or, equivalently, if and only if X and Y are both null.

IV. *Solution by Roy Dubisch, Fresno State College.* We use the equivalence between the algebra of the class of all subsets of any set and the algebra of statements to rewrite the given equality as:

$$t \text{ equivalent to } (x \wedge \sim t) \vee (y \wedge \sim t) \vee (\sim x \wedge t \wedge \sim y).$$

Ogilvy, J. K. Patterson, J. D. Riley, L. A. Ringenberg, Stewart Robinson, Azriel Rosenfeld, David Rothman, J. P. Scholz, A. J. Tingley, Chih-yi Wang, David Zeitlin, and the proposer.

Schell included the following interesting note with his solution:

"Mr. Richard Ash, of the Programming Research Group of Remington Rand Univac Division, has devised a set of machine instructions which enables one to settle the validity of propositions in symbolic logic on the Univac, which is a large scale general purpose electronic computer. Because of the ease in stating Birnbaum's problem as a proposition in symbolic logic, I asked Mr. Ash to try it out using his machine instructions, which he kindly undertook to do. The problem was translated into symbolic logic form using the Polish parentheses-free notation, which is the convention adopted for the machine instructions. After the instructions were read into the machine, it typed out the query, 'How many variables?' For this problem the response '3' was typed in. The machine then typed out the request, 'Please type in problem.' The appropriate translation of Birnbaum's problem was then typed in. In something less than a second, the machine typed out the response, 'Result: True.' Mr. Ash informs me that the present program handles up to ten variables, which is an arbitrary limitation he imposed in writing the machine instructions, and an extension to a larger number of variables could readily be made.

"I believe your contributors will be impressed as I was with the speed with which the machine solved the problem, even though it pursued what appears to be a round about manner; namely an examination of all possible combinations of the truth variables."

An Application of Fundamental Continuity Theorems

E 1204 [1956, 120]. *Proposed by A. J. Goldman, Princeton University*

Let C be the circle $x^2+y^2=1$, $f(x, y)$ a continuous real-valued function defined on C, and A an angle such that $0<A\leqq\pi$. Show that there are two radii of C which form an angle A and have endpoints at which $f(x, y)$ has the same value.

Solution by L. R. Ford, Illinois Institute of Technology. Put $f(x, y)=F(\theta)$, where θ is the central angle. The periodic function $F(\theta)-F(\theta+A)\geqq 0$ at the maximum of $F(\theta)$ and $\leqq 0$ at the minimum; hence $=0$ somewhere.

Also solved by Louis Brickman, C. N. Campopiano, Peter Crawley, David Ellis, David Freedman, D. S. Greenstein, H. J. Hamilton, Vern Hoggatt, A. R. Hyde, M. S. Klamkin, Joseph Lehner, D. C. B. Marsh, J. C. Mathews, C. S. Ogilvy, C. F. Pinzka, J. D. Riley, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, David Rothman, J. A. Tierney, Jože Ulčar, Chih-yi Wang, J. V. Whittaker, David Zeitlin, and the proposer.

Special Case of an Inequality Due to Borg

E 1205 [1956, 121]. *Proposed by H. J. Cohen, City College of New York*

If $f''(x)$ is continuous on $[0, 1]$ and if $f(0)=f(1)=0$, show that

$$\int_0^1 |f''(x)/f(x)|\,dx > 4.$$

Solution by Martinus Esser, University of Maryland. There exist, by Rolle's Theorem, a point (X, Y) at which $|f(x)|$ is maximum, and, by the Mean Value Theorem, two abscissas a, b such that

$$0 < a < X < b < 1, \qquad f'(a) = Y/X, \qquad f'(b) = -Y/(1-X).$$

Then

$$\int_0^1 |f''(x)/f(x)|\,dx > \int_0^1 |f''(x)/Y|\,dx > \left|\int_a^b f''(x)dx\right| \Big/ |Y|$$
$$= |f'(b) - f'(a)| / |Y| = 1/X(1-X) \geqq 4.$$

The given inequality is the best possible because for $f(x)=x$ on $[0, 1/2-\epsilon]$, $f(x)=1-x$ on $[1/2+\epsilon, 1]$ and $f''(x)\leqq 0$ on $[0, 1]$ we have

$$\int_0^1 |f''(x)/f(x)|\,dx < 4/(1-2\epsilon),$$

which is arbitrarily close to 4.

Also solved by Paul Chessin, N. J. Fine, A. J. Goldman, H. Kestelman, M. S. Klamkin, Viktors Linis, T. F. Mulcrone, D. J. Newman, John Selfridge, R. E. Shafer, Chih-yi Wang, Albert Wilansky, and David Zeitlin.

Several solvers pointed out that the proposed inequality is a special case of an inequality due to A. Beurling, which in turn is a special case of an inequality due to G. Borg. See this MONTHLY [1956, 27]. Chessin pointed out that in the *Quarterly Journal of Applied Mathematics*, Oct. 1955, p. 335, is a reference to the proposed inequality with an indication that it is due to Liapounoff and is the best estimate.

It should be noted that $f(x)\equiv 0$ is to be ruled out, since in this case the integral does not exist.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4703. *Proposed by R. E. Bellman, Rand Corporation*

Show that the conditions (a) $u_n \geqq u_{n-1} - a_{n-1}$, $a_n \geqq 0$, $\sum_{n=0}^{\infty} a_n < \infty$, (b) $u_n \leqq M$, imply the convergence of the sequence $\{u_n\}$.

4704. *Proposed by L. E. Clarke, University College of the Gold Coast*

If m is a positive integer greater than 1, show that a necessary and sufficient condition that m be prime is that, for every integer n,

$$\binom{n}{m} \equiv [n/m] \pmod{m}.$$

4705. *Proposed by Albert Wilansky, Lehigh University*

It is possible to construct an additive but discontinuous real function of one real variable. Is it possible to construct one with the property that $f(x) = f(y)$ implies $x = y$? (This would give an algebraic isomorphism of the additive group of the reals which is not continuous.)

4706. *Proposed by S. W. Golomb, Cambridge, Mass.*

If $f(x)$ and $g(x)$ are real periodic functions of the real variable x, having incommensurable periods, is it possible for $f(x)+g(x)$ to ~~be~~ periodic?

4707. *Proposed by Charles Fox, McGill University*

A magic matrix is one whose elements are the numbers of a magic square, *i.e.*, every row, column and diagonal has the same sum. (a) Show that a 3 by 3 magic matrix inverts into a magic matrix. (b) Can this result be extended to magic matrices of higher order?

SOLUTIONS

A Summation of Cosines

4636 [1955, 259]. *Proposed by Emma Lehmer, Berkeley, California*

Show that, if n is a prime of the form $8m+5$, then

$$\sum_{\nu=0}^{n-1} \cos \frac{2\pi\nu^4}{n} = \sqrt{n}.$$

For what other values of n is this true?

Solution by Leonard Carlitz, Duke University. A. Let p be a prime $\equiv 5 \pmod 8$ and let r be a primitive root (mod p). Put

$$\eta_k = \sum_{s=0}^{m-1} \epsilon^{r^{4s+k}} \qquad (k = 0, 1, 2, 3)$$

where $p = 4m+1$ and $\epsilon = e^{2\pi i/p} = \cos(2\pi/p) + i \sin(2\pi/p)$. Next define the Lagrange resolvents (see, *e.g.*, Bachmann, *Die Lehre von der Kreisteilung*, p. 168)

$$(i, \epsilon) = \eta_0 + i\eta_1 - \eta_2 - i\eta_3,$$
$$(-i, \epsilon) = \eta_0 - i\eta_1 - \eta_2 + i\eta_3.$$

We have also

$$\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1,$$
$$\eta_0 - \eta_1 + \eta_2 - \eta_3 = \sqrt{p},$$

the last from the evaluation of a Gauss sum. It follows that

$$1 + 4\eta_0 = (i, \epsilon) + (-i, \epsilon) + \sqrt{p},$$
$$1 + 4\eta_2 = -(i, \epsilon) - (-i, \epsilon) + \sqrt{p}.$$

Noting that η_2 and η_0 are conjugates, and that $1^4, 2^4, \cdots, (p-1)^4$ give the set of biquadratic residues counted four times, we get

$$\sum_{s=0}^{p-1} \cos \frac{2\pi s^4}{p} = 1 + 2(\eta_0 + \eta_2) = \sqrt{p}, \tag{1}$$

the stated result.

It also follows from the above that

$$1 + 2\eta_1 + 2\eta_3 = -\sqrt{p},$$

so that

$$\sum_{s=0}^{p-1} \cos \frac{2\pi k s^4}{p} = -\sqrt{p}, \tag{2}$$

where k is any quadratic non-residue of p. We may combine (1) and (2) in the form

$$\sum_{s=0}^{p-1} \cos \frac{2\pi a s^4}{p} = \left(\frac{a}{p}\right)\sqrt{p}, \qquad (a \not\equiv 0). \tag{3}$$

For $p \equiv 1 \pmod 8$, the numbers $1+4\eta_0$, $1+4\eta_2$ are determined by means of

$$\sqrt{p} + \sqrt{2(p + a\sqrt{p})}, \tag{4}$$

where $p=a^2+b^2$, $a\equiv -1 \pmod 4$, (see Fricke, *Lehrbuch der Algebra*, v. 1, pp. 441–443), but it is not known which value of (4) corresponds to $1+4\eta_0$. Since for $p\equiv 1 \pmod 8$,

$$1 + 4\eta_0 = \sum_{s=0}^{p-1} \cos \frac{2\pi s^4}{p},$$

it is clear from (4) that (1) does not hold.

B. To find values of n such that

$$\sum_{s=0}^{n-1} \cos \frac{2\pi s^4}{n} = \sqrt{n}, \tag{5}$$

we make use of some known results concerning

$$S_\rho = \sum_{s=0}^{n-1} \rho^{s^4},$$

where ρ is a primitive nth root of unity (see Landau, *Vorlesungen über Zahlentheorie*, v. 1, Th. 281, 307, 312):

(a) $$S_{\rho_1\rho_2} = S_{\rho_1}S_{\rho_2},$$

where ρ_1 is an n_1th root and ρ_2 an n_2th root of unity, $(n_1, n_2)=1$,

(b) $$S_\rho = p^{l-1} \qquad (n = p^l,\ p > 2,\ 2 \leqq l \leqq 4),$$

(c) $$S_\rho = p^3 S_{\rho^{p^4}} \qquad (n = p^l,\ l > 4).$$

First, if $n=m_1^2$, where m_1 is quadratfrei and odd, it follows from (a) and (b) that $S_\rho = m_1$.

Second, if $m_2=p_1 \cdots p_{2r}$ where the p's are distinct primes $\equiv 3 \pmod 4$, then every square (mod m) is necessarily congruent to a fourth power, so that

$$S_\rho = \sum_{s=0}^{m_2-1} \rho^{s^2} = \left(\frac{k}{m_2}\right)\sqrt{m_2} \qquad (\rho = e^{-2\pi k i/m}),$$

Third, for $m_3=4$, we have $S_i = 1+i$, $S_{-i}=1-i$.

Fourth, for $m_4=p\equiv 5 \pmod 8$ we have the case treated in A above.

It follows at once that (5) holds in each of these four cases. Moreover we may combine by means of (a) to get the following possibilities:

$$\begin{aligned} n &= m_1^{2e_1} m_2^{e_2} m_3^{e_3}, \\ n &= m_1^{2e_1} m_2^{e_2} m_4^{e_4}, \end{aligned} \tag{6}$$

where each $e_j=0$, 1 and $(m_1, m_2)=(m_1, m_3)=1$. For example, when $n=m_2m_4 = m_2p$, let $up+vm_2=1$, $\rho=e^{2\pi i/n}$, $\rho_2=e^{2\pi iu/m_2}$, $\rho_4=e^{2\pi iv/p}$. Then

$$S = S_2S_4 = \left(\frac{u}{m_2}\right)\sqrt{m}S_4,$$

$$\sum_{s=0}^{n-1} \cos \frac{2\pi s^4}{n} = \left(\frac{u}{m_2}\right)\sqrt{m_2}\left(\frac{y}{p}\right)\sqrt{p} = \left(\frac{u}{m_2}\right)\left(\frac{v}{p}\right)\sqrt{m_2 p}.$$

But since $up \equiv 1 \pmod{m_2}$ and $vm_2 \equiv 1 \pmod{p}$,

$$\left(\frac{u}{m_2}\right)\left(\frac{v}{p}\right) = \left(\frac{p}{m_2}\right)\left(\frac{m_2}{p}\right) = +1,$$

by the quadratic reciprocity theorem. Hence (5) holds in this case.

Presumably (6) enumerates all the cases in which (5) holds.

Also solved (part A only) by J. H. Hodges, R. E. Shafer, and the Proposer.

Integral Points on a Sphere

4643 [1955, 446]. *Proposed by Norman Anning, Alhambra, California*

If r is a prime of the form $4k+1$ or is the product of such primes, show that the number of points with integral coordinates on the sphere

$$x^2 + y^2 + z^2 = r^2$$

is $6r$.

Remarks by C. D. Olds, San Jose State College. In a letter written in 1884 to Ch. Hermite, T. J. Stieltjes proved by means of elliptic functions that if $n = p^k$, $p \equiv 1 \pmod 8$, p prime, then $N_3(n^2) = 6p^k$, where $N_3(n^2)$ denotes the number of representations of n^2 as the sum of three squares. ["Lettre 45, *Correspondence d'Hermite et de Stieltjes*, vol. 1, Paris, 1905, pp. 89–94.] In 1907 this formula was generalized by A. Hurwitz [*Mathematische Werke*, vol. 2, Basel, 1933, p. 751.]

The general formula is as follows: Let

$$n = 2^k m = 2^k PQ, \quad P = \prod_{\nu=1}^{r} p_\nu^{a_\nu}, \quad Q = \prod_{\nu=1}^{s} q_\nu^{b_\nu},$$

where each p_ν is a prime $\equiv 1 \pmod 4$, and each q_ν is a prime $\equiv 3 \pmod 4$, then

$$N_3(n^2) = 6P \prod_{\nu=1}^{s}\left[q_\nu^{b_\nu} + 2\,\frac{q_\nu^{b_\nu} - 1}{q_\nu - 1}\right].$$

Arithmetical proofs have been given by G. Pall [*Journal of the London Mathematical Society*, vol. 5 (1930), pp. 102–105], and by C. D. Olds [*Bulletin of the American Mathematical Society*, vol. 47 (1941), pp. 499–503.]

Similar discussions were submitted by Leonard Carlitz, M. S. Klamkin, A. Walfisz, and Chih-yi Wang.

Isomorphic Groups

4644 [1955, 447]. *Proposed by Azriel Rosenfeld, Columbia University*

Prove that in no field K can the additive group K^+ be isomorphic to the

multiplicative group K^*.

Solution by R. E. Priest, Bell Telephone Laboratories, New York City. If the characteristic of K is not equal to 2, then K^+ has precisely one element of order 2, *viz.* 0, and K^* has precisely two elements of order 2, *viz.* 1 and -1.

If the characteristic of K equals 2, then all elements of K^+ are of order 2 while K^* has precisely one element of order 2, *viz.* $1 = -1$.

In no case, therefore, can K^+ and K^* be one to one.

Also solved by W. J. Blundon, G. U. Brauer, Leonard Carlitz, R. C. Courter, Helen F. Cullen, M. P. Drazin, David Ellis, M. P. Epstein, V. D. Gokhale, A. J. Goldman, William Googe, R. D. Gordon, C. D. Gorman, D. S. Greenstein, B. Gross, A. S. Hendler, J. Horváth, H. H. Johnson, Tung-Po Lin, D. C. B. Marsh, E. C. Milner, T. O. Moore, F. D. Parker, G. B. Robison, H. D. Ruderman, J. J. Schäffer, and W. R. Scott.

A Determinant Divisible by a Prime

4645 [1955, 447]. *Proposed by Leonard Carlitz, Duke University*

Let $0 \leqq r \leqq p-1$, where p is prime, and let $a_0, a_1, \cdots, a_r$ be integers not divisible by p. Define the determinant of order $r+1$

$$\Delta_r = \left| a_i^{p^j} \right| \qquad (i, j = 0, 1, \cdots, r).$$

Show that

$$\Delta_r \equiv 0 \pmod{p^{r(r+1)(r+2)/6}}.$$

Solution by the Proposer. We shall prove the following stronger result. Let $m \geqq 0$ and define

$$\Delta_r^{(m)} = \left| a_i^{p^{m+j}} \right| \qquad (i, j = 0, 1, \cdots, r).$$

Then we show that

$$\Delta_r^{(m)} \equiv 0 \pmod{p^e}, \tag{1}$$

where $e = mr(r+1)/2 + r(r+1)(r+2)/6$.

Define

$$\Delta a^{p^n} = \frac{a^{p^{n+1}} - a^{p^n}}{p^{n+1}}, \qquad \Delta^2 a^{p^n} = \frac{\Delta a^{p^{n+1}} - \Delta a^{p^n}}{p^{n+1}}$$

and generally

$$\Delta^{s+1} a^{p^n} = \frac{\Delta^s a^{p^{n+1}} - \Delta^s a^{p^n}}{p^{n+1}}.$$

In $\Delta_r^{(m)}$ subtract the jth column from the $(j+1)$th for $j = 0, 1, \cdots, r-1$.

Then

$$\Delta_r^{(m)} = p^{mr+r(r+1)/2} \left| a_i^{p^m} \Delta a_i^{p^m} \cdots \Delta a_i^{p^{m+r-1}} \right| \quad (i = 0, 1, \cdots, r).$$

Again subtracting the jth column from the $(j+1)$th for $j=1, 2, \cdots, r-1$, we get

$$\Delta_r^{(m)} = p^c \left| a_i^{p^m} \Delta a_i^{p^m} \Delta^2 a_i^{p^m} \cdots \Delta^2 a_i^{p^{m+r-2}} \right|,$$

where $c=m(2r-1)+r(r+1)/2+r(r-1)/2$. Continuing in this way we finally get

$$\Delta_r^{(m)} = p^e \left| \Delta^j a_i^{p^m} \right| \qquad (i, j = 0, 1; \cdots, r), \tag{2}$$

where

$$\begin{aligned} e &= m\{r + (r-1) + \cdots + 1\} + \{r(r+1)/2 + (r-1)r/2 + \cdots + 1\} \\ &= mr(r+1)/2 + r(r+1)(r+2)/6. \end{aligned}$$

Now Schur (*Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 1933, pp. 145–151) has proved that for all $(a, p)=1$, $r<p$, the numbers $\Delta^r a^{p^n}$ are integral for all $n \geqq 0$. Consequently (2) implies (1).

Also solved by F. W. Ponting.

Two Limits Suggested by Probabilistic Considerations

4646 [1955, 447]. *Proposed by Oliver Gross, the Rand Corporation*

Let $F_1(x)=G_1(x)=x$ for x real, and for $n>1$ define F_n, G_n recursively by

$$\begin{aligned} F_n(x_1, \cdots, x_n) &= \max(x_1, G_{n-1}(x_2, \cdots, x_n)), \\ G_n(x_1, \cdots, x_n) &= \min(x_1, F_{n-1}(x_2, \cdots, x_n)). \end{aligned}$$

Are the following relations true:

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 F_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n = \frac{\pi\sqrt{3}}{9},$$

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 G_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n = 1 - \frac{\pi\sqrt{3}}{9}?$$

Solution (abridged) by Chih-yi Wang, University of Minnesota. The given limits are correct. Define

$$\int_0^1 \cdots \int_0^1 [F_n(x_1, \cdots, x_n)]^r dx_1 \cdots dx_n = I(r, n),$$

$$\int_0^1 \cdots \int_0^1 [G_n(x_1, \cdots, x_n)]^s dx_1 \cdots dx_n = J(s, n).$$

It can be easily verified that for $n>2$:

$$(1)\qquad I(r, n) = \frac{1}{r+1} + \frac{r}{r+1} I(r+1, n-2) - \frac{r}{r+2} I(r+2, n-2),$$

$$(2)\qquad J(r, n) = I(r, n-1) - \frac{r}{r+1} I(r+1, n-1),$$

$$(3)\qquad I(r, 1) = \frac{1}{r+1}, \qquad I(r, 2) = \frac{1}{r+1} + \frac{r}{(r+1)(r+2)}.$$

Case I ($n=2m+1$). By applying (1) successively and by aid of (3) we obtain, omitting details,

$$I(1, n) = \int_0^1 \int_0^y \frac{1 - x^{m+1}(1-x)^{m+1}}{1 - x + x^2}\, dxdy.$$

In this integral, the part which is independent of m yields the required result $\pi\sqrt{3}/9$, while the other part can be so estimated that the lower bound of the denominator is $3/4$ and the upper bound of the numerator is 4^{-m-1} which will $\to 0$ as $m \to \infty$.

Case II ($n=2m$). By applying (1) successively and by aid of (3) we obtain, again omitting details,

$$I(1, n) = \int_0^1 \int_0^y \frac{1 - x^m(1-x)^m}{1 - x + x^2}\, dxdy + \int_0^1 \int_0^y x^m(1-x)^{m-1} dxdy.$$

The first integral converges to the same limit $\pi\sqrt{3}/9$ as stated in case I, and the integrand of the second integral is dominated by 4^{-m+1} which will $\to 0$ as $m \to \infty$.

The second proposed statement can be verified by (2) and the first result.

The result of the problem can be generalized as follows: For any given positive integers r and s, we have

$$\lim_{n\to\infty} I(r, n) = \int_0^1 \int_0^y \frac{rx^{r-1}}{1 - x + x^2}\, dxdy,$$

$$\lim_{n\to\infty} J(s, n) = \int_0^1 \int_0^y \frac{sx^{s-1}(1-x)}{1 - x + x^2}\, dxdy,$$

which can be verified in a straightforward manner.

Note by the Proposer. The following heuristic argument is of interest. Interpret the x_i's as independent random variables from the uniform distribution over $[0, 1]$. The integrals in question represent the expected values of the two random variables F_n, G_n. But since, if x_1 and x_2 are independent random variables, it is easy to see that

$$P\{\max(x_1, x_2) \leqq x\} = P\{x_1 \leqq x\} \cdot P\{x_2 \leqq x\},$$

$$P\{\min(x_1, x_2) \leqq x\} = 1 - \{1 - P(x_1 \leqq x)\}\{1 - P(x_2 \leqq x)\},$$

whence we can set up a pair of simultaneous equations for the limiting distributions f, g of F_n, G_n (which can be shown to exist), namely,

$$f(x) = x \cdot g(x), \qquad g(x) = 1 - (1 - x)(1 - f(x)), \qquad x \in [0, 1].$$

Then

$$f(x) = \frac{x^2}{1 - x + x^2} \quad \text{and} \quad g(x) = \frac{x}{1 - x + x^2}.$$

The limits required by the problem are then given by

$$\int_0^1 x df(x) \quad \text{and} \quad \int_0^1 x dg(x)$$

which give the numerical values as stated.

Also solved by Richard Courter, M. L. Freimer and A. L. Tritter, R. D. Gordon, and E. C. Milner.

RECENT PUBLICATIONS

All new material for this section should be sent to the incoming editor, Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

Elementary Topology. By D. W. Hall and G. L. Spencer, II. New York, John Wiley & Sons, Inc., 1955. 7+303 pages. $7.00.

As point set topology has become firmly established in a position of fundamental importance in much of modern mathematics, it is inevitable that attention be given to the feasibility and desirability of introducing the subject to the student of mathematics at an early stage in his education. At many schools, it is difficult, if not impossible, to teach an elementary course, unless one has available an adequate text. Those who feel that topology should be taught early will probably welcome the book by Hall and Spencer, for this book is intended as a text for an introductory course in point set topology for students who have not necessarily had courses in advanced calculus and modern algebra. The book does not treat algebraic topology.

The material treated in the first half of the book (149 pages) is elementary and should prove to be accessible to undergraduates. The second half of the book, which is primarily concerned with the proofs of some of the gems of topology, is, in this reviewer's opinion, too difficult for the typical undergraduate mathematics major. Nevertheless, the authors hope that it will be accessible, and therefore spell out all proofs in great detail.

The book begins with a concise chapter on set theory, which, with minor exceptions, meets the prevailing standards of rigor for non-axiomatic set theory.

Chapter II is devoted to a treatment of the basic properties of the real number system. Concepts such as connectedness and continuity are defined and developed in such a way as to prepare the student for the corresponding abstractions he will meet in later chapters. Presumably in an attempt to keep the book elementary, no definition of the real numbers is given. It is assumed that each real number has a "representation" as an infinite decimal. The reviewer believes that some students will feel uncomfortable with this treatment.

Chapters III and IV, which comprise about 100 pages, are devoted to topological and metric spaces. The basic concepts are here introduced, almost always with their most modern definitions, and the fundamental theorems are carefully proven. There are a large number of exercises, some of which are routine, others of which are thought-provoking. Some topics not treated in the text, notably the theory of category and of Moore-Smith convergence, are developed in a series of exercises. The contents of Chapters III and IV will probably be the heart of any introductory course in point set topology.

The remainder of the book is concerned with some beautiful but deep and difficult concepts and theorems. The unit interval and circle are given topological characterizations along the lines first done by Janiszewski. The famous theorem of Hahn and Mazurkiewicz which characterizes the continuous image of the closed unit interval, as well as Zippin's characterization of the simple closed surface are presented. Also proven are the Jordan curve theorem, the Jordan-Schoenflies theorem, and theorems concerning complex concepts introduced by Sierpinski and Bing. The last chapter of the book deals with transfinite reasoning, *i.e.*, the Axiom of Choice, Zorn's lemma, well-ordering, and Tychonoff's theorem.

Some readers of the book may be critical of some aspects of the authors' style. For example, some may feel that the authors should have presented Hilbert's proof of Peano's startling discovery that the closed unit square is the continuous image of the unit interval in order to elucidate the statement and proof of the more general theorem of Hahn and Mazurkiewicz. Others, however, may feel grateful that the authors left to the instructor pleasant opportunities to delight and enlighten his class. In any event *Elementary Topology* is a welcome addition to the relatively few books suitable as texts for courses for advanced undergraduates in modern mathematics.

L. E. DUBINS
Carnegie Institute of Technology

Through the Mathescope. By C. S. Ogilvy. New York, Oxford University Press 1956. 162 pages. $4.00.

The current importance of mathematics is generally acknowledged: it is an indispensable adjunct to technology. But to the public this merely signifies that a familiar evil has become a necessary one. The greater therefore is the mathematicians' wish "to proclaim to the world that their beloved mathe-

matics is not the dry-as-dust, ugly, routine affair that so many people think it is."

The author, like others before him, accepts the challenge. But he is unlike them in at least some of these respects: he speaks briefly and on few but well-chosen topics; he does not assume that a reader with little mathematical preparation may safely be treated like a fool; he provides elementary proofs when available and none otherwise; he distinguishes nicely between mathematical truth and physical reality; he exploits some of the simpler paradoxical situations; he makes a single concept do multiple duty whenever possible—thus providing a happy continuity between otherwise apparently unrelated subjects; he injects lively historical observations; he stresses the artistic element; and by his breezy style no less than the remarkable recency of many of his references he makes mathematics live.

The "mathescope," says the author (after crediting invention of the word to Watson Davis), is the mathematician's counterpart of the scientists' telescope and microscope, "with reason for its pedestal and inspiration for its lenses." Subjects viewed through the mathescope are organized into 11 intriguingly titled chapters (*e.g.*, "Wheels within Wheels" for the one on cycloids) and include (though not in this order) integral, rational, irrational, imaginary, and transfinite numbers; Pythagoras's Theorem, the regular polyhedra, coordinate geometry, the conic sections, and cycloids; infinite series and continued fractions; fallacies arising from algebraic misdemeanors, pictorial proofs, and immoderate enthusiasm for circle-squaring and angle-trisecting; probability, with the binomial and normal distributions; and elementary topology.

An appendix of 23 pages provides a running commentary on the 129 pages of main text, with additional references, proofs of slightly higher order of difficulty than those presented in the earlier section, and what might be termed "lay research questions".

The only statement in the book to which the reviewer took serious exception was this: "Essentially what the calculus does is integrate (sum) an infinite number of infinitely small elements of this type" (p. 86). However, the author is still apologizing for it two pages later. (One recalls with a shudder the presentations of the calculus offered by authors like Hogben!) Misprints noted were "x^2" for "p^2" on p. 143, missing subscripts on a couple of "V's" on p. 144, and a superfluous "the" (within a quotation) on p. 118. Slips in reasoning, terminology, *etc.* which were observed were few in number and were generally slight.

Perhaps the strongest feature of the book is its success in conveying the author's personal enthusiasm for mathematics. It is hard to imagine anyone's reading a single page without concluding that mathematics is a subject in which at least Professor Ogilvy is passionately interested! The more laymen who can be tempted to read *Through the Mathescope* the better.

H. J. HAMILTON
Pomona College

Methods in Numerical Analysis. By Kaj L. Nielsen. New York, The Macmillan Company, 1956. xiii+382 pages. $6.90.

It has become increasingly evident that numerical analysis may now be regarded as having a *raison d'être* in the modern mathematics curriculum.

Dr. Nielsen has sought to write *Methods in Numerical Analysis* as an elementary textbook and has approached his objective in a careful and effective manner. As the title implies, the subject matter stresses methodology in the art of computation rather than mathematical sophistication. Appropriate emphasis is placed on those modern computational procedures arising in the analysis of tabulated data and approximate solutions of equations which lend themselves to the use of tables included in the appendix, or are easily adapted to automatic desk calculators for problem laboratory purposes.

The first five chapters are concerned primarily with classical material from the theory of errors, accuracy of approximate calculations, finite differences, interpolation theory, numerical differentiation and integration. Chapters six and seven deal principally with existent procedures for numerical solutions of sets of linear and nonlinear algebraic equations, and an introduction to techniques for handling differential, difference, and partial differential equations. Chapters eight and nine portray some recent developments in the representation of empirical functions and smoothing of data employing the methods of least squares, Nielsen-Goldstein, orthogonal polynomials, and autocorrelation functions.

The author apparently believes firmly in systematic procedures and utilizes a multitude of calculating forms and schematics in the topical illustrations which are worked out in detail.

In summary, it is the reviewer's opinion that the author has succeeded in writing an effective elementary text and convenient reference on up-to-date methods in the field of numerical analysis, useful for a wide variety of situations encountered by students and workers in applied mathematics.

W. E. RESTEMEYER
University of Cincinnati

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ORACLE APPLICATIONS PROGRAM

The Oak Ridge Institute of Nuclear Studies has announced that it will make available to universities throughout the region the services of the ORACLE, its

high-speed electronic digital computer. Dr. T. W. Hildebrandt will direct the new program. One object of the program is to aid in extending to university campuses a wider awareness of the capabilities of modern high-speed computing equipment. The program will also assist university research programs.

For further information write to Dr. T. W. Hildebrandt, ORACLE Applications, University Relations Division, Oak Ridge Institute of Nuclear Studies, P.O. Box 117, Oak Ridge, Tennessee.

THE UNIVERSITY OF ILLINOIS PROGRAM OF HIGH SCHOOL MATHEMATICS

The University of Illinois has announced that a grant of $277,000 from the Carnegie Corporation of New York will be used by a University committee to continue its work of devising a new mathematics course for each of the four secondary years, preparing classroom text materials and teachers' manuals, and conducting training courses to acquaint teachers with the new teaching techniques to be developed. This committee consists of representatives of the colleges of education, engineering, and liberal arts and sciences. The Illinois project has the goal of bringing the high school mathematics curriculum up to date by including topics from modern mathematics and by presenting the subject as an integrated body rather than as a group of isolated courses.

PERSONAL ITEMS

Professor Richard Courant, head of New York University's Graduate Mathematics Department and scientific director of the Institute of Mathematical Sciences, has been elected to foreign membership in the Royal Netherlands Academy of Sciences and Letters.

Professor S. C. Kleene of the University of Wisconsin has been elected President of the Association for Symbolic Logic for a three-year term beginning January 1, 1956.

Assistant Professor M. D. Marcus of the University of British Columbia has been awarded a Postdoctoral Research Associateship for advanced study at the National Bureau of Standards.

Professor J. J. Stoker, Jr., associate director of the Institute of Mathematical Sciences, New York University, has been awarded the Dannie Heineman Prize of $5,000 for his two books, "Non-Linear Vibrations" and "Theory of Water Waves".

California Institute of Technology announces the following: Professor H. F. Bohnenblust has been appointed Dean of Graduate Studies; Associate Professor C. R. DePrima has been promoted to a professorship in the Department of Applied Mechanics; Assistant Professor T. M. Apostol has been promoted to an associate professorship.

At Purdue University: Professor C. Y. Pauc of the Faculté des Sciences de Rennes, France, has been appointed to a visiting professorship; Dr. H. S. C. Sharp, previously chairman of the Department of Mathematics of the U. S. Coast Guard Academy, has been appointed to an associate professorship at

the Hammond Extension Center; Dr. Melvin Carter, formerly a research assistant at North Carolina State College, Professor Philip Dwinger of the University of Indonesia, Assistant Professor Imanuel Marx of the University of Michigan, and Dr. R. F. Williams, previously a visiting lecturer at the University of Wisconsin, have been appointed to assistant professorships; Dr. M. J. Mansfield, previously a research assistant at the University, has been appointed to an instructorship.

Seattle University announces that Mr. C. C. Chang and Mr. A. L. Yandl have been appointed to instructorships.

University of California, Davis, reports: Assistant Professor Anne C. Davis of the University of Washington has been appointed to an assistant professorship; Professor G. A. Baker has been named Faculty Research Lecturer.

University of Washington (Seattle) announces: Assistant Professors M. G. Arsove, F. H. Brownell, and E. A. Michael have been promoted to associate professorships; Instructors H. A. Forrester, J. H. Walter, and R. L. Vaught have been promoted to assistant professorships; Associate Professor V. L. Klee, Jr. has been awarded a Sloan Foundation Fellowship; Assistant Professor R. L. Vaught has been awarded a Fulbright Fellowship for study during 1956–57 at the University of Amsterdam; Associate Professor E. A. Michael will spend the year 1956–57 at the Institute for Advanced Study; J. Butler of the University of California at Los Angeles, R. K. Getoor of Princeton University, T. J. McMinn of the University of California at Berkeley, and A. Nijenhuis of the University of Chicago have been appointed to assistant professorships; R. M. Blumenthal of Cornell University, G. Helmberg of Innsbruck, Austria, and R. T. Ives of the University of Washington have been appointed to instructorships; J. M. G. Fell of the University of Chicago has been appointed to a research assistant professorship under a grant from the Office of Naval Research.

Wayne State University announces: Professor S. T. Hu of the University of Georgia and Professor Wallace Givens of the University of Tennessee have been appointed to professorships; Assistant Professor Felix Haas, University of Connecticut, and Assistant Professor M. T. Wechsler of the State College of Washington have been appointed assistant professors.

Dr. S. S. Abhyankar, recently an associate at Columbia University, has been appointed to a visiting assistant professorship at Columbia University.

Assistant Professor A. V. Banes of Trinity University has a position as a programmer analyst with the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. C. R. Berndtson, previously a teacher at Vinalhaven High School, Maine, is employed as an applied research mathematician by the Avco Manufacturing Corporation, Stratford, Connecticut.

Dr. F. G. Brauer, formerly a graduate student at Massachusetts Institute of Technology, has been appointed to an instructorship at the University of Chicago.

Professor T. H. Brown, Graduate School of Business Administration, Har-

vard University, has been appointed to an emeritus professorship in business statistics.

Dr. R. G. Buschman of the University of Colorado has been appointed to an associate professorship at the University of Wichita.

Assistant Professor Lorenzo Calabi of Boston College has been promoted to an associate professorship.

Mr. M. N. Chase, previously a mathematician at Battelle Memorial Institute, has accepted a research position with Technical Operations, Inc., Arlington, Massachusetts.

Mr. T. E. Cheatham, Jr., formerly a research assistant at Purdue University, is Head of the Computer Division, Technical Operations, Inc., Washington, D. C.

Dr. John Christopher of Fresno State College has a position as a mathematician with the ElectroData Corporation, Pasadena, California.

Assistant Professor L. J. Cote of Purdue University has been appointed to an assistant professorship at Syracuse University.

Mr. D. F. Coulter, Jr. of Grays Harbor College has been appointed to an instructorship at Hartnell College.

Mr. D. L. Daly of Missouri School of Mines and Metallurgy has been appointed to an assistant professorship at Illinois Wesleyan University.

Mr. D. C. Davis, previously a development engineer for McDonnell Aircraft Corporation, St. Louis, Missouri, is employed now as an engineer by Chance-Vought Aircraft, Dallas, Texas.

Dr. Bernard Dimsdale of Raytheon Manufacturing Company has accepted a position as Director of the Univac Computing Center, Remington Rand Division of Sperry-Rand Corporation, Los Angeles, California.

Dr. Jesse Douglas of Columbia University has been appointed to a professorship at the City College of the City of New York.

Dr. H. T. Engstrom, formerly vice-president for Research and Engineering, Engineering Research Associates, Washington, D. C., is Vice-President of Remington-Rand, Division of Sperry Rand Corporation, New York, New York.

Assistant Professor M. P. Epstein of Johns Hopkins University has received an appointment as mathematician at Bell Telephone Laboratories, Whippany, New Jersey.

Mr. D. O. Etter, Jr., of International Lubricant Corporation, New Orleans, Louisiana, has accepted a position as a research engineer with North American Aviation, Los Angeles, California.

Visiting Assistant Professor G. H. F. Gardner of Carnegie Institute of Technology has a position as a mathematical physicist with Gulf Research and Development, Pittsburgh, Pennsylvania.

Mr. L. T. Gardner, formerly an actuarial analyst for the Travelers Insurance Company, Hartford, Connecticut, is teaching at Oakwood School, Poughkeepsie, New York.

Professor A. W. Goodman of the University of Kentucky is on leave at the Institute for Advanced Study.

Dr. Edison Greer, previously a design engineer at Aerojet General, Azusa, California, and professor of mathematics at Mount San Antonio College, has a position as a research scientist on the applied mathematics staff of Missile Systems Division Research Laboratories, Lockheed Aircraft Corporation, Palo Alto, California.

Professor I. R. Hershner, Jr., formerly chairman of the Department of Mathematics of the University of Vermont, has accepted a position with the Office of Chief of Research and Development, U. S. Army, Washington, D. C.

Professor Ralph Hull, head of the Department of Mathematics of Purdue University, has accepted a position as a staff member of the Ramo-Wooldridge Corporation, Los Angeles, California.

Associate Professor P. E. Irick of Purdue University has accepted a position with the Highway Research Board's Road Test Project.

Assistant Professor H. G. Jacob, Jr., of Louisiana State University has been promoted to an associate professorship.

Mr. Edgar Karst, formerly an electronic computing analyst with Great Lakes Pipe Line Company, Kansas City, Missouri, now has a position as a computing engineer with the IBM Scientific Computation Laboratory, Endicott, New York.

Assistant Professor N. D. Kazarinoff of Purdue University has been appointed to an assistant professorship at the University of Michigan.

Mr. J. P. Line of the University of Rochester has been appointed to an assistant professorship at Georgia Institute of Technology.

Mr. G. E. Mahoney of Computer Control Company, Inc., Point Mugu, California, has accepted a position as Director of the Computing Bureau, SPICA, Inc., Somerville, Massachusetts.

Dr. P. J. McCarthy of the College of the Holy Cross has been appointed to an assistant professorship at Florida State University.

Mr. D. S. McManus of the University of Buffalo has been appointed Assistant Professor of Mechanical Engineering at Norwich University.

Mr. D. D. Morrison, previously a research associate in the Computation Laboratory, Wayne State University, has a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Dr. Margaret Owchar, formerly a researcher and statistician for the Manitoba Cancer Relief and Research Institute, Winnipeg, Manitoba, has been appointed to an assistant professorship at Bemidji State Teachers College.

Associate Professor P. M. Pepper of Ohio State University is now Director of the Mapping and Charting Research Laboratory of the University.

Assistant Professor A. J. Perlis of Purdue University has been appointed to an associate professorship at Carnegie Institute of Technology.

Mr. Arthur Reetz, Jr., of Missouri School of Mines has a position as a nu-

clear engineer with Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Assistant Professor J. B. Roberts of Reed College is on leave of absence for the academic year 1956–57 as Visiting Assistant Professor at Wesleyan University.

Dr. Berthold Schweizer has been appointed to an instructorship at Illinois Institute of Technology.

Mr. Seymour Sherman of Research and Advanced Development Division, Avco Manufacturing Company, Stratford, Connecticut, has accepted a position as staff member of Dian Laboratories, New York, New York.

Professor C. N. Shuster, head of the Department of Mathematics of State Teachers College, Trenton, New Jersey, has retired from this position. He is teaching at Trenton Junior College and at The Pennington School, Pennington, New Jersey.

Mr. W. K. Spears of the Electronics Corporation of America has accepted a position as an engineer with the General Electric Company, Utica, New York.

Assistant Professor E. A. Sturley of Allegheny College has been promoted to an associate professorship.

Professor Irving Sussman of California State Polytechnic College has been appointed Professor and Chairman of the Department of Mathematics at the University of Santa Clara.

Professor H. L. Turrittin of the University of Minnesota was awarded a Fulbright Fellowship and was Visiting Professor at the University of Innsbruck, Austria, during 1955–1956.

Associate Professor W. R. Utz, Jr., of the University of Missouri is on leave of absence for the year 1956–57 and is spending the year at the Institute for Advanced Study.

Dr. Hermann von Schelling, formerly a biomathematician for the Naval Medical Research Laboratory, U. S. Submarine Base, New London, Connecticut, has a position as a consultant mathematician with the General Electric Company, Schenectady, New York.

Miss Ruth A. Wagner of the Rand Corporation has accepted a position as a mathematician with the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor L. E. Ward, Jr., of the University of Utah has accepted a position as a mathematician with the U. S. Naval Ordnance Test Station, China Lake, California.

Dr. Jack Warga of Consolidated Engineering Corporation, Pasadena, California, is employed as a mathematician by the ElectroData Corporation, Pasadena.

Mr. C. S. Wolfe of the University of Maryland has been appointed to an assistant professorship at the U. S. Naval Academy.

Mr. F. W. Wolock of Iona College has been appointed to an instructorship at St. John's University.

Professor E. J. Lowry, head of the Department of Mathematics of Hastings College, died on February 18, 1956.

Professor J. A. McKee of Beaver College died on February 7, 1956.

Dean Emeritus T. G. Rodgers of New Mexico Highlands University died on May 19, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 84 persons have been elected to membership by the Board of Governors on applications duly certified.

J. G. ALEX, A.B. (Sacramento S.C.) Math., University of California Radiation Lab.

L. J. ANDREWS, B.S. (U.C.L.A.) Project Engr., National Cash Register Co., Hawthorne, Calif.

G. C. ARROWSMITH, Student, Oregon State College.

A. C. BARBEAU, Student, University of Montreal.

R. J. BEIL, M.S. (Okla. A.&M.C.) Asst. Professor, Vanderbilt University.

JOAN BLEWITT, Student, Seton Hill College.

C. O. BLOOM, M.S. (Pennsylvania) Math. Teacher, Central School, Ellenville, N. Y.

D. M. BLOOM, Student, Columbia University.

M. L. BROUSSARD, Student, Southwestern Louisiana Institute.

PATRICIA A. BROWN, Student, Trinity University.

D. J. C. BURES, Student, Queen's University.

P. B. BURT, Student, University of Tennessee.

E. W. CAMPBELL, A.B. (San Jose S.C.) Math., University of California Radiation Lab.

F. J. CARTER, Student, LeMoyne College.

C. C. CONLEY, Student, Wayne University.

W. C. CREHL, M.A. in Ed. (Arizona S.C., Flagstaff) Math., Holloman Air Force Base, N. Mex.

C. F. DANIEL, Student, Oregon State College.

J. K. DIEI, Student, Ottawa University.

THADDEUS DILLON, M.S. (John Carroll) Instr., John Carroll University.

IRMA B. ESRIG, Student, City College of the City of New York.

J. C. EZYK, A.B. (St. Anselm's) Asst. Professor, St. Anselm's College.

J. B. FLANSBURG, B.S. (New Hampshire) Chairman, Department of Mathematics, St. John's School, Houston, Texas.

F. E. GINSBERG, Instr., Brooklyn Polytechnic Institute.

ALVA M. GIRDLEY, Student, Cardinal Stritch College.

W. G. GOELLER, Central Office Tech., New York Telephone Co., N. Y.

W. E. GOULD, Student, Rutgers University.

E. F. GRIFFIN, II, Student, Georgia Institute of Technology.

R. E. GRISWOLD, Student, Stanford University.

D. E. HARRIS, JR., B.S. (Southern Methodist) Staff Member, University of California, Los Alamos Scientific Lab., N. Mex.

J. B. HARVILL, JR., B.A. (North Texas S.C.) Teaching Fellow, North Texas State College.

F. E. HAZLETT, B.A. (U.C.L.A.) Computer Applications Engr., Librascope, Glendale, Calif.

H. G. HENRY, Student, University of Oregon.

W. H. HOLTER, B.S. (Franklin and Marshall) Applied Math., Atlantic Research Corp., Alexandria, Va.

E. L. HUBBARD, Employee, Wright Studio and Camera Shop, Midland, Mich.

R. W. HUNT, Student, West Texas State College.

MEYER JORDAN, Ph.D. (N.Y.U.) Lecturer, Brooklyn College.

D. S. KAHN, Student, Princeton University.

J. L. KATZ, M.S. (Union C.) Engr., General Electric Co., Schenectady, N. Y.

DOROTHEA A. KELLER, Student, University of Rochester.

J. W. KENNEDY, B.S. (Purdue) Wisconsin High School, University of Wisconsin.

W. J. LEE, M.A. (Carleton) Asst. Teacher, Hunter College High School, New York, N. Y.

GERALD LEIBOWITZ, Student, City College of the City of New York.

I. J. LIEBERMAN, M.A. (Southern California) Sr. Res. Scientist, Litton Industries, Beverly Hills, Calif.

MARSHALL LUBAN, Student, Yeshiva University.

H. B. LYNCH, Student, Colorado School of Mines.

KURT MAHLER, D.Sc. (Manchester) Professor, Manchester University, England.

R. A. MAIER, B.A. (U.C.L.A.) Sr. Lofting Math., North American Aviation, Santa Monica, Calif.

E. V. MARTIN, Student, University of Cincinnati.

J. G. MAXWELL, M.A. (Kent S.U.) Teacher, John Marshall High School, Cleveland, Ohio.

REV. L. W. MCCALL, S.A.C., M.S. (Marquette) Supervisor, Pius XI High School, Milwaukee, Wisc.

M. C. MCCORD, Student, University of Tennessee.

M. A. MCKIERNAN, Ph.D. (Illinois I.T.) Instr., Illinois Institute of Technology.

MADELINE D. MESSNER, M.A. (Montclair S.T.C.) Teacher, Abraham Clark High School, Roselle, N.J.

R. L. MILLS, M.S. (Southern Methodist) Sr. Computing Engr., North American Aviation, Los Angeles, Calif.

CAROLEN MONCURE, M.A. (Fisk) Crystal Springs, Mississippi.

R. A. NEUMANN, Student, Knox College.

R. L. NORRIS, Student, Eastern Kentucky State College

BETTY OPPENHEIM, B.A. (Brooklyn C.) Math., Brookhaven National Lab., Upton, N. Y.

MR. JINGYAL PAK, Student, Kent State University.

H. W. PALCZEWSKI, A.M. (Columbia) Instr., City College of the City of New York.

C. W. PECK, B.S. (Trinity U.) Stock Clerk, Southwest Research Institute, San Antonio, Texas.

P. L. PHIPPS, M.S. (M.I.T.) Math., Remington Rand Univac, St. Paul, Minn.

E. A. PORTER, B.A. (Michigan) Vice President and Chief Actuary, Manhattan Life Insurance Co., New York, N. Y.

PHILBURN RATOOSH, Ph.D. (Columbia) Asso. Professor of Psychology, Ohio State University.

ALFRED REICHENTHAL, M.A. (Columbia) Res. Engr., Hughes Aircraft Co., Los Angeles, Calif.

MICHAEL ROSEN, Student, Brandeis University.

L. P. ROSENBERRY, B.S. (Pennsylvania S.U.) Applied Science Representative, International Business Machines, Detroit, Mich.

D. G. RUSSELL, M.A. (Oklahoma) Exploitation Engr., Shell Oil Co., Odessa, Texas.

R. E. SCHLEA, M.A. (Bowling Green S.U.) Instr., Ohio Northern University.

HERMANN SCHMIDT, Ph.D. (Jena) Professor, Universität Wurzbürg, Germany.

JANET H. SCOTT, B.S. (Limestone) Grad. Asst., University of South Carolina.

J. J. SEGEDY, Student, Queens College.

C. P. SEGUIN, Student, University of Detroit.

INEZ SHAPIRO, Student, Northeastern University.

E. A. SHULMAN, Student, McGill University.

A. J. STRACCIOLINI, B.A. (Temple U.) Math., Ballistic Research Lab., Aberdeen, Md.

E. H. SWIFT, M.A. (Southern California) Chairman, Department of Mathematics, El Cerrito High School, Calif.

MYLES TIERNEY, Student, Brown University.

T. L. TRUEMAN, Student, Dartmouth College.

R. C. WARTH, Student, University of Rochester.

MR. CHIEN WENJEN, Ph.D. (U.C.L.A.) Asst. Professor, Texas Technological College.

T. L. WILLIAMS, M.S. (Utah) Instr., Idaho State College.

R. J. WINKELMAN, Student, University of Rochester.

B. A. YALE, Student, University of Buffalo.

THE APRIL MEETING OF THE KANSAS SECTION

The forty-first annual meeting of the Kansas Section of the Mathematical Association of America was held at the University of Wichita, Wichita, Kansas, on April 21, 1956, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 130 persons registered, including 40 members of the Association. Sessions of the Association were held on Saturday morning and afternoon. Professor Ferna Wrestler, Chairman, presided.

The following officers were elected for one-year terms: Chairman, Professor W. R. Scott, University of Kansas; Vice-Chairman, Professor L. E. Laird, Kansas State Teachers College, Emporia; Secretary-Treasurer, Miss Helen F. Kriegsman, Kansas State Teachers College, Pittsburg.

By invitation of the Executive Committee, Professor B. W. Jones of the University of Colorado delivered an address at the morning session entitled "Observations on English Schools and Colleges."

The following short papers were presented at the afternoon session:

1. *Quality control*, by Professor F. C. German, Kansas State Teachers College, Pittsburg, introduced by the Chairman.

As tolerances narrow and the demands of the general public become more exacting, the demand for a higher and more uniform quality and for more refined engineering procedures grows at an astounding rate. Statistical quality control is the tool used to maintain and improve this quality. The established distributions such as the normal, Poisson's approximation, or the log normal, have wide enough practical applicability to approximate a vast number of situations that arise. In other instances a distribution must be established empirically in which case its validity is restricted to the particular type of procedure being studied. It is an obligation imposed on the control operator that he keep in constant touch with the frontier of mathematical development as it pertains to his field.

2. *Divided differences and linear differential equations*, by Dr. A. H. Kruse, University of Kansas.

Let $\xi_x(m)=e^{mx}$ for all real x and complex m, let $m_0, m_1 \cdots m_n$ be complex numbers, and let $[m_0, \cdots, m_n; \xi_x]$ be the nth order divided difference of ξ_x at $(m_0, m_1, \cdots, m_n)$. If g is a real-valued continuous function on $a \leqq x \leqq b$, then $y(x)=\int_0^{x-a}[m_0, m_1, \cdots, m_n; \xi_u]g(x-u)du$ is that solution of the differential equation $(d/dx-m_0)(d/dx-m_1)\cdots(d/dx-m_n)y(x)=g(x)$ on $a \leqq x \leqq b$ such that $y^{(j)}(a)=0$ for $0 \leqq j \leqq n$. The divided difference in the integrand leads to the inequality $|y(x)| \leqq 1/n! \int_0^{x-a} u^n e^{ru} |g(x-u)|\,du$, where r is the maximum of the real parts of $m_0, m_1, \cdots, m_n$.

3. *Geodesics in metric spaces*, by Professor W. L. Stamey, Kansas State College.

A geodesic in a metric space is a locally congruent map of the real axis. In a finitely compact, convex metric space in which each metric segment admits a unique local prolongation beyond each end point, each two points are joined by a geodesic. The nature of the geodesics in some such spaces is discussed, and some general properties of geodesics are pointed out.

4. *A note on teaching statement problems*, by Professor Paul Eberhart, Washburn University.

On a test in intermediate algebra it was found that most of the students could not, or did not, complete correctly a statement of the form "Let $x=$———," as the first step in a statement

problem. In spite of this, correct numerical answers were generally obtained. It is then fairly obvious why more difficult problems cause trouble. However, with some extra stress on the formulation of problems much better results were obtained.

5. *Report of Board of Governors Meetings*, by Professor C. B. Read, University of Wichita.

Professor Read reported on meetings of the Board of Governors held in Ann Arbor on August 29, 1955, and in Houston on December 29, 1955. Topics mentioned included the continuation of the program of visiting lecturers; the employment register; policies relative to solicitation of funds; Carus Monographs; omitting publication of names of members attending meetings; revised policies relative to payment of expenses of sections and of section officers attending annual meetings; high school contests; time of annual meetings.

LAURA Z. GREENE, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The fortieth annual meeting of the Ohio Section of the Mathematical Association of America was held at Oberlin College, Oberlin, Ohio, on April 14, 1956. Professor R. R. Stoll, Chairman of the Section, presided at the morning and afternoon sessions.

One hundred twenty persons registered in attendance including eight-five members of the Association.

The following officers were elected for the coming year: Chairman, Professor P. V. Reichelderfer, Ohio State University; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor W. E. Deskins, Ohio State University; Program Committee: Chairman, Professor H. D. Lipsich, University of Cincinnati, Professor E. B. Leach, Case Institute of Technology, Professor Sam Selby, University of Akron.

Professor C. H. Yeaton submitted a tribute to the late Professor W. D. Cairns, which was unanimously adopted by the Section.

The following papers were presented:

1. *Sperner's lemma*, by Professor R. R. Stoll, Oberlin College (Chairman's address).

Several fundamental theorems of topology can be proved in an entirely elementary way using the following combinatorial lemma due to Sperner: If the simplex S with vertices $P_0, P_1, \cdots, P_n$ is divided simplicially and each vertex e of every subsimplex T is assigned a number $n(e)$ such that if e lies on the face of S with vertices $P_{i_0}, P_{i_1}, \cdots, P_{i_k}$ then $n(e)$ is one of $i_0, i_1, \cdots, i_k$, then there exists an odd number of T's having vertices labeled with $0, 1, \cdots, n$. This result was proved and then applied to prove Lebesgue's Covering Theorem and Brouwer's Fixed Point Theorem.

2. *A proof of the composite function theorem for matric functions*, by Mr. D. W. Robinson, Case Institute of Technology.

The composite function theorem for matric functions was stated and demonstrated by R. F. Rinehart in this MONTHLY, vol. 62, 1955. The proof contained in that paper is rather lengthy and tedious. The present note provides a simpler proof, which depends upon the combinatorial properties of matric functions.

3. *Some applications of stochastic processes*, by Professor W. R. Van Voorhis, Fenn College.

Most of the developments in the theory of stochastic processes have taken place during the past 25 years. In certain applications, dynamic probability models of physical processes have been developed and described mathematically. Professor Van Voorhis pointed out that these models, used extensively in the field of Operations Research, arise out of sets of postulates concerning probabilities associated with a stochastic variable and characterizing the process. Examples were given of the use of the theory in problems involving contagion, "birth and death", queueing, and replacement.

4. *The theory of verniers on non-uniform graduated scales*, by Professor P. M. Pepper, The Ohio State University.

Since the time of Pierre Vernier (1580–1637) the application of a small auxiliary scale along a uniformly graduated scale to obtain refined reading has been in widespread use. Because of the non-uniformity it has been thought impossible to obtain a vernier action for any scales excepting those with equal graduations. The author shows that a vernier mechanism can be constructed for any scale generated by a strictly monotone continuous function. Such a vernier mechanism will be operable with all of the accuracy of an ordinary vernier. The nearest prior approach to verniers of non-uniform scales was disclosed in a British patent issued in 1944, to an inventor who laid out a scale for a logarithmic function on a suitably chosen logarithmic spiral for which the scale graduations have equal intervals of arc between them. Suitably spaced vernier lines were then drawn on a transparent pivoted arm in such a way that these lines cut off equal arc length intervals of the original scale line suitable for forming a vernier at each stage. The present author shows that this restriction, namely, of adapting the scale line to the function scale, is inessential to the construction of a correct vernier. A number of possible applications of the vernier device to instruments are cited, and the possibility of constructing a five significant figure slide rule with verniers for all numerical and trigonometrical scales is demonstrated. The author holds U. S. Patent No. 2,564,227 which covers, among other things, the application of verniers to non-uniform scales.

5. *The role of the mathematician in a computing laboratory*, by Mr. E. F. Ormsby, Applied Science Division, International Business Machines Corporation.

The rapid growth of computing laboratories in the past few years has opened wide opportunity for employment of mathematicians. A typical laboratory will employ from 25 to 200 mathematicians or physicists. All levels of professional qualification are required, from the new college graduate with a B.A. to the post graduate mathematician. Today's universities must expand their horizons to meet this challenge.

6. *The geometric method in modern algebra*, by Professor Ernst Snapper, Miami University. (By invitation.)

The geometric method in modern algebra consists of the geometric use of the notion of n-dimensional vector space with an arbitrary field as scalar domain. A good example of this method is Witt's theory of quadratic forms. A quadratic form is a metric space. The space is an n-dimensional vector space R with a commutative field F as scalar domain, where F does not have characteristic 2. The metric of R is given by an inner product for the vectors of R. Witt's theorem concerning the extension of isometries between subspaces of a nondegenerate R to isometries of R was discussed. Many other examples of the geometric method can, of course, be given. The talk ended with a discussion of how the fact, that the notion of n-dimensional vector space has become so important in algebra, should influence our undergraduate teaching.

7. *Some current problems of teachers of mathematics in elementary and secondary schools*, by Dr. Herschel Grime, Supervisor of Mathematics, Cleveland Public Schools. (By invitation.)

Due to the critical shortage of elementary teachers and mathematics teachers in the secondary schools, it is extremely difficult to maintain the integrity of mathematics courses. By assigning our best teachers to college preparatory mathematics classes this danger can be lessened to some extent. Careful guidance by the secondary schools insures that pupils having ability in mathematics elect courses best fitted to their ability and probable future needs. The courses in college preparatory mathemtics in the secondary schools are very much in need of reform. The necessary revisions and additions should be made by committees including both college and secondary teachers.

8. *Three courses in the elements of modern mathematics*, by Professors Gaylord Merriman and H. D. Lipsich, University of Cincinnati.

The authors discussed three courses to be given at the University of Cincinnati, designed to meet the problems raised by (a) the desire of local high schools to modernize their curricula, (b) the increasing number of graduates of honors programs in mathematics from the local high schools, (c) the insistence of social scientists on a revision of traditional freshman mathematics curriculum. These courses are respectively: (a) Fundamental Modern Concepts in Mathematics, For Teachers. —Some of the modern concepts now fundamental in the structure of secondary school and college mathematics prior to calculus, such as, Boolean algebra, basic algebraic structures, finite and non-euclidean geometries; matrix solution of systems of linear equations; (b) Tutorial Readings in Modern Mathematics.—Continuation of analytic geometry and calculus, introduction to the algebra of sets, logic and postulational systems, groups and matrices; (c) Fundamentals of Modern Mathematics.—Basic modern concepts now important in mathematics, science, and some social sciences, such as, logic and postulational systems, sets, groups, matrices; selected topics in algebra, trigonometry, analytic geometry, statistics and the ideas of calculus.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the New Mexico College of Agriculture and Mechanic Arts, State College, New Mexico, on April 27 and 28, 1956. Professor R. L. Westhafer, Chairman of the Section, presided at the afternoon session on April 27 and also at the morning session on April 28.

There were 76 persons in attendance including 31 members.

The following officers were elected: Chairman, Professor R. B. Lyon, Arizona State College at Tempe; Vice-Chairman, Dr. R. C. Hildner, Sandia Corporation, Sandia Base, Albuquerque, New Mexico; Secretary-Treasurer, Mr. W. W. Mitchell, Jr., Phoenix College.

The following papers were presented.

1. *The role of mathematics in present day technology*, by Dr. S. M. Ulam, Los Alamos Scientific Laboratory. (By invitation).

2. *A study on generalized implication*, by Professor D. L. Webb, University of Arizona.

The operations of implication, logical sum, logical product, and negation were generalized to cover n-values. A study was then made to see which of sixteen methods of proof listed by J. Barkley Rosser in his *Logic for Mathematicians* held in the generalized systems.

3. *First removal time in a general stochastic epidemic*, by Professor P. W. M. John, University of New Mexico.

In Bailey's model for a general stochastic epidemic a population of size n is considered, of which r are susceptibles, s are infectives (in circulation), and the remaining $z=n-r-s$ are either removed or immune. The probability that in an element of time dt a susceptible will become infected is $\beta rsdt+o(dt)$, and that an infective will be removed is $\gamma sdt+o(dt)$. The number of infectives in circulation is thus a birth and death process with $\lambda_s=\beta rs$ and $\mu_s=\gamma_s$. An expression is obtained for the expected time elapsed until the first removal after the population has reached the above state; the result is illustrated in the case of a small intra-household epidemic.

4. *Two basic methods in extended analytic geometry*, by Professor R. S. Underwood, Texas Technological College.

This paper describes briefly a flexible system for plotting on a plane the locus of an equation in n variables. When X and Y are specified single-valued functions of the variables, each set of coordinates determines a point (X, Y). Two types of plotting rules are found effective in solving simultaneous equations. (1) Simple rules may be used with which are associated many readily-found loci. (2) In a specific problem an artificial rule designed to make one locus a curve is often feasible, since the remaining loci may then be found by a routine method.

5. *A metric space defined on the subsets of* (0, 1), by Professor B. C. Meyer, University of Arizona.

Nikodym defined a metric space on the (Lebesgue) measurable subsets of (0, 1). This paper was concerned with extending his space to include nonmeasurable sets and examining the properties of the resulting space.

6. *The potential with Legendre polynomials for bodies with axis of symmetry*, by Professor O. B. Ader, New Mexico College of Agriculture and Mechanic Arts.

If a material body has an axis of symmetry, it is generally possible to express the potential $U(P)$, at any point P on this axis, as a function of r, the distance of the point from a fixed origin. When $U(P)$ is expanded in powers of r, a theorem from potential theory gives the potential at any point of free space as a series of Legendre polynomials. The theorem is illustrated for the case of a straight line segment of constant linear density, and the series compared with the closed form of the solution.

7. *Points at which the osculating conic hyperosculates a plane curve*, by Professor Louis Child, New Mexico College of Agriculture and Mechanic Arts.

At a point P of a plane curve Γ, the osculating conic γ_5 exists and then exists uniquely if and only if at P

(a) The radius of curvature ρ and its first two derivatives ρ_1, ρ_2 with respect to the arc-length s exist and are finite, and $\rho\neq 0$.

The conic γ_5 contacts Γ at P in at least six points if and only if (a) is true and also

(b) $$q \equiv 36\rho_1 + 9\rho^2\rho_3 - 9\rho\rho_1\rho_2 + 4\rho_1^3 = 0.$$

The conic γ_5 contacts Γ at P in at least $n+6$ points if and only if (a) and (b) hold and also

(c) $$\frac{d^j q}{ds^j} = 0 \qquad 1 \leqq j \leqq n.$$

The derivatives (c) are the formal derivatives of (b), and (a), (b), (c) are all evaluated at P.

8. *More about the altitude quadric*, by Professor J. H. Butchart, Arizona State College at Flagstaff.

Professor Butchart showed that the quadric on which are the four altitudes of a tetrahedron is characterized by the properties that a section normal to any ruling is an equilateral hyperbola and that its asymptotic cone contains three mutually perpendicular generators. Analytically, a necessary and sufficient condition for a hyperboloid of one sheet to be an altitude quadric is that the sum of the coefficients of the square terms is zero. If the Monge point is equidistant from any three of the altitudes and orthocentric lines, it is equidistant from all, and these distances all lie in the equatorial plane of the quadric.

9. *Some mathematical problems investigated through computations on electric machines*, by Dr. S. M. Ulam, Los Alamos Scientific Laboratory. (By invitation).

As example of the heuristic value of calculations on fast computing machines, two problems were mentioned:

1. A study of the vibrating string with non-linear forces. The distribution of the energy among the various modes for the corresponding linear problem was studied as a function of time. A strange almost periodic behavior of the system was described. (A Los Alamos report by Fermi, Pasta, and Ulam in 1955 and a forthcoming one by Pasta and Ulam).

2. A study of a mathematical game by statistical experiments. (A paper by P. Stein and S. Ulam in the proceedings of the Louisiana Conference on Computors in 1955, and a forthcoming article by P. Walden in Volume 3 of Contributions to Theory of Games.)

10. *Making electrons count*, by Professor R. L. Westhafer, New Mexico College of Agriculture and Mechanic Arts.

Professor Westhafer showed a film on the Whirlwind Computer by courtesy of the Digital Computation Laboratory of the Massachusetts Institute of Technology.

11. *The down hill method of solving a polynomial $f(x)=0$, for real and imaginary roots*, by Dr. J. A. Ward, Holloman Air Force Base. (By invitation).

The author shows a numerical method of determining to any desired accuracy the real and imaginary roots of a polynomial equation with real coefficients. The technique of using this method on a small computer is also given.

12. *Some equations over complete Boolean algebras*, by Professor H. D. Sprinkle, University of Arizona, introduced by the Secretary.

The purpose of this paper is to establish as a main result necessary and sufficient conditions for β (β any ordinal number) equations in β unknowns to have a solution, not necessarily unique. These conditions will be stated in terms of the coefficients of the unknowns.

Other results—the existence of a minimal solution, a disjoint solution, and a unique solution—are stated for $\beta=2$, and the question as to generalizations is left open.

13. *Normal subgroups of direct products*, by Professor Ralph Crouch, New Mexico College of Agriculture and Mechanic Arts.

Let $\Sigma_m(H)$ be the monomial group of H over U where H is a group, U is a finite set. The normal subgroups of $\Sigma_m(H)$ are known (see: O. Ore, *Theory of Monomial Groups*, Trans. Amer. Math. Soc., vol. 51, 1942, pp. 15–64). The set of all monomial substitutions with permutation component in the alternating group is a subgroup $\Sigma_{A,m}(H)$ and the normal subgroups have been determined (see: R. B. Crouch, *Monomial Groups*, Trans. Amer. Math. Soc., vol. 80, pp. 187–215). If $m=2$, $\Sigma_{A,m}(H)$ is simply the direct product $H\times H$ and the structure theorems for the normal subgroups become a special case for the known theorems of Mazzoni (see: *Annali Della R. Scuola*

Normale Di Pisa, vol. 14) and Remak (see: *Journal fur die Reine und Angewändte Mathematik*, vol. 163, 1930.

14. *Intermediate algebra and an experiment*, by Professor F. C. Gentry, University of New Mexico.

Dr. Gentry pointed out that in view of the ever increasing number of students entering college who are unable to pass traditional courses in college algebra without remedial courses, it seems wise to find ways and means of removing these deficiencies as early as possible. He discussed an experiment with a combined course in intermediate and college algebra followed by a combined course in trigonometry and analytic geometry.

15. *An application of least squares*, by Mr. W. W. Mitchell, Jr., Phoenix College.

The method of least squares is used to determine a set f_i of n numbers and two other numbers, A and B, such that a set of n numbers, a_i, is approximated by Af_i, a set of n numbers, b_i, is approximated by Bf_i, and a third set of n numbers, c_i, is approximated by the set f_i, the approximations being the "best possible."

W. W. MITCHELL, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at San Marcos, Texas, on April 20–21, 1956, sponsored by Southwest Texas State Teachers College. Sessions were held Friday afternoon and Saturday morning, Professor M. E. Mullings presiding. At the banquet Friday night Dr. Don Cowan of Texas Christian University gave a speech entitled "Education for the Nuclear Age."

There were 108 persons in attendance and 78 of these were members.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Don Cude, Southwest Texas State Teachers College; Vice-Chairman, Professor Martin Wright, University of Houston; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The program consisted of the following papers:

1. *Algebraic properties of classes of functions*, by Mr. B. T. Goldbeck and Professor A. A. Grau, The University of Oklahoma, presented by Mr. Goldbeck.

Any function of a complex variable, $f(x)$, has a unique expansion in terms of functions of type (n, k) for each n, $k=0, 1, 2, \cdots, n-1$, where a function of type (n, k) is a function $f(x)$ having the property that $f(\omega x)=\omega^k f(x)$, ω being a primitive nth root of unity. Furthermore the classes of functions, (n, k), $k=0, 1, 2, \cdots, n-1$, form a ring isomorphic to the ring of the residue classes of integers, modulo n.

2. *The singularities of a function defined by a Taylor series with gaps*, by Mr. Alan Wilson, The Rice Institute, introduced by the Secretary.

J. M. Whittaker proved that if $f(z)=\sum_{n=1}^{\infty} a_{\lambda_n} z^{\lambda_n} (a_{\lambda_n} \neq 0)$ with radius of convergence one has exactly k isolated singularities (poles or essential) on $|z|=1$ and no other singularities there, then $\overline{\lim}_{n\to\infty} \lambda_n/n \leqq k$. S. Mandelbrojt presented without proof essentially the following improvement of Whittaker's result: let a be the length of the smallest arc of $|z|=1$ which contains the k singularities of $f(z)$ and put $\omega=[2\pi/a]$. Then for each integer $p (0 \leqq p < \omega)$ the sequence $\{\mu_n^{(p)}\}$ consisting

of those $\lambda_n \equiv p \pmod{\omega}$ satisfies $(1/\omega)\ \overline{\lim}_{n\to\infty} \mu_n^{(p)}/n \leqq k$. The present paper contains a proof of the latter theorem along lines suggested by Mandelbrojt.

3. *Euler's diagrams showing all 2^n-minterms*, by Professor R. E. Greenwood, The University of Texas.

The familiar Euler diagram has three intersecting circles showing all $2^3=8$ distinct intersections or regions into which objects can be classified as to whether or not they possess three different attributes. These distinct intersections or regions have been named as minterms by Serrell. It is possible to modify Euler's diagram for case of four (or a higher number of) attributes, but the symmetry disappears. However, Euler's circles may be replaced by a quadrant diagram in which symmetry is retained, particularly for an even number of attributes. Some properties of the quadrant diagram were discussed.

4. *Extremal elements of a convex cone of concave functions*, by Professor E. K. McLachlan, Baylor University.

Let D_n be a convex compact subset of Euclidean n-space, E^n, that contains a sphere of E^n. Let C_n be the set of all real valued nonnegative concave functions f on D_n. A function f is *concave* on D_n if, for x and y in D_n, $f(tx+(1-t)y) \geqq tf(x)+(1-t)f(y)$ for $0 \leqq t \leqq 1$. The set C_n is closed under addition and positive scalar multiplication and hence C_n is a convex cone. A function $f \neq 0$ is an *extremal element* of C_n if $f=f_1+f_2$ where f_1 and f_2 belong to C_n implies that f_1 (and f_2) is proportional to f. The extremal elements of C_1 were characterized. Partial results were given for $n \geqq 2$.

5. *Special series expansions of three Jacobian inverse elliptic functions*, by Mr. R. P. Kelisky, University of Texas.

Power series expansions in x of $\mathrm{sn}^{-1}(x, k)$, and $\mathrm{tn}^{-1}(x, k)$ are obtained in which the coefficients are expressed in terms of the Legendre polynomial $P_n(\lambda)$ where λ is a function of k and the complementary modulus k'. These infinite series are generalizations of known series for $\sin^{-1} x$, $\cos^{-1} x$, and Gregory's series for $\tan^{-1} x$. For $x=1$, series expansions of the complete elliptic integral K in terms of Legendre polynomials may be obtained. In obtaining these expansions, several known formulas relating the complete elliptic integrals E and K to the Legendre functions of orders $1/2$ and $-1/2$ are easily obtained.

6. *Concerning two theorems of R. L. Moore*, by Mr. J. R. Boyd, Chance Vought Aircraft, Dallas.

The primary purpose of this paper is to prove the theorem: *if a collection of fundamental sequences of regions has the hypothesis property, then a T-space satisfies R. L. Moore's Axioms* 0, 1(1), 1(2) *and* 1(3).

With point and region as undefined terms, a *T-space* is a collection of points and regions satisfying Axiom 0 and Theorem 2 as given in R. L. Moore's *Foundations of Point Set Theory*, New York, A.M.S., 1932. A *collection of fundamental sequences of regions* is the collection of regions satisfying Theorem 2 for each point of the space. The definition of *hypothesis property* is a modification of the hypothesis of Theorem 17′ of the same publication.

7. *Spirals and their whirl points and windings*, by Professor R. L. Moore, The University of Texas. (By invitation).

Definitions of the whirl points and windings of spirals in the plane were given.

8. *Growth of entire functions*, by Professor G. R. MacLane, The Rice Institute.

9. *A non-modular compact connected topological lattice*, by Professor D. E. Edmondson, Southern Methodist University.

An example is constructed of a compact connected topological lattice which is not modular. This is done by suitably defining an order on a portion of the unit cube.

10. *Geometric interpretations related to the mean value theorem*, by Dr. R. C. Osborn, The University of Texas.

In differential calculus, insufficient attention is given to geometric interpretations of the Mean Value Theorem (hereafter called MVT). The so-called generalized MVT may be obtained as a specialized form of the ordinary MVT by taking the function in parametric form—a fact which is not new, but which is insufficiently emphasized. The MVT may be taught more meaningfully if the generating function for the MVT is interpreted geometrically. Many generating functions are available, each with its own interpretation, and they lead to various forms of the MVT, some of which are unusual.

11. *Operational analysis of certain nonlinear systems*, by Professor W. T. Guy, Jr., The University of Texas.

Certain solutions of the differential equations of van der Pol and of Duffing are obtained using integral transform techniques under rather mild hypotheses regarding the form of the solution.

12. *On the contingent of G. Bouligand at the points of a line*, by Professor H. C. Parrish, North Texas State College.

If E is a planar set, the *contingent* $T(A, E)$ of E at a point A consists of the set of all rays (A, θ), of origin A in direction θ, such that every open sector $S(A; \theta)$, of vertex A and containing θ, contains a point of E. It is shown that if E is an arbitrary set in a half-plane K bounded by a line L, then at every limit point A of E in L not belonging to a set which is exhaustible and of measure 0, either $T(A, E)$ is the line L (both directions) or $T(A, E)$ is the half-plane K (all directions in K). Several partial converses of this result are given.

13. *Approximation of functions which have harmonic support*, by Mr. Guy Johnson, Jr., The Rice Institute.

Consider a real valued function $v(z)$ defined in a domain D of the complex z-plane. Define the class (hs), functions which have harmonic support in D, to be functions v such that for each $z_0 \in D$, $\exists h(z)$ harmonic and single valued in D satisfying $h(z_0) = V(z_0)$ and $h(z) \leqq v(z)$ in D. The following theorem is proved. If $v \in (hs)$ in D then $\exists$ a sequence of functions $\{u_n\}$, where each u_n is in (hs) and infinitely differentiable, such that $u_n \to v$ subuniformly in D.

14. *College mathematical training for students specializing in agriculture*, by Professor W. I. Layton, Stephen F. Austin State College.

The study included seventy-five colleges that offer training in agriculture. These colleges were of varying size and type in forty states. Requirements for agricultural engineers were excluded, but requirements were included for those preparing for Smith-Hughes work in agriculture and all others who major or minor in agriculture.

Recommendations of departments of agriculture indicated that there is a growing interest in requiring agricultural mathematics instead of college algebra and/or trigonometry in the curricula for college students specializing in agriculture.

15. *Texas Tech's special committee for the advancement of the study of the sciences*, by Professor J. W. Lindsay, Texas Technological College.

At Texas Technological College, the Special Committee for the Advancement of the Study of the Sciences was appointed to study the current scientific manpower shortage in the U. S. The Committee's recommendations will help determine the College's role in stimulating more interest in the study of mathematics and the natural sciences and improving the quality of science teaching in these subjects. The shortage of teachers and mathematicians was illustrated.

16. *Revisions in the 1956 engineering curricula at Texas Tech*, by Professor E. R. Heineman, Texas Technological College.

Entering engineering students will take a placement test in mathematics. Those making unsatisfactory scores will take a non-credit course in intermediate algebra. Those making satisfactory scores (roughly, the highest 40%) will take a dual course (algebra-trigonometry). For the first half of the fall semester, they will take a three-credit course in algebra meeting six times a week. In the second half, trigonometry will be given in the same way. The spring semester's work consists of a three-credit course in analytic geometry, followed by a three-credit calculus course. Two more three-credit courses in calculus and one in differential equations follow.

17. *Types of mathematics used in aeronautical engineering*, by Dr. Nat Edmonson, Jr., Convair, Fort Worth.

The kinds of mathematical analysis in daily use in the research and engineering design groups of a large airplane company engaged in the development and production of weapons systems were described by means of slides and examples. It was pointed out that such fields of mathematics as ordinary and partial differential equations, matrices and determinants, tensor analysis, integral equations, statistical analysis, and numerical analysis are necessary tools in the development of the highly complex modern airplane.

C. R. SHERER, *Secretary*

THE APRIL MEETING OF THE UPPER NEW YORK STATE SECTION

The twelfth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Alfred University, Alfred, New York, on April 28, 1956. The Chairman of the Section, Professor C. E. Rhodes of Alfred University, presided at the morning session, and the Vice-Chairman, Professor A. J. Coleman of the University of Toronto, presided at the afternoon session.

There were 97 persons in attendance, including 63 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor A. J. Coleman, University of Toronto; Vice-Chairman, Col. W. W. Bessell, Jr., U. S. Military Academy; Secretary, Professor N. G. Gunderson, University of Rochester.

The morning session was devoted to a panel discussion on "The Secondary School—College Transition." Professor A. J. Coleman, University of Toronto, was moderator, and the members of the panel were: Professor H. M. Gehman, University of Buffalo; Mr. C. W. Pflaum, John Marshall High School, Roches-

ter; Dean R. S. Fisk, School of Education, University of Buffalo; Professor C. W. Munshower, Colgate University.

The following papers were presented at the afternoon session:

1. *Mathematics in industry*, by Dr. T. C. Fry, Bell Telephone Laboratories. (By Invitation.)

The substance of this talk appeared under the title, *Mathematics as a profession today in industry*, in this MONTHLY, vol. 63, 1956, pp. 71–80.

2. *A fundamental limit and its applications*, by Professor M. J. Pascual, Siena College.

An elementary geometric proof that the limit of the ratio of arc length to chord length, as each approach zero, is one for smooth curves, was given. This limit was then used to establish several formulas of elementary calculus, which are usually derived by more involved means.

3. *Some special determinants*, by Professor F. R. Olson, University of Buffalo.

Let $x^{(n)}=x(x-1)(x-2)\cdots(x-n+1)$. The Stirling numbers of the first and second kind, S_j^n and T_j^n, are defined as the coefficients in the expansions

$$x^{(n)}=\sum_{j=1}^{n} S_j^n x^j \quad \text{and} \quad x^n=\sum_{j=1}^{n} T_j^n x^{(j)}.$$

It was shown that

$$\sum_{j=1}^{n} S_j^i T_k^{n-1+j}=\begin{cases}0, & k<i\\ i^{n-1}, & k=i,\end{cases}$$

and as a consequence that the determinant

$$\left|T_k^{n-1+j}\right|=(n!)^{n-1}, \qquad j,k=1,2,\cdots,n.$$

This parallels the known result

$$\left|S_k^{n-1+j}\right|=[(n-1)!]^n.$$

4. *False solutions in elementary extremum problems*, by Professor C. S. Ogilvy, Hamilton College.

Examples from elementary calculus were used to illustrate "false" solutions of two types. (1) A maximum is sought and only a minimum is delivered by the machinery of differentiation, because of an unsuitable choice of independent variable. (2) Two "solutions" appear, but one is extraneous, even though both may lie within the admissible range of the problem. The false solution here usually has an interesting geometric interpretation. In general, if two problems requiring different answers lead at some stage to the same equation, then this equation must deliver at least two solutions, one of which will be extraneous to each problem.

5. *Mathematics for statisticians*, by Professor J. E. Freund, Virginia Polytechnic Institute. (By invitation.)

The speaker, who is Chairman of the Section on Training of the American Statistical Association, discussed some of the problems of planning graduate curricula in statistics with special reference to the type of mathematics courses to be included in such programs. He also illustrated briefly what topics in modern algebra and advanced calculus are essential for an M.S. degree in statistics.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-third annual meeting of the Indiana Section of the Mathematical Association of America was held jointly with the Indiana Council of Teachers of Mathematics at Wabash College, Crawfordsville, Indiana, on May 5, 1956. Professor R. O. Virts of Central High School and the Purdue University Center at Fort Wayne, Chairman of the Section, presided at the general session and at the sectional meetings of the Association which followed.

Professor K. O. May, Chairman of the Mathematics Department of Carleton College, guest speaker for the hour lecture, addressed the general session on the topic, "A Modern Introduction to Mathematics."

There were 133 in attendance including 51 members of the Association.

The following officers were elected: Chairman, Professor C. H. Brumfiel, Ball State Teachers College; Vice-Chairman, Professor C. B. Gass, DePauw University; Secretary-Treasurer, Professor J. C. Polley, Wabash College.

Professor P. D. Edwards, Chairman of the Committee on Awards, reported that five Association medals had been awarded during the year for high mathematical achievement in the Indiana Science Talent Search.

The following short papers were presented:

1. *Non-linear recurrence relations for classical orthogonal polynomials*, by Professor M. S. Webster, Purdue University.

Some of the known and some new non-linear relations involving Hermite, generalized Laguerre, and ultraspherical polynomials were given. A typical proof was presented showing that such a relation essentially characterizes polynomials of the type given.

2. *A geometry experiment*, by Professor C. F. Brumfiel, Ball State Teachers College, and Professor M. E. Shanks, Purdue University, presented by Professor Shanks.

The purpose of this paper was to call the attention of college teachers to shortcomings in high school geometry texts and to encourage the writing of better texts. Defects in Euclid's geometry have long been known but current texts do not remove these flaws. In fact, they add errors of their own. Besides being incomplete, these texts are lacking in logic and give as postulates what are really theorems or definitions. A critique of Euclid's postulates was given and Hilbert's postulates were described briefly. A modified version of Hilbert's postulates was described, a version which is being taught at the Burris High School in Muncie, Indiana in a course for which the textual material has been prepared by the authors of the paper.

3. *Discontinuities for the classroom*, by Professor H. L. Hunzeker, DePauw University.

Graphical illustrations of discontinuous functions for classroom use were obtained from considering common machines as mathematical systems.

4. *An operator identity for* $(D^2+a^2)y=f(x)$, by Professor R. E. MacKenzie, Indiana University, introduced by Professor H. E. Wolfe.

A solution of the differential equation $(D^2+a^2)y=f(x)$ by quadratures was effected by means of an operator identity using trigonometric functions.

5. *On a function defined by means of an infinite radical*, by Professor G. N. Wollan and Mr. D. M. Mesner, Purdue University Center, Fort Wayne, presented by Professor Wollan.

If $0<x\leqq 1$, then x has a non-terminating binary representation $x=a_1a_2\cdots a_n\cdots$. Let $\alpha_n=(-1)^{a_n}$ and $f_n(x)=\sqrt{k+\alpha_1\sqrt{k+\alpha_2\sqrt{k+\cdots+\alpha_n\sqrt{k}}}}$, $n=1, 2, 3, \cdots$, and let I denote the interval $0<x\leqq 1$. When $k>2+\sqrt{2}$, then $\lim_{n\to\infty} f_n(x)$ exists and is real for every x in I and this defines a function $f(x)=\lim_{n\to\infty} f_n(x)$. This function is discontinuous at every point in I having a terminating binary representation and is continuous elsewhere in I. The function is not monotone in any sub-interval of I and yet has a derivative equal to zero at each point of a dense set of points of I and has a left derivative equal to zero at every point of discontinuity.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Maryland, on May 5, 1956. Professor F. E. Johnston, Chairman of the Section, presided at the morning and afternoon sessions.

There were 99 persons in attendance, including 75 members of the Association.

The following officers were elected to serve for a period of one year: Chairman Professor R. C. Yates, College of William and Mary; Vice-Chairmen, Professor J. E. Freund, Virginia Polytechnic Institute and Professor D. B. Lloyd, District of Columbia Teachers College; Secretary, Professor R. P. Bailey, U. S. Naval Academy; Treasurer, Professor T. W. Moore, U. S. Naval Academy.

The following papers were presented.

1. *The coloring of maps*, by Professor R. W. Rector, U. S. Naval Academy.

The speaker presented a historical survey of the four color problem with particular attention to the method of Birkhoff and Lewis for constrained chromatic polynomials for the n-ring. Certain steps were indicated leading to the solution of the constrained chromatic polynomials of the seven-ring.

2. *Aliquot sequences (preliminary report)*, by Mr. G. A. Paxson, U. S. Army, Fort Meade, Maryland.

Let $S(n)=\sum_{d|n} d$, $0<d<n$. $S(n)$ is called the sum of the *aliquot parts* of n. Put $S^0(n)=n$, $S^1(n)=S(n)$, and $S^{k+1}(n)=S(S^k(n))$. The infinite sequence n, $S(n)$, $S^2(n)$, $\cdots$ is called the *aliquot sequence* with leader n.

A sequence is *eventually periodic* if any term of the sequence recurs.

Conjecture: Every sequence is eventually periodic. Every sequence examined for this property, except a few, displayed it upon computation of a sufficient number of terms. The few involve terms of great size, rendering continued hand computation impracticable. All sequences with leaders $n\leqq 10{,}000$ are being computed until discovered to be periodic or until a term with more than 20 digits is reached. Computation is being done on the IBM Type 650 Magnetic Drum Data-Processing Machine at the Watson Scientific Computing Laboratory, 612 West 116th Street, New York, New York.

3. *A graphical method of determining the instantaneous center of zero acceleration of a rigid body having plane motion*, by Professor J. P. Hoyt, U. S. Naval Academy.

If the acceleration of one point on a rigid body having plane motion is known together with the angular velocity and angular acceleration of lines in the body, the instantaneous center of zero acceleration is found as the intersection of two circles. A property of these circles is considered for a special case. Certain general results and applications are discussed.

4. *A set of predictor and corrector formulas for the numerical solution of ordinary differential equations*, by Dr. C. H. Frick and Mr. J. E. Mulligan, Jr., U. S. Naval Proving Ground, Dahlgren, Virginia, presented by Mr. Mulligan.

The use of

$$hy_n' = 12y_{n-2} - 12y_{n-1} + h(5y_{n-2}' + 8y_{n-1}') + \tfrac{1}{2}h^4y^{iv} \tag{1}$$

or

$$hy_n' = 33y_{n-3} + 24y_{n-2} - 57y_{n-1} + h(10y_{n-3}' + 57y_{n-2}' + 24y_{n-1}') + \frac{33}{180}h^6y^{vi} \tag{2}$$

for predicting the slope, and

$$y_n = 0.2y_{n-2} + 0.8y_{n-1} + h(0.8y_{n-1}' + 0.4y_n') - \frac{1}{30}h^4y^{iv} \tag{3}$$

or

$$y_n = 0.1y_{n-3} + 1.8y_{n-2} - 0.9y_{n-1} + h(0.9y_{n-2}' + 1.8y_{n-1}' + 0.3y_n') - \frac{1}{200}h^6y^{vi} \tag{4}$$

for estimating y_n was discussed. The estimated value of y_n can be entered into the differential equation $y'=f(x, y)$ and corrected values of y' and y computed.

5. *A Carnegie Foundation project in modern mathematics for college sophomores (preliminary report)*, by Professor R. V. Andree, University of Oklahoma and Haverford College.

The Carnegie Corporation is sponsoring an experiment in the inclusion of some mathematics other than calculus in the college sophomore course and the inclusion of examples from fields other than physics. It is suggested that about three-fourths of the academic year be spent on calculus, utilizing the (polynomial) calculus contained in many freshman courses. The remaining one-fourth academic year will be spent on material mainly from abstract algebra.

6. *The national status of mathematics contests in secondary schools*, by Professor D. B. Lloyd, District of Columbia Teachers College.

The purposes of this study were two-fold: (1) to survey the mathematics contests now being given in high schools of the United States; and (2) to ascertain the prevailing sentiment for or against the launching of a nation-wide contest. Some 125 questionnaires were distributed to all states through state representatives and others prominent in mathematics education. Thirty-six contests were discovered in 26 states, involving some 3900 schools. As to the desirability of a single nation-wide contest, the enormity of the problem, and the impracticability of attempting to adapt a single contest to the heterogeneous local needs, were cited from some areas. However, some 22 states reported no contests at all,—thus indicating fertile virgin soil for introducing a well-planned contest program under the auspices of a national body. The study revealed the feasibility of a nationally sponsored contest, provided that interference with certain local contests was carefully avoided.

7. *Commutativity of finite matrices*, by Dr. Olga Taussky-Todd, National Bureau of Standards, introduced by the Secretary.

Various generalizations of the concept of commutativity of a pair A, B of finite matrices are being discussed. One such generalization consists in assuming that all polynomials $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$ where α_i, β_i are the eigenvalues of A and B taken in a special order. A more recent generalization assumes that only all linear combinations $\lambda A+\mu B$ have as eigenvalues $\lambda\alpha_i+\mu\beta_i$. A further generalization is obtained by studying the matrices in the pencil $\lambda A+\mu B$ which have multiple characteristic roots. The vanishing of the higher commutators of A and B also plays a big role. If in particular $B=A^*$ (the complex conjugate and transpose of A) then the vanishing of the higher commutators sheds light on the nature of A itself.

R. P. BAILEY, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-ninth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Utah, Salt Lake City, Utah, on May 4 and 5, 1956. Professor C. R. Wylie, Jr., Chairman of the Section, presided at all three sessions.

There were 77 persons registered for the meeting, including 48 members of the Association.

Officers elected at the meeting for 1956–1957 were: Chairman, Professor R. R. Gutzman, Colorado School of Mines; Vice-Chairman, Professor J. S. Leech, Colorado College; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *Connectedness in partially ordered sets*, by Professor L. E. Ward, Jr., University of Utah.

A partially ordered space is a triple $(X, T, <)$ where (X, T) is a topological space, $(X, <)$ is a partially ordered set, and there obtains some harmonious relation between T and $<$. Toward the end of characterizing various partially ordered spaces theoretically, results of the following type are of interest: order hypothesis→topological conclusion. Sample theorem: if the set of $a \leqq x$ is compact for each $x \in X$, if the partial order is continuous and dense, and if each pair of points of X have a common predecessor, then X is connected.

2. *Convergence of sequences of linear fractional transformations*, by Professor W. J. Thron, University of Colorado.

Let $\{T_n(z)\}$ be a sequence of nonsingular linear fractional transformations. Let Z be the set in the complex plane, such that $\{T_n(z)\}$ converges for all $z \in Z$. Finally, let $T(z)$ be the function to which the sequence converges on Z. Possible domains of convergence Z and corresponding limit function $T(z)$ are determined. Among other results it is proved that $T(z)$ must either be a constant for all $z \in Z$, or a constant for all but one value of Z, or a linear fractional transformation (in this case Z is the whole plane).

3. *Orthonormal functions used for the approximate solution of integro-differential equations*, by Professor F. M. Stein, Colorado Agricultural and Mechanical College.

Under certain conditions the integro-differential equation $U(u) \equiv L(u) - \int_a^b h(x, t)u(t)dt = f(x)$

with boundary conditions $U_i(u)=0 (i=1, 2, \cdots, m)$ has a unique solution of the form $u(x) = \int_a^b H(x, t)f(t)dt$, where $H(x, t)$ is the Green's function of the problem. Let $S_n(x)$ be a sum of functions of the complete set $\phi_i(x)$ which are orthonormal with respect to the weight function $\rho(x)$ and which satisfy the boundary conditions. Under the criterion that $\int_a^b |f(x) - U[S_n(x)]|^r dx$ shall be least for $r>0$, this paper examines the sufficient condition that $S_n(x)$ as well as its first m derivatives be uniformly convergent to the corresponding derivative of the solution $u(x)$ as n becomes infinite.

4. *The Dirichlet series transformation*, by Professor J. R. Britton, University of Colorado.

Let $F(t)$ be a complex valued function defined for nonnegative integral values of t. If a is a positive constant, the Dirichlet series $\sum_{t=0}^{\infty} a^{-st}F(t)$ is the Dirichlet series transform, $D\{F(t)\}$. The series converges, for example, if $F(t)=\theta(k^t)$, $k>0$, $t>t_0$. Simple properties of the transform were developed, in particular, the convolution theorem

$$[D\{F(t)\}][D\{G(t)\}] = D\left\{\sum_{t=0}^{\tau} F(t-\tau)G(\tau)\right\}.$$

Application was made to "summation" equations of the convulution type $\sum_{t=0}^{\tau} F(t-\tau)G(\tau) = H(t)$, where $G(t)$ and $H(t)$ are given functions, and $F(t)$ is to be found.

5. *Series solution of linear differential equations*, by Mr. C. A. Grimm, South Dakota School of Mines and Technology.

By the use of the exponential shift formula,

$$P(\Delta)e^{mt}f(t) = e^{mt}P(\Delta + m)f(t)$$

and the substitution $x-a=e^t$, one easily arrives at the formula $P(\Delta)(x-a)^m = P(m)(x-a)^m$. Then by writing (if possible) a linear differential equation in the form $f(\Delta)y+(x-a)^k g(\Delta)y=0$ with the assumed solution $y=\sum_{n=0}^{\infty} c_n(x-a)^{\lambda+n}$, $c_0\neq 0$, the recurrence relation, standardly found by the method of Frobenius, is readily obtained.

6. *Homogeneous continua*, by Professor C. E. Burgess, University of Utah.

A brief history of results on homogeneous continua was given, and some related unsolved problems were mentioned.

7. *Finite geometries and difference sets*, by Professor B. W. Jones, University of Colorado.

Cyclic finite geometries were defined and it was shown that any such geometry leads to a perfect difference set. Conversely, any perfect difference set leads to a finite geometry. One or two generalizations were indicated.

8. *Decimal expansions*, by Dr. Bodo Volkmann, University of Mainz, Visiting Professor, University of Utah. (By invitation.)

Borel proved in 1909 that almost all real numbers are normal, *i.e.*, their decimal expansion involves each of the possible digits with the same asymptotic frequency. A number of generalizations of Borel's theorem to the digit distribution of non-normal numbers were obtained by Besicovitch, Knichal, Eggleston, and the speaker. These theorems describe certain sets of real numbers in terms of their Hausdorff (or fractional) dimension. Among the unsolved problems on decimal expansions are questions such as whether numbers like e, π, or $\sqrt{2}$ are normal in any scale, and what conclusions can be drawn from the normality of a real number in a given scale about its digits distribution in any other scale.

9. *Some constants associated with the Riemann zeta-function*, by Dr. W. E. Briggs, University of Colorado.

The Riemann zeta-function has the representation

$$\zeta(s) = \frac{1}{s-1} + \sum_{0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (s-1)^n$$

where

$$\gamma_n = \lim_{n \to \infty} \left[\sum_{t=1}^{N} \frac{\log^n t}{n} - \int_1^N \frac{\log^n x}{x} dx \right]$$

and satisfies the functional equation $\zeta(s) = 2^s \pi^{s-1} \sin s\pi \Gamma(1-s)\zeta(1-s)/2$. By letting $s = 1-2m$, where m is a positive integer, it can be shown that infinitely many γ_n are positive, and infinitely many are negative. From a representation of γ_n as an infinite series, it can be shown that $\gamma_n = O\{(n/2e)^n\}$. Using the fact that $\zeta(s) - 1/(s-1)$ is an entire function of order one, it can be shown that if $\epsilon > 0$, then $n^{-\epsilon n} < |\gamma_n| < n^{\epsilon n}$ is true for infinitely many n.

10. *Production delay period*, by Professor E. A. Davis, University of Utah.

In the production of economic goods a "production delay period" elapses between the time factors are joined in a firm's productive mechanism and the time resulting output materializes. The delay period, "time", acts somewhat as a productive factor whose cost is expressed through the interest rate. The author considered adjustments of a firm attempting to maximize "profit" integrals, strict competition being assumed, under the assumption that the production delay period for the firm was: a) of zero length; b) of fixed length; c) of length subject to entrepreneur control. The latter two cases lead naturally to a theory involving producer expectations.

11. *Summing of series with missing terms*, by Dr. H. J. Fletcher, Brigham Young University.

A method is presented to sum many series of the form $\sum a_n b_n$ where the series $\sum a_n$ is known and b_n is a periodic function of n. The method consists of expressing b_n as a finite sum of sines and cosines, and summing the corresponding trigonometric series. In particular, if the series $\sum a_n \sin nx$ can be summed to a known function, then the corresponding sum which has terms missing periodically can be summed to a known function.

12. *Fitting empirical equations to data from small orifice fluid meters*, by Professor S. R. Smith, University of Wyoming.

Two empirical equations, $C = R/(a+bR)$ and $C = d + k/\sqrt{R}$ were fitted to the orifice meter data (C, R data was determined by Professor Eric Lindahl, University of Wyoming). Water was the fluid used in each case. C is the coefficient of discharge of the meter and R its corresponding Reynolds number, both dimensionless.

13. *Errors in linear systems*, by Professor C. A. Hutchinson, University of Colorado. (Presented by title.)

14. *Coordination of college and high school mathematics topics*, by Professor O. M. Rasmussen, University of Denver.

The author expressed the need for coordination of mathematics programs in the high schools and colleges, especially with respect to topics of modern mathematics. This need was shown by a brief review of some of the present activity concerning modern topics in mathematics. It was emphasized that coordination should take place on national, regional, and local levels with each individual aiding in any manner possible.

F. M. CARPENTER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Spring, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS

KENTUCKY, Berea College, Berea, April 20, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College of William and Mary, Williamsburg, Virginia, December 1, 1956.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, March 23, 1957.

MINNESOTA, Concordia College, Moorhead, October 6, 1956.

MISSOURI, Southeast Missouri State College, Cape Girardeau, Spring, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 27, 1957.

NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.

NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.

OHIO

OKLAHOMA, Oklahoma City University, October 26, 1956.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, Muhlenberg College, Allentown, Pennsylvania, November 24, 1956.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, Spring, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May, 1957.

THE CARUS MATHEMATICAL MONOGRAPHS

A new Carus Monograph has been published in 1956. The title of this Monograph is:

MONOGRAPH 11: *Irrational Numbers,* by Ivan Niven.

Each member of the Association may purchase one copy of this Carus Monograph at the special price of $1.75. Orders should be addressed to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies of Monograph 11 for members and copies for nonmembers may be purchased at $3.00 from John Wiley and Sons, 440 Fourth Avenue, New York 16, New York.

Professor Niven's Monograph on *Irrational Numbers* provides an exposition of some central results on irrational, transcendental, and normal numbers. There is a complete treatment by elementary methods of the irrationality of the exponential, logarithmic, and trigonometric functions with rational arguments. The approximation of irrational numbers by rationals, up to such results as the best possible approximation of Hurwitz, is also given with elementary techniques. The last third of the monograph treats normal and transcendental numbers, including the transcendence of π and its generalization in the Lindemann theorem, and the Gelfond-Schneider theorem.

INTERMEDIATE ANALYSIS

AN INTRODUCTION TO THE THEORY OF THE FUNCTIONS OF ONE REAL VARIABLE

By John M. H. Olmsted, ***University of Minnesota***

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CAMPANOLOGICAL GROUPS

T. J. FLETCHER, Sir John Cass College

The art of change ringing has always been considered highly mathematical, but little has been said about the precise mathematical notions involved. This may be because the art has been restricted almost entirely to English-speaking countries, and so has not attracted the attentions of algebraists on the continent of Europe. This paper contains examples of some of the methods used by bell-ringers, and shows that they are decompositions of a symmetric group into Lagrange cosets. In consequence they provide useful practical illustrations of a topic which most textbooks treat in an entirely abstract fashion. We also see that a knowledge of the elementary notions of group theory might have enabled campanologists to construct peals without the tedious empirical methods which seem to have been the practice in the past.

The problem which a team of ringers undertakes is to produce all the permutations (changes) on a set of bells, proceeding according to certain rules. The highest bell, number one, is called the *Treble*, and the lowest is called the *Tenor*. When the bells ring down the scale, from Treble to Tenor, they are said to be in *rounds*. The rules to which a peal must conform are:

i) the peal must begin and end in rounds,
ii) no bell may move more than one place between successive changes,
iii) no bell may lie still for more than two successive changes.

(The last rule is relaxed occasionally.)

A peal may be composed of any number of bells up to twelve, and as it takes about 28 hours to ring all the changes on eight bells alone, the ringers cannot hope to ring all the possible changes if the number of bells is larger. Then they restrict attention to ringing those changes which are regarded as the most musical.

The six changes on three bells may be rung as follows:

1 2 3
2 1 3
2 3 1
3 2 1
3 1 2
1 3 2
1 2 3

They could equally well be rung in the reverse order, but only these two ways conform to the rules. The course of the Treble is particularly simple—it *hunts up* and then *hunts down*. The other bells follow a similar *hunting course* but naturally the phase is different.

Notation. We will denote changes as ringers do by numbering the bells from Treble to Tenor and giving the order in which they ring in the change. The symbols which result could be regarded as the elements of a symmetric group, but it is more strictly logical to regard the transitions between changes as constituting the group, and we will denote these by the familiar cycle notation. Thus the transition

$$\begin{bmatrix} 213654 \\ 514263 \end{bmatrix}$$

will be denoted by the symbol (145)(36). (Rule ii of course prohibits the carrying out of this transition in one move.) The notation means that the bell in first place moves to fourth place, and the bell in fourth place moves to fifth place, and the bell in fifth place moves to first place; while the bells in third and sixth place change over. The cycle symbol is an operator which transforms the first change to the second, and such operators multiply together according to well-known rules. We will use capital letters to denote these transitions. Initially they will denote transitions between adjacent changes, but later on they will be used to denote transitions from one change to another which may occur elsewhere in the peal. Because of Rule ii, capital letters denoting transitions between adjacent changes are operators of period two. That is to say they are their own inverses. It is important to avoid confusing the change 1342 with the change generated by applying the operator (1342) to rounds, because this is 2413.

Singles. A peal on four bells is called *Singles*. If four bells hunt they produce the eight changes in the first block of Table I. A further move of a similar kind would restore rounds, so, in ringers' language, "the bell that the treble takes from lead makes second place and leads again; and the other bells dodge at the back-stroke lead of the treble." The mysteries of "hand-stroke" and "back-stroke" do not concern mathematicians; they arise because a bell swings two ways and successive strokes have to be rung with a different action. The meaning of "dodge" should be sufficiently clear from the table.

TABLE I

SINGLES		
1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3

The process of hunting consists of employing the two operators $A = (12)(34)$ and $B = (23)$ alternately. These generate a group of order eight, the first eight

changes. In general hunting on n bells generates a group of order $2n$. $AB = (1342)$ and this element has period four. It is convenient to call these eight elements the *hunting sub-group* and to denote it by $\mathfrak{H}$. The last of these changes is 1324, and the peal continues by following this with the irregular move $C = (34)$, instead of employing B which would produce rounds. The second block therefore consists of the hunting sub-group premultiplied by $B^{-1}C = (243)$. This can be denoted by $(243)\mathfrak{H}$. After it has been rung the irregular move C has to be performed again, and the third block consists of $(243)^2\mathfrak{H}$. When the last change is followed once more by C we return to rounds because $(243)^3 = I$.

This method of ringing therefore displays the decomposition into cosets:

$$\mathfrak{G} = \mathfrak{H} + (243)\mathfrak{H} + (243)^2\mathfrak{H}.$$

Other methods are possible on four bells but their structure is not quite so clear as in this example.

Doubles. A peal on five bells is called *Doubles*. For five bells or more it is usually necessary to have a conductor who calls *bobs* and *singles* as required. That is to say he does the multiplications which change from one coset to the next—or to put the matter more correctly he does some of them, because the ringers engaged on any particular peal proceed under a set of standing orders which ensure that a number of the transitions from one coset to the next are done automatically. The first five bell method which we will consider is called *Grandsire Doubles*. The hunting sub-group of ten changes is generated by the two operations $A = (23)(45)$ and $B = (12)(34)$. In Grandsire Doubles the number of changes which the ringers can ring without further instructions is increased to 30 by using the further operation $C = (12)(45)$. In practice this irregular move is performed at the very beginning, before hunting commences; and the peal begins as shown in Table II.

TABLE II

GRANDSIRE DOUBLES

1 2 3 4 5		
2 1 3 5 4	2 1 5 4 3	2 1 4 3 5
2 3 1 4 5	2 5 1 3 4	2 4 1 5 3
3 2 4 1 5	5 2 3 1 4	4 2 5 1 3
3 4 2 5 1	5 3 2 4 1	4 5 2 3 1
4 3 5 2 1	3 5 4 2 1	5 4 3 2 1
4 5 3 1 2	3 4 5 1 2	5 3 4 1 2
5 4 1 3 2	4 3 1 5 2	3 5 1 4 2
5 1 4 2 3	4 1 3 2 5	3 1 5 2 4
1 5 2 4 3	1 4 2 3 5	1 3 2 5 4
1 2 5 3 4	1 2 4 5 3	1 2 3 4 5

These 30 changes are called *plain course*. It is easy to see why the ringers prefer to begin with the operation C. If this is done the return of the Treble to the lead can be taken as the cue to perform the irregular operation C again,

whereas if they commence with hunting no such clear indication is given them. It is now necessary to introduce some means whereby the ringers may pass from one plain course to another; and to do this the conductor calls bobs and singles. A *single* is a change which interrupts the regular work of an even number of bells, and a *bob* is a change which interrupts the regular work of an odd number. The distinction is again not one which a mathematician would have chosen. In some cases a bob or a single alters the parity of the changes which are being rung, whereas the other does not, but this is not the case in general.

Grandsire uses the single

$$\begin{bmatrix} 15432 \\ 15423 \end{bmatrix} = (45),$$

and Rule iii is relaxed at this point in this method of ringing. To ring a full peal of Grandsire, four bobs and two singles are necessary. Grandsire Doubles is a decomposition of the symmetric group on five symbols into twelve cosets of ten, masked by a pre-multiplication by C. Without the pre-multiplication plain course may be written:

$$\mathfrak{C} + (354)\mathfrak{C} + (354)^2\mathfrak{C}.$$

$(354) = B^{-1}C$ and is obviously of period three. This leads one to suspect that it is possible to generate a longer plain course by employing some move different from C. This can be done, and Plain Bob employs $D = (23)$ instead of C. $B^{-1}D = (1342)$, and so plain course consists of 40 changes, and the entire peal can be rung with fewer calls.

One of the most pleasing methods from a mathematical point of view is *Stedman's Doubles*. This method was invented round about 1640, and it displays a striking knowledge of decomposition into cosets very nearly one hundred years before Lagrange was born. Troyte [1] describes the method as follows: "Three bells go through the three bell changes while the other bells dodge behind; at the completion of each six changes one bell coming down from behind to take its part in the changes, and one going up behind to take its part in the dodging." In the Grandsire the work of the Treble is different from that of the other bells, as he pursues a hunting course the whole of the time, whereas the other bells modify their hunting course now and again. In Stedman's method the work of every bell is the same, and the symmetric group is decomposed into cosets of six. By tradition the method begins in the middle of a set, so the decomposition into cosets is pre-multiplied by three operations, or alternatively it is carried out from the reference order 32415 instead of 12345. The peal is shown in Table III. It will be noticed that the sets of six changes are rung as in our very first example, but taken forwards and backwards alternately.

The method requires two singles, and in this case they change the parity of the permutations being rung. The singles are indicated on the table by the letter S. They interchange some of the members of the cosets in which they occur, and they also decompose the whole group in another way—into two cosets of 60.

All the members of the alternating sub-group occur in the peal before and after the singles, while the members of the other coset occur between them.

TABLE III

STEDMAN'S DOUBLES

	1	2	3	4	5														
	2	1	3	5	4														
	2	3	1	4	5														
	3	2	4	1	5	5	2	3	4	1	3	1	4	2	5	5	1	3	4	2
	2	3	4	5	1	5	3	2	·	·	1	3	4	·	·	5	3	1	·	·
	2	4	3	1	5	3	5	2	·	·	1	4	3	·	·	3	5	1	·	·
	4	2	3	5	1	3	2	5	·	·	4	1	3	·	·	3	1	5	·	·
	4	3	2	1	5	2	3	5	·	·	4	3	1	·	·	1	3	5	·	·
	3	4	2	5	1	2	5	3	1	4	3	4	1	5	2	1	5	3	2	4
	4	3	5	2	1	5	2	1	3	4	4	3	5	1	2	5	1	2	3	4
S	4	5	3	1	2	2	5	1	·	·	*S* 4	5	3	2	1	1	5	2	·	·
	5	4	3	1	2	2	1	5	·	·	5	4	3	2	1	1	2	5	·	·
	5	3	4	2	1	1	2	5	·	·	5	3	4	1	2	2	1	5	·	·
	3	5	4	1	2	1	5	2	·	·	3	5	4	2	1	2	5	1	·	·
	3	4	5	2	1	5	1	2	4	3	3	4	5	1	2	5	2	1	4	3
	4	3	2	5	1	1	5	4	2	3	4	3	1	5	2	2	5	4	1	3
	3	4	2	1	5	1	4	5	·	·	3	4	1	·	·	2	4	5	·	·
	3	2	4	5	1	4	1	5	·	·	3	1	4	·	·	4	2	5	·	·
	2	3	4	1	5	4	5	1	·	·	1	3	4	·	·	4	5	2	·	·
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	1	2	4	3	5	5	3	1	·	·	2	1	4	·	·	5	3	2	·	·
	1	4	2	5	3	3	5	1	·	·	2	4	1	·	·	3	5	2	·	·
	4	1	2	3	5	3	1	5	·	·	4	2	1	·	·	3	2	5	·	·
	4	2	1	5	3	1	3	5	2	4	4	1	2	5	3	2	3	5	1	4
	2	4	5	1	3	3	1	2	5	4	1	4	5	2	3	3	2	1	5	4
	4	2	5	3	1	3	2	1	·	·	4	1	5	·	·	3	1	2	·	·
	4	5	2	1	3	2	3	1	·	·	4	5	1	·	·	1	3	2	·	·
	5	4	2	3	1	2	1	3	·	·	5	4	1	·	·	1	2	3	4	5
	5	2	4	1	3	1	2	3	·	·	5	1	4	·	·					
	2	5	4	3	1	1	3	2	4	5	1	5	4	3	2					

The number of changes generated by some particular method can be calculated quite easily. Consider for example *Plain Bob Major*. (A peal on eight bells is called *Major*.) The complete 40,320 changes on eight bells were first rung at Leeds, Kent, in 1761, by relays of men who took 27 hours to complete the task. Plain Bob Major generates the hunting sub-group by means of $A=(12)(34)(56)(78)$ and $B=(23)(45)(67)$, and when the Treble returns to the lead, rounds are avoided by using $C=(34)(56)(78)$ which means the second coset commences with $B^{-1}C=(2468753)$ which is of period 7. Plain course therefore consists of 7

cosets of 16, which is 116 changes. This method uses the bob (23)(56)(78) and the single (56)(78), and it is interesting to work out how many more changes these will generate and where they must be used.

Treble Bob. The *Treble Bob* methods of ringing are more complicated than the plain methods because the bells do not follow a simple hunting course, but dodge on their way up and down. This means that a "lead" (the set of changes between successive appearances of the Treble as the first bell) is twice as long as in the plain method, and that the changes of a lead do not possess the group property, being usually a complex generated by three operations employed according to some scheme such as $ABACABAC\cdots$. Even here group properties are very apparent in the tables which are given in books on campanology. The move at the end of a lead in Treble Bob is arranged so that when the Treble returns to the front the other $n-1$ bells are cyclically permuted; in this way plain course consists of as many leads as possible.

We have so far pointed out only that the three blocks of Table I are a decomposition into left-hand cosets. If we read the table by lines we see a decomposition into right-hand cosets using a sub-group of order three. This can also be seen in Table II although it is a little obscured by the irregular way in which the peal begins, and it is a general feature of Plain and Treble Bob methods.

As we have indicated earlier most of the methods of campanology are empirical, *ad hoc* methods devised some hundreds of years ago long before the invention of formal group theory, but the subject has not been without its great theorists. One of the finest of these was W. H. Thompson, who was not a practicing ringer but a civil servant in India who made a hobby of solving change ringing problems. As far as can be ascertained he was not a mathematician and his writings [3] do not suggest that he realized that the calculations which he was performing were in fact pieces of group theory. He was concerned in particular with the following problem: "Is it possible to ring all the changes on seven bells (Triples) using the Grandsire method with only plain and bob leads?" He devised his own notation for this problem, and it bears striking analogies to the notation invented by the celebrated Irish mathematician Hamilton for another problem [4] a few years earlier.

Because the work in each lead is perfectly regular (being the hunting subgroup), the analysis of a complicated peal is greatly simplified by considering only the "lead-ends." Putting this in a way which seems more natural to mathematicians we need only consider the first member of each coset. The effect of a plain lead in Grandsire Triples is to send bells which have just rung a lead starting with rounds into the next lead in the order 1253746. We may therefore write $P=(34675)$. The effect of a bob lead is to send bells which have just rung a lead starting with rounds into the next lead in the order 1752634, and so we put $B=(247)(365)$.

There are 7! changes in Grandsire Triples and each lead contains 14. The total number of leads is therefore 6!/2 and the initial members of the cosets form the alternating group on six symbols. Denoting these six symbols by 2, 3, 4,

5, 6, 7 (because bell number 1 is always in front at the beginning and end of a lead) the question which Thompson was discussing was: "Will P and B generate the alternating group on six symbols, and furthermore generate it unicursally?"

The answer was "No," but the interest is in Thompson's methods. Using the notion of lead-ends he drew up a chart in such a way that the unicursal generation of the group became a matter of the unicursal description of a polytope whose edges represented the leads of the peal. The problem is remarkably like the problem of the game of visiting the vertices of an icosahedron which Hamilton had discussed a few years before [4]. Hamilton's problem is by far the simpler of the two, and it is not exactly the same as he is concerned with visiting the vertices of the polytope and Thompson is concerned with traversing the edges, but the two employ an algebraic notation which is almost identical. The solution of Hamilton's game presumably gives a peal on six bells (or rather a *touch*, because the 20 vertices of the icosahedron correspond to only 40 leads, whereas 60 leads are necessary to ring a full peal in Minor).

After constructing the polytope Thompson argues as follows. If the result of following x with a plain ending is the same as the result of following y with a bob, then clearly the leads x and y must be treated in the same way, that is both must be plain or both must be bob. Now if $xP=yB$ then $y=xPB^{-1}$. Therefore if x is plain or bob, $x(PB^{-1})$ must be the same. But the operator PB^{-1} is of period five, and so all the leads may be split up into sets of five which must all be treated the same way. Thompson called these Q-sets, and they are of course cosets in the strict meaning of the term.

If all the 360 leads are plain they are split up into 72 *round blocks* since P is of period five. On the other hand if they are all bobs they are split up into 120 round blocks since B is of period three. The peal can be rung if we may start at one of these extremes and by bobbing or plaining suitable Q-sets convert the whole 360 leads into one round block. This would be denoted by a chain of letters of the form *PPBPBPPPB etc*, where the whole product was equal to the identity but no sub-section was. Thompson showed this to be impossible by proving that bobbing or plaining any Q-set always results in the loss or gain of an even number of round blocks. Hence the number of round blocks, which is 72 or 120 to start with, can never be reduced to one. The beauty of the proof is marred by the fact that the stage showing that the number of round blocks lost or gained is always even is carried out by a long and tedious process of enumeration. But it is very difficult to see any means by which this could have been avoided. The enumeration of the cosets of a group of large order is inevitably tedious, and modern processes do not seem to offer any way of reducing Thompson's labours to any marked extent.

The study of campanology therefore shows how bell-ringers acquired a considerable amount of empirical knowledge about groups long before the theoretical basis was established by mathematicians, and it also provides instructive examples of the abstract theorems. The main problem of campanology is a problem of Group Theory to which no practical solution seems to have been

found even yet: "Given a particular set of elements of a group, will they generate the whole group, and will they generate it unicursally?"

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POLYNOMIALS DEFINED BY GENERATING RELATIONS

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1. Introduction. Many familiar sets of polynomials $p_n(z)$ have generating relations of the form

$$\sum_{n=0}^{\infty} w^n p_n(z) = A(w)\psi(zg(w)), \tag{1}$$

where $A(w)$, $\psi(t)$, $g(w)$ are power series. These sets include such general classes as Appell polynomials ($g(w)=w$, $\psi(t)=e^t$), Sheffer polynomials† ($\psi(t)=e^t$), Brenke polynomials [1] ($g(w)=w$), and such particular sets as certain constant multiples of the Jacobi,‡ Laguerre and Hermite polynomials. The generating relation (1) appears not to have been discussed before in full generality. Here we shall characterize the polynomials defined by (1) by means of both an explicit formula and a recursion relation; the explicit formula reduces to known results for Appell and Brenke polynomials, but otherwise seems to be new. We also give a brief discussion of a simple special subclass that is interestingly related to the class of Appell polynomials.

2. An explicit formula. Let $A(w)$, $\psi(t)$, $g(w)$ be the formal power series

$$A(w) = \sum_{n=0}^{\infty} a_n w^n, \qquad a_0 \neq 0; \tag{2}$$

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† Sheffer [5], [6] calls them sets of type zero; Steffensen [9], [10] calls them poweroids; the Bateman Manuscript Project [2] calls them sets of generalized Appell type.

‡ The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ have a generating relation of the form (1) only when $\alpha-\beta=-1$, 0, or 1 (Smith [8]), but $P_n^{(\alpha,\beta)}(x+1)$ do have a generating relation (1) (Erdélyi [3], p. 264).

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n t^n, \tag{3}$$

with $\psi(t)$ not a polynomial;

$$g(w) = \sum_{n=0}^{\infty} b_n w^n. \tag{4}$$

We are actually concerned only with the algebraic relations among coefficients implied by relations among formal power series. In establishing them we may operate with the power series as if they were convergent; for a discussion of this point see Sheffer [5]. In all the applications of which we are aware, the power series have positive radii of convergence.

THEOREM 1. *In* (1), *the* $p_n(z)$ *are polynomials if and only if* $b_0=0$; *if this is the case,* $p_n(z)$ *is of degree* n *if and only if* $b_1\gamma_n\neq 0$; *moreover,*

$$p_n(z) = \sum_{j=0}^{n} z^j\gamma_j \sum a_{k_0}b_{k_1}b_{k_2}\cdots b_{k_j}, \tag{5}$$

where the inner summation extends over all sets of $j+1$ *non-negative integers* $\{k\}$ *such that* $k_0+k_1+\cdots+k_j=n$.

In particular, if $g(w)=w$ this reduces to Brenke's formula [1]

$$p_n(z) = \sum_{j=0}^{n} z^j\gamma_j a_{n-j},$$

which in turn reduces to the well-known formula for Appell polynomials,

$$p_n(z) = \sum_{j=0}^{n} z^j a_{n-j}/j!, \tag{6}$$

if in addition $\psi(t)=e^t$. In the general case with $\psi(t)=e^t$ we have the following formula for Sheffer polynomials:

$$p_n(z) = \sum_{j=0}^{n} \frac{z^j}{j!} \sum a_{k_0}b_{k_1}\cdots b_{k_j}, \qquad k_0+k_1+\cdots+k_j=n.$$

This could also be deduced from Sheffer's integral expression for these polynomials [7].

Let (1) hold with A, ψ, g from (2), (3), (4), and write $K(z, w)=A(w)\psi(zg(w))$. Let $p_n(z)=\sum_{j=0}^{\infty} p_{nj}z^j$. Then

$$K(z, w) = \sum_{n=0}^{\infty} w^n \sum_{j=0}^{\infty} p_{nj}z^j = \sum_{j=0}^{\infty} z^j \sum_{n=0}^{\infty} p_{nj}w^n,$$

so that

$$k! \sum_{n=0}^{\infty} p_{nj} w^n = \left. \frac{\partial^j K}{\partial z^j} \right|_{z=0}.$$

We also have

$$\frac{\partial^j K}{\partial z^j} = A(w)\{g(w)\}^j \psi^{(j)}(zg(w)),$$

and so

$$\sum_{n=0}^{\infty} p_{nj} w^n = \gamma_j A(w)\{g(w)\}^j. \tag{7}$$

Now if $g(0)=0$, the right-hand side has a power series beginning with $\gamma_j a_0 b_1^j w^j$, so that $p_{nj}=0$ at least for $n \leqq j$ and so $p_n(z)$ is a polynomial of degree n at most. On the other hand, if $g(0)=b_0 \neq 0$, it follows from (7) that, for an infinity of integers j, $p_{0j}=\gamma_j a_0 b_0^j$ does not vanish, so that $p_0(z)$ is not a polynomial.

Now suppose that $g(0)=0$ and write (7) in the form

$$\sum_{n=j}^{\infty} p_{nj} w^n = \gamma_j A(w)\{g(w)\}^j. \tag{8}$$

Picking out the coefficient of w^k on both sides, we have $p_{jj}=\gamma_j a_0 b_1^j$, which shows that $p_j(z)$ is of degree j if and only if $\gamma_j b_1 \neq 0$. Similarly, p_{nj} is the coefficient of w^n in the product of the $j+1$ power series on the right-hand side of (8), and so

$$p_{nj} = \gamma_j \sum a_{k_0} b_{k_1} \cdots b_{k_j}, \qquad k_0 + k_1 + \cdots + k_j = n.$$

This completes the proof of Theorem 1.

With any polynomial $p_n(z)=c_0 z^n + c_1 z^{n-1} + \cdots + c_n$ we can associate another polynomial $p_n^*(z)=c_0+c_1 z+ \cdots + c_n z^n = z^n p_n(1/z)$; we call this the reversed polynomial, for lack of a generally accepted term. For the special case of Brenke polynomials we have the following result.

THEOREM 2. *If $p_n(z)$ are Brenke polynomials with $\gamma_0 \neq 0$ and $A(w)$ not a polynomial, $p_n^*(z)$ are also Brenke polynomials.*

In fact, we have

$$A(w)\psi(zw) = \sum p_n(z) w^n,$$

and writing $1/z$ for z and then zw for w we obtain

$$A(zw)\psi(w) = \sum p_n^*(z) w^n.$$

3. Characterization by properties of the generating function. We next obtain a characterization of polynomial sets which are generated by relations of the form (1). Given any sequence $\{p_n(z)\}$ with p_n of degree n, we form the series

$$\sum_{n=0}^{\infty} w^n p_n(z) = K(z, w).$$

Our first characterization is in terms of properties of K.

THEOREM 3. *A necessary and sufficient condition that* $K(z, w) = A(w)\psi(zg(w))$ *with* A, ψ, g *as in* (2), (3), (4) *and* $b_0 = 0$, $b_1 = 1$, *is that there exist power series*

$$\alpha(w) = \sum_{n=0}^{\infty} \alpha_n w^n, \qquad \beta(w) = 1 + \sum_{n=1}^{\infty} \beta_n w^n, \tag{9}$$

such that

$$K_2(z, w) = \alpha(w)K(z, w) + zw^{-1}\beta(w)K_1(z, w), \tag{10}$$

where the subscripts denote partial derivatives.

If $K(z, w)$ has the specified form we can verify by differentiation that (10) holds with $\alpha(w) = A'(w)/A(w)$, $\beta(w) = wg'(w)/g(w)$.

Conversely, if (10) holds, choose $A(w)$ of the form (2) so that $A'(w)/A(w) = \alpha(w)$. Set $H(z, w) = K(z, w)/A(w)$. Since $K_1(z, w) = A(w)H_1(z, w)$ and $K_2(z, w) = A'(w)H(z, w) + A(w)H_2(z, w)$, (10) implies that

$$H_2(z, w) = zw^{-1}\beta(w)H_1(z, w). \tag{11}$$

Now choose $g(w)$ of the form (4) so that $\beta(w) = wg'(w)/g(w)$, and so that $g(0) = 0$, $g'(0) = 1$. Then (11) states that

$$H_2(z, w) = \{zg'(w)/g(w)\}H_1(z, w).$$

Replacing z by $z/g(w)$, we obtain

$$H_2\{z/g(w), w\} - zg'(w)\{g(w)\}^{-2}H_1\{z/g(w), w\} = 0,$$

which is to say

$$\frac{\partial}{\partial w} H\{z/g(w), w\} = 0.$$

Thus $H\{z/g(w), w\}$ is independent of w, and so has the form $\psi(z)$ for some ψ. In other words,

$$K(z, w) = A(w)H(z, w) = A(w)\psi(zg(w)).$$

4. Characterization by a recursion relation. Theorem 3 can be expressed more directly in terms of the $p_n(z)$ as follows.

THEOREM 4. *If* $p_n(z)$ *is a polynomial of degree* n, *the sequence* $\{p_n(z)\}$ *has a generating relation of the form* (1) *with* A, ψ, g *as in* (2), (3), (4) *and* $b_0 = 0$, $b_1 = 1$, *if and only if there are two sequences of numbers* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *such that*

$$z^{n+1}\frac{d}{dz}\{z^{-n}p_n(z)\} + U_0[p_{n-1}(z)] + U_1[p_{n-2}(z)] + \cdots + U_{n-1}[p_0(z)] = 0, \tag{12}$$

where U_k is the linear differential operator

$$U_k = \alpha_k + z\beta_{k+1}\frac{d}{dz}.$$

More explicitly, (12) states that

$$\begin{aligned}
z^2[p_1(z)/z]' &= -\alpha_0 p_0(z),\\
z^3[p_2(z)/z^2]' &= -[\alpha_1 p_0(z) + \alpha_0 p_1(z)] + \beta_1 z p_1'(z),\\
z^4[p_3(z)/z^3]' &= -[\alpha_2 p_0(z) + \alpha_1 p_1(z) + \alpha_0 p_2(z)] + z[\beta_2 p_1'(z) + \beta_1 p_2'(z)],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
z^{n+1}[p_n(z)/z^n]' &= -\sum_{k=0}^{n-1} a_{n-k-1}p_k(z) + z\sum_{k=1}^{n-1}\beta_{n-k}p_k'(z).
\end{aligned} \tag{13}$$

Relation (12) can be used to calculate the $p_n(z)$ recursively. Theorem 4 gives no information about the form of ψ and so cannot be used to characterize Appell polynomials. The Brenke polynomials, for which $g(w)=w$, are characterized by

$$z^{n+1}[z^{-n}p_n(z)]' = -\sum_{k=0}^{n-1}\alpha_{n-k-1}p_k(z),$$

since in this case all the β's, except β_0, are 0.

To establish Theorem 4 we merely observe that (13) is equivalent to (11). In fact, if $K(z, w)=\sum p_n(z)w^n$, (11) is

$$\sum p_n(z)nw^{n-1} = \alpha(w)\sum p_n(z)w^n + zw^{-1}\beta(w)\sum p_n'(z)w^n, \tag{14}$$

and (13) follows if we express $\alpha(w)$ and $\beta(w)$ by (9) and equate coefficients of w^n. Conversely, if (13) holds, (14) follows, and so $K(z, w)$ satisfies (11).

5. Analogues of Appell polynomials. The Appell polynomials arise from the Sheffer polynomials, defined by

$$A(w)e^{zg(w)} = \sum_{n=0}^{\infty} p_n(z)w^n \tag{15}$$

by the specialization $g(w)=w$. We note that the left-hand side of (15) can be written as $H\{a(w)+zg(w)\}$, where $H(t)=e^t$.

THEOREM 5. *The only functions H with $H(0)=1$ such that $K(z, w)=H\{a(w)+zb(w)\}$ is of the form* (1) *are $H(t)=e^t$ and $H(t)=(1-t)^{-\lambda}$.*

To prove this, we obtain another necessary condition on $K(z, w)$ for it to have the form $A(w)\psi(zg(w))$. By differentiation, we have

$$\begin{aligned}
K(z, w) &= A(w)\psi(zg(w)),\\
K_1(z, w) &= A(w)g(w)\psi'(zg(w)),\\
K_{11}(z, w) &= A(w)\{g(w)\}^2\psi''(zg(w)),
\end{aligned}$$

so that, for $z=0$, $KK_{11}=c\{K_1\}^2$, where c is a constant. When $K(z, w)=H\{a(w)+zb(w)\}$, this leads to the equation

$$H(a(w))H''(a(w))\{b(w)\}^2 = c\{H'(a(w)\}^2\{b(w)\}^2.$$

The only solutions of this with $H(0)=1$ are $H(t)=e^{\alpha t}$ and $H(t)=(1+\alpha t)^{-\lambda}$. The number α can be absorbed into the functions $a(w)$ and $b(w)$, so that we may take $\alpha=1$ or $\alpha=-1$, respectively.

Thus, in addition to the Sheffer polynomial sets, which correspond to the first choice $H(t)=e^t$, one is naturally led to consider a second class of sets whose generating functions are $\{1-a(w)-zb(w)\}^{-\lambda}$. Putting this into the standard form (1) we have

$$A(w)\{1 - zg(w)\}^{-\lambda} = \sum_{n=0}^{\infty} p_n(z)w^n.$$

The special case $g(w)=w$ yields polynomial sets which are in many ways analogous to the Appell polynomials. They are defined by

$$A(w)(1 - zw)^{-\lambda} = \sum_{n=0}^{\infty} P_n(z)w^n \tag{16}$$

and satisfy the mixed equation

$$P'_{n+1}(z) = \lambda P_n(z) + zP'_n(z).$$

To see the connection between these polynomials and those of Appell, set $Q_n(z)=P_n(z/\lambda)$. Then

$$Q'_{n+1}(z) = Q_n(z) + (z/\lambda)Q'_n(z),$$

so that $\lim_{\lambda\to\infty} Q_n$ is the Appell polynomial of degree n associated with the function $A(w)$.

Some other interconnections may be mentioned. For example, formula (5) gives the explicit formula

$$P_n(z) = \sum_{j=0}^{n} a_{n-j}\binom{j+\lambda-1}{\lambda-1}z^j,$$

where $A(w)=\sum a_k w^k$. The reversed set, $P_n^*(z)=z^nP_n(1/z)$, is recognized to be exactly the nth Cesàro partial sum of order $\lambda-1$ (if $\lambda>0$) of the power series for $A(w)$ (see [3], p. 96). In particular, when $\lambda=1$, $P_n^*(z)=\sum_{k=0}^{n} a_k z^k$, so that $P_n(z)=\sum_{j=0}^{n} a_{n-j}z^j$ (compare (6)). In this case, there is a further formal connection with the Appell polynomials. The polynomials P_n obey the relation

$$\frac{P_{n+1}(z) - P_{n+1}(0)}{z} = P_n(z),$$

which is the special case $r=0$ of the formula

$$\frac{P_{n+1}(z) - P_{n+1}(rz)}{z - rz} = P_n(z). \tag{17}$$

When $r=1$, (17) becomes $P'_{n+1}(z)=P_n(z)$, the recursion relation for Appell polynomials. Thus the polynomial sets characterized by (17) form a family which contains both the Appell sets and the sets $\{P_n\}$ of (16) for $\lambda=1$. They have been investigated in detail by Sharma and Chak [4]; the recursion (17) was suggested to us by T. S. Motzkin several years ago. These polynomials are also a subclass of the general polynomials defined by (1); indeed, they are the Brenke polynomials with

$$\psi(w) = \sum_{n=0}^{\infty} \frac{w^n(r-1)^n}{(r^n-1)(r^{n-1}-1)\cdots(r-1)}.$$

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A NOTE ON QUASI-IDEMPOTENT MATRICES*

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1. Introduction. Let A be a square matrix of order p over the field C of complex numbers, and $F(x)$ a polynomial matrix of the same order p over the

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ring $C[x]$ of polynomials in the indeterminate x. Huff [1] defines A as quasi-idempotent if $A^r = F(r)$ for all positive integral r.

He states that by substituting fractional and negative values for r in $F(r)$ we can obtain the corresponding fractional and negative powers of A. It is, therefore, of some interest to know what matrices can possibly be quasi-idempotent.

Two necessary and sufficient conditions to determine a quasi-idempotent matrix are given by Huff. A few more results about such matrices are obtained in this note.

2. Definitions. Huff has given some definitions which are stated below for the sake of convenience.

(i). If A is q.i.p. (read quasi-idempotent) such that $A^r = F(r)$, then $F(x)$ is called "an exponential polynomial matrix" associated with A.

(ii). Clearly $F(x) = \sum A_k x^k$ where A_k are square matrices over C of the same order as that of A. Thus $F(x)$ is a polynomial in the indeterminate x with *matric* coefficients. If we replace x by a square matrix X we get a *matric* polynomial with *matric* coefficients. Huff calls the latter the *image* of the first. There is a natural isomorphism between $(C[X])_p$ and $C_p[X]$. Thus if $F(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_k x^k$, then the image $F^*(X)$ of $F(x)$ is given by

$$F^*(X) = A_0 + A_1 X + A_2 X^2 + \cdots + A_k X^k,$$

and $F^*(X)$ is called "the exponential matric polynomial" associated with A.

(iii). If A is q.i.p. a necessary and sufficient condition proved by Huff is $A(A-I)^k = 0$ for some positive integer k, I being the unit matrix of the same order as A. Further, if n is such that $A(A-I)^n \neq 0$ but $A(A-I)^{n+1} = 0$, then n is called the *index* of the q.i.p. matrix A.

(iv). If A is q.i.p., Huff has shown that there is an "identity" matrix B_0 for A such that $AB_0 = B_0A = A$ and $(A - B_0)^{n+1} = 0$. In case A is non-singular, B_0 equals I.

3. Uniqueness and existence.

THEOREM 1. *Given a q.i.p. matrix A, there is a unique exponential polynomial matrix $F(x)$ associated with A, and conversely, given an exponential polynomial matrix $F(x)$, there is a unique q.i.p. matrix A associated with $F(x)$.*

Proof. For a given q.i.p. matrix A the existence of a unique $F^*(X)$ is stated by Huff. Now there is a natural isomorphism between $(C[X])_p$ and $C_p[X]$. Hence uniqueness of $F^*(X)$ implies the uniqueness of $F(x)$.

The converse is obvious. For, if possible, let there be two q.i.p. matrices A and B associated with the same exponential polynomial matrix $F(x)$. Hence $A^r = F(r) = B^r$ for all positive integers r. In particular, when r equals 1 we get $A = B$.

4. The minimum equation of a Q.I.P. matrix.

THEOREM 2. *The minimum equation of every q.i.p. matrix has only two possible distinct roots viz.* 0 *and* 1.

Proof. Let A be q.i.p. Then for some positive integer k we have:

$$A(A - I)^k = 0.$$

Thus A satisfies $\lambda(\lambda-1)^k=0$. Hence if $m(\lambda)=0$ is the minimum equation of A then $m(\lambda)\,|\,\lambda(\lambda-1)^k$. Hence $m(\lambda)$ can have only two distinct factors λ and $(\lambda-1)$. Hence the possible distinct roots of $m(\lambda)=0$ are zero and unity.

THEOREM 3. *The zero root of the minimum equation of a q.i.p. matrix of index n is at most simple and the multiplicity of the unit root is* $(n+1)$.

Proof. If A is q.i.p. and n its index then by the definition of the index $A(A-I)^n\neq 0$ and $A(A-I)^{n+1}=0$. So that $m(\lambda)\nmid\lambda(\lambda-1)^n$ but $m(\lambda)\,|\,\lambda(\lambda-1)^{n+1}$. Hence $m(\lambda)$ can at best have one nonrepeated factor λ and the factor $(\lambda-1)$ is repeated $(n+1)$ times. This proves the theorem.

It has been observed by Huff that if a q.i.p. matrix A is nonsingular then its identity matrix B_0 is the universal identity matrix I. This also follows from the above theorem. For if A is nonsingular and q.i.p. then the minimum equation $m(\lambda)=0$ of A has no zero root. Hence $(\lambda-1)^{n+1}=0$ corresponds to $(A-I)^{n+1}=0$. But also $(A-B_0)^{n+1}=0$. So $B_0=I$.

5. Transforms of Q.I.P. matrices.

THEOREM 4. *Let A be q.i.p. of index n and order p and let* $B=HAH^{-1}$ *where H is a* $p\times p$ *nonsingular matrix over C. Then B is q.i.p. of index n.*

Proof. If $f(x)$ is any polynomial with coefficients in C then $f(B)=f(HAH^{-1})=Hf(A)H^{-1}$. Put $f(x)=x(x-1)^{n+1}$. Since A is q.i.p. so $f(A)=0$. Hence $f(B)=0$. Then B is q.i.p. Again as $A(A-I)^n\neq 0$ we have $B(B-I)^n=HA(A-I)^nH^{-1}\neq 0$ since H is not a divisor of zero. Hence the index of B is n.

In the above theorem H is nonsingular. This theorem can be slightly generalized even if H is singular.

Let H be singular but q.i.p. with $F(x)$ as its associated exponential polynomial matrix and G_0 as its identity. Further let G_0 be commutative with A. Now H^{-1} can be defined as $H^{-1}=F(-1)$ and $HH^{-1}=F(0)=G_0$. Hence $f(B)=f(HAH^{-1})=Hf(A)H^{-1}$. So that if A is q.i.p. so is B. However, the index of B may not be the same as that of A since H may be a divisor of zero.

THEOREM 5. *If B is the transform of a q.i.p. matrix A, the matrix of transformation being H, then the exponential matric polynomial* $G^*(X)$ *of B is also the transform by H of the exponential matric polynomial* $F^*(X)$ *of A.*

Proof. Let $F^*(X)=F_0+F_1X+\cdots+F_kX^k$ and let $G^*(X)=G_0+G_1X+\cdots+G_hX^h$. Let $B=HAH^{-1}$. Hence $H^{-1}BH=A$. Now $A^r=F(r)$ for all

positive integers r. So $B^r = HF^*(rI)H^{-1}$. But the exponential polynomial matrix is unique by Theorem 1. Hence $G(r) = HF^*(rI)H^{-1}$. Thus the degree of $G(x)$ is the same as the degree of $F(x)$ *i.e.*, $h=k$. Further $G_i = HF_iH^{-1}$ for $i=0, 1, 2, \cdots, h=k$. Hence we have

$$G(x) = H(F_0 + F_1x + \cdots + F_kx^k)H^{-1}$$

so that

$$\begin{aligned} G^*(X) &= H(F_0 + F_1X + \cdots + F_kX^k)H^{-1} \\ &= HF^*(X)H^{-1} \end{aligned}$$

which proves the theorem.

6. Characteristic roots of Q.I.P. matrices. The minimum equation of a q.i.p. matrix is not in general its characteristic equation. For example consider the unit matrix I of order p. As I is idempotent it is also q.i.p. The minimum equation of I is $\lambda - 1 = 0$, while its characteristic equation is $(\lambda - 1)^p = 0$.

THEOREM 6. *The possible distinct characteristic roots of a q.i.p. matrix A are zero and unity. The multiplicity of the unit root equals the rank of A while the multiplicity of the zero root equals its nullity.*

Proof. The distinct characteristic roots of a matrix are the distinct roots of its minimum equation. The minimum equation of a q.i.p. matrix has only two distinct roots *viz.*, zero and/or unity. So the distinct characteristic roots of a q.i.p. matrix are possibly zero and unity only.

If A is q.i.p. of order n and nonsingular then there is no zero root of the characteristic equation. Hence the multiplicity of the unit root is n, the rank of A.

If A is singular of rank $\rho = n - \sigma$, where σ is its nullity, then the characteristic roots of A contain σ zero roots and the remaining roots are non-zero. Now as A is q.i.p. all the remaining roots are unity, their number being ρ, which proves the theorem.

The converse of the above theorem is not necessarily true. A matrix may have distinct characteristic roots zero and/or unity and yet it may not be q.i.p.

For example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

has distinct characteristic roots zero and unity. Now the minimum equation of A is $\lambda^2(\lambda - 1) = 0$, and so by Theorem 3 is not q.i.p.

Reference

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ALGEBRAIC PROPERTIES OF CLASSES OF FUNCTIONS

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The concepts of odd and even functions are well known. An even function is a function $f(x)$ such that $f(-x)=f(x)$, and an odd function is a function $f(x)$ such that $f(-x)=-f(x)$. It is also well known that any function may be uniquely expressed as the sum of an odd and even function [1].

A simple generalization of these concepts to n types of functions is furnished by the nth roots of unity. Let ω be a primitive nth root of unity. The distinct nth roots of unity are ω^j, $j=0, 1, \cdots, (n-1)$ [2].

Consider a function, $f(x)$, of a complex variable, x, to be of type (n, k) with $0 \leqq k < n$ if

$$f(\omega x) = \omega^k f(x).$$

It follows, that for each n, $f(x)$ may belong, at most, to one of these types; for let

$$f(\omega x) = \omega^k f(x),$$

and

$$f(\omega x) = \omega^s f(x).$$

Then

$$(\omega^k - \omega^s) f(x) = 0,$$

or

$$\omega^k = \omega^s.$$

First, a lemma is proved.

LEMMA. *If ω is a primitive nth root of unity, then the determinant $D \neq 0$, where*

$$D = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega & \omega^2 & & \omega^{(n-2)} & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & & \omega^{2(n-2)} & \omega^{2(n-1)} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \omega^{n-2} & \omega^{2(n-2)} & \cdots & \omega^{(n-2)^2} & \omega^{(n-2)(n-1)} \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-2)} & \omega^{(n-1)^2} \end{vmatrix}.$$

Proof. It may be verified that

$$D^2 = \begin{vmatrix} n & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & 0 \cdots 0 & n \\ 0 & 0 & 0 \cdots n & 0 \\ \cdot & \cdot & \cdot \quad\quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad\quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad\quad \cdot & \cdot \\ 0 & 0 & n \cdots 0 & 0 \\ 0 & n & 0 \cdots 0 & 0 \end{vmatrix}$$

$$= (-1)^{(n-2)(n-1)/2} n^n \neq 0.$$

This result determines the value of D to within sign. This cannot be predetermined, since this depends upon the primitive root of unity used.

THEOREM 1. *If $f(x)$ is any complex valued function, it may be expressed uniquely as*

$$f(x) = f_0(x) + f_1(x) + f_2(x) + \cdots + f_{n-1}(x),$$

where $f_k(x)$ is of type (n, k).

Proof. If $f(x)$ has an expansion of this type, then

$$f(\omega^j x) = f_0(x) + \omega^j f_1(x) + \omega^{2j} f_2(x) + \cdots + \omega^{(n-1)j} f_{n-1}(x),$$

for $j = 0, 1, 2, \cdots, (n-1)$.

Since the determinant of the coefficients of $f_0(x), f_1(x), \cdots, f_{n-1}(x)$ is D and therefore non-zero by the lemma, these functions have a unique expansion in terms of $f(x), f(\omega x), \cdots, f(\omega^{(n-1)}x)$ and are, therefore, unique.

It may be easily verified that these expansions are:

$$f_j(x) = \frac{1}{n}\left[f(x) + \omega^j f(\omega x) + \omega^{2j} f(\omega^2 x) + \cdots + \omega^{(n-1)j} f(\omega^{n-1} x)\right]$$

for $j = 0, 1, 2, \cdots, (n-1)$.

The following isomorphism can be established between a system of non-empty classes, ${}_n\mathfrak{F}_k$, of the functions of type (n, k), and the ring of residue classes of integers modulo n.

With suitable definitions of operations, the classes ${}_n\mathfrak{F}_k$ form a ring. Let ${}_n\mathfrak{F}_k$ be the set of all functions of type (n, k), and let ${}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s$ denote the set of all functions which arise as products of functions in ${}_n\mathfrak{F}_k$ and ${}_n\mathfrak{F}_s$. Also, if $f(x) \in {}_n\mathfrak{F}_k$, $g(x) \in {}_n\mathfrak{F}_s$, and $f[g(x)] \in {}_n\mathfrak{F}_{ks}$, it is possible, then, to define the *composition* of the classes ${}_n\mathfrak{F}_k$ and ${}_n\mathfrak{F}_s$, ${}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s$ to be the class ${}_n\mathfrak{F}_{ks}$.

Let $f(x) \in {}_n\mathfrak{F}_k$ and $g(x) \in {}_n\mathfrak{F}_s$; then

$$f(\omega x) \cdot g(\omega x) = \omega^k f(x) \cdot \omega^s g(x)$$
$$= \omega^{k+s} f(x) g(x).$$

Hence,

$$ {}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s \subset {}_n\mathfrak{F}_{k+s}. $$

Conversely, if $f(x) \in {}_n\mathfrak{F}_{k+s}$, then

$$ f(x) = x^k [x^{-k} f(x)], $$

and

$$ x^k \in {}_n\mathfrak{F}_k $$
$$ [x^{-k} f(x)] \in {}_n\mathfrak{F}_s. $$

Therefore, ${}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s \supset {}_n\mathfrak{F}_{k+s}$ and

$$ {}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_m, \qquad (k + s \equiv m \bmod n). $$

Then, if $f(x) \in {}_n\mathfrak{F}_k$ and $g(x) \in {}_n\mathfrak{F}_s$,

$$ f[g(\omega x)] = f[\omega^s g(x)] $$
$$ = \omega^{ks} f[g(x)]. $$

From this and the definition, it follows:

$$ {}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_m, \qquad (ks \equiv m \bmod n). $$

THEOREM 2. *The ring of classes of functions, ${}_n\mathfrak{F}_k$, is isomorphic to the ring of residue classes of integers modulo n.*

Proof. Define a mapping h, such that

$$ h: {}_n\mathfrak{F}_k \leftrightarrow k \bmod n. $$

Then,

$$ h: {}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_{k+s} \leftrightarrow (k + s) \bmod n; $$
$$ h: {}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_{ks} \leftrightarrow (ks) \bmod n. $$

Therefore, h is an isomorphism between the ring of classes of functions, ${}_n\mathfrak{F}_k$, and the ring of residue classes of integers modulo n.

In view of the fact that the residue classes modulo 2 are the odd and even integers, this isomorphism lends additional justification to the use of the terminology: odd and even.

It may be pointed out that if the total set of functions includes the kth roots of x, then the *composition* defined herein coincides with the definition of composition where ${}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s$ is taken as the set of all functions $f[g(x)]$ which arise from the composition of functions in ${}_n\mathfrak{F}_k$ and ${}_n\mathfrak{F}_s$, where $f(x) \in {}_n\mathfrak{F}_k$ and $g(x) \in {}_n\mathfrak{F}_s$.

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COLLEGE MATHEMATICS FOR NON-SCIENCE STUDENTS

A report of the Special Committee on College Mathematics for Non-Science Students, a Subcommittee of the California Committee for the Study of Education.

This committee has undertaken a study of College Mathematics for General Education, defined as that mathematics which is studied in the freshman or sophomore year as part of a liberal education. It is not a course planned for engineers, professional mathematicians, physical or biological scientists, or even for the modern social scientist who needs specialized mathematics peculiar to his field of research.

I. Current Practices in the Field of College Mathematics for General Education

In order to gain a knowledge of current practices in this field as to curricula, related teaching problems and their solutions, questionnaires were sent to about 250 representative colleges throughout United States and Canada.

Of 110 colleges making replies on the questionnaires, 67 were offering special courses in mathematics for non-science majors, 16 expressed a desire to give such courses, although at present they did not do so. The remaining 27 implied that the usual sequence of mathematics courses was adequate for all of their students.

Most colleges reported that the students who took these special courses had two years of high school mathematics, but in many schools the students had less preparatory mathematics and in a few they had more.

Thirty-three different textbooks were being used, and in six colleges instructors used their own manuscripts. The books used most frequently by those replying on the questionnaires were the textbooks by Richardson; Newsom and Eves; Leonhardy; Cooley, Gans, Kline and Wahlert; Trimble, Peck, Bolser; Jones; Griffin. Relatively few instructors were completely satisfied with any available text indicating that there is essentially no agreement on the topics to be included in a course in mathematics for general education.

Colleges devoted different amounts of time and credit to the course, varying from two to ten semester units. The majority offered either a one semester three credit course or a two semester course, three units each semester. These were in about equal numbers. Most of these courses were required, or they met a general college requirement that could also be satisfied in some other way.

The nature of the course varied greatly in different institutions. In some of the large state universities where they were obliged to take students with little or no high school mathematics, the presence of poor students, large classes and inexperienced teachers made their courses chiefly basic skills courses with emphasis on computation and techniques. They were unhappy about the situation and saw the need for improvement.

In most colleges these courses in mathematics for general education used a historical, intuitive and cultural approach with emphasis on ideas rather than on manipulation or computation. There was a wide range of difficulty. It varied from courses planned for students with no high school mathematical preparation to courses on a high level which emphasized modern mathematics and tried to encourage thinking of an original nature.

Problem solving was the chief type of homework for these courses, but essays and oral reports were frequently required. Practically all organized course activity was limited to the classroom, with few field trips.

The questionnaires also sought information concerning the attitudes of the students, their emotional reactions, what topics they liked most and what they liked least. The replies on these questions were difficult to tabulate, but they indicated (1) that a large proportion of the students who took these courses had had some unfortunate mathematical experience which frequently left them with a negative attitude toward mathematics; and (2) that there is some agreement on topics which inspire enthusiasm and those which antagonize.

The following were mentioned as experiences which caused most enthusiasm: applications; bizarre, new or unusual results; modern implications; problems which can be visualized; problems which the student can solve; challenging situations; discovery on one's own; reasoning; elementary logic; number scales; higher dimensions; transfinite numbers; maximum and minimum problems; problems involving rates; learning how to read stated problems; variation problems; percentage; applications in the field of business; annuities; graphs; statistics and probability; starting at the student's level of understanding and enabling him to achieve success.

Among experiences which seemed to antagonize students the following were mentioned: new definitions; difficult ideas with no applications; not obviously useful or necessary abstractions; logic; topology; proofs, especially detailed tricky proofs; abstract geometry; formal algebra; word problems; lengthy computations; having to give reasons; careful reasoning; thinking; routine.

From an overall examination of the replies on the questionnaires the following generalizations may be drawn:

(1) There is not a single course that meets the needs of all students and all instructors. Successful courses are taught at different levels of mathematical difficulty and maturity.

(2) It is just as important to consider the instructor as it is the students in preparing a course.

(3) The choice of topics is not as important for the success of the course as are the attitude and interests of the teacher, and the attitudes and emotions of the students.

II. Textbooks and Literature on Mathematics for General Education

Much has been published in recent years on the subject of mathematics for general education. The committee has compiled a limited annotated bibliog-

raphy of some textbooks and articles from leading journals which are pertinent to the subject of this study. Copies of this bibliography may be obtained by writing to Professor May M. Beenken, Immaculate Heart College, 2021 North Western Avenue, Los Angeles 27, California.

III. Committee Recommendations

The committee does not recommend a single course to meet the needs of all students and all instructors. Successful courses may be constructed for students of different degrees of mathematical maturity.

The following are guiding principles to consider in constructing a course in mathematics for general education. It should include mathematics that is useful in the home, business and everyday occupations, mathematics which the student will use in other college courses and the mathematics which throws light on the culture of our society. The course should be taught with an adult approach, from a mature and reasonable point of view.

Emphasis in this course will be on critical thinking and understanding rather than technique. But since real understanding must come through application of principles, it must be a course in mathematics and not just about mathematics. Appeal will be made through the beauty of mathematics as well as its utility.

Students who take mathematics for general education are frequently those who have emotional blocks toward the study of mathematics. It will then be not so much the content as the attitude and teaching approach that must distinguish this course from other courses in mathematics. We must meet students where they are and lead them along the path of mathematics as a meaningful aid in precise thinking. This course requires a teacher who is interested in helping students to overcome their blocks and to acquire a liking for mathematics and a feeling of confidence in their ability to do mathematics. The teacher should have a thorough knowledge of mathematical theory and a wide acquaintance with applications to the arts and sciences. He should be well versed in the history of mathematics and its cultural significance. Further qualifications of the successful teacher are patience and an understanding of the learning process.

Beyond such essential topics as the number system, operations with numbers, arithmetic of measurement, functions, graphs, equations and formulas, logical reasoning, the selection of topics should be influenced by the desires and abilities of the students, the interests of the teacher, and the amount of available time. Other topics which may be included are introduction to the calculus, probability and statistics, number theory and some elementary aspects of modern mathematics. In this work the objective will be more an attitude or point of view than a definite amount of knowledge or skill. Some history of mathematics, of important mathematical concepts and the role of mathematics in a world of scientific achievement should be interspersed throughout the entire course.

Upon completion of the course the student should be more critical and exact in his thinking, be better able to meet quantitative situations that arise in everyday living, and he should have a wholesome respect for the power and beauty of

mathematics as a living subject that is constantly growing to meet the needs of an advancing civilization.

A standard course in mathematics for general education should carry three units of credit for each of two semesters. Some colleges may wish to devote more time and credit to this course; others may be restricted to less. Since the success of the course in any college requires the support of all members of the faculty and administration, a major effort should be made to enlist their cooperation through an adequate interpretation of the course to them.

IV. Universal Mathematics

A committee of the Mathematical Association of America on the Undergraduate Mathematical Program has made a thorough study, and advocates a universal freshman course. A first effort at writing a text for such a course has been published as *Universal Mathematics* by the 1954 Summer Writing Group of the University of Kansas.

Universal Mathematics requires a higher level of mathematical preparation than is assumed for the course discussed in this report. This committee urges college mathematics teachers to read the reports of the Committee on the Undergraduate Mathematics Program, and to experiment with the ideas suggested. We agree that students interested in technical mathematics, whether for natural science, social science or the arts, might well take the same course, which should be different from the traditional course. But this is not the class of students for whom a course in mathematics for general education, as defined in this report, is designed.

We feel that it would be inadvisable to give the technically able students, with superior mathematical background of high school mathematics, the same course as average liberal arts students pursuing non-scientific courses, who have had little training in secondary mathematics.

Acknowledgments

This committee wishes to give special recognition to Professor Paul B. Johnson who was responsible for constructing the questionnaire and analyzing the replies; also to Professor Paul A. White who served as chairman of the committee for the year, 1954–55, leaving at the end of the academic year to spend a sabbatical leave in Europe. He relinquished the chairmanship to Professor May M. Beenken for the academic year, 1955–56.

This report was presented April 20, 1956 by Committee:

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON WALLIS' FORMULA

JOHN GURLAND, Iowa State College

1. Introduction. It is well known that the number π satisfies the following inequalities

$$\frac{2}{2k+1}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2 < \pi < \frac{1}{k}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2, \qquad k = 1, 2, \cdots, \tag{1}$$

where

$$(2k)!! = 2\cdot 4\cdot \cdots \cdot (2k-2)(2k)$$
$$(2k-1)!! = 1\cdot 3\cdot 5 \cdots (2k-3)(2k-1).$$

This result is due to Wallis (*cf.* de la Vallée Poussin [1]).

It is interesting that a basic theorem in mathematical statistics concerning unbiased estimators with minimum variance leads to the following inequalities which yield a closer approximation to π than that afforded by (1), namely

$$\frac{4k+3}{(2k+1)^2}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2 < \pi < \frac{4}{4k+1}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2, \qquad k = 1, 2, \cdots. \tag{2}$$

A statement and a sketch of the proof of this theorem is given below in section 2. Following this, a simple application is given in section 3 which leads to the result (2). In addition to providing this result, the method may be of some interest in itself.

2. A theorem in mathematical statistics. Although a more general formulation of this theorem is possible (*cf.* Cramér [2], Lehmann [3]), the following form of the theorem will be adequate for the purpose considered. The notation $Ef(X)$ denotes, as usual, the mathematical expectation of $f(X)$.

THEOREM. *Let $(X_1, X_2, \cdots, X_n)$ be a random sample from a population with probability density $p(x|\theta)$, where the unknown parameter θ is a number in some open interval I. Suppose $T(X_1, X_2, \cdots, X_n)$ is an unbiased estimator of θ, and assume the regularity conditions* (C) *hold. Then the variance σ_T^2 satisfies the inequality*

$$\sigma_T^2 \geqq \frac{1}{nE\left\{\dfrac{\partial \log p(X|\theta)}{\partial\theta}\right\}^2}. \tag{3}$$

The equality can occur if and only if $T(X_1, X_2, \cdots, X_n)$ *is a linear function of*

$$\sum_{i=1}^{n} \frac{\partial \log p(X_i \mid \theta)}{\partial \theta}.$$

The regularity conditions referred to are the following:

$$C\begin{cases} \text{(i)} & \dfrac{\partial \log p(x \mid \theta)}{\partial \theta} \text{ exists for all } x, \text{ and all } \theta \in I \\ \text{(ii)} & \displaystyle\int_{-\infty}^{\infty} \frac{\partial p(x \mid \theta)}{\partial \theta}\, dx = 0 \\ \text{(iii)} & \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T(x_1, x_2, \cdots, x_n) \sum_{i=1}^{n} \frac{\partial \log p(x_i \mid \theta)}{\partial \theta}\, dx_1 \cdots dx_n = 1 \\ \text{(iv)} & \displaystyle E\left\{\frac{\partial \log p(X \mid \theta)}{\partial \theta}\right\}^2 > 0. \end{cases}$$

Proof. The method of proof is briefly indicated here. Let

$$\phi(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} \frac{\partial \log p(X_i \mid \theta)}{\partial \theta}.$$

Then the regularity conditions (C) imply that $E\phi(X_1, X_2, \cdots, X_n) = 0$ and $ET(X_1, \cdots, X_n)\phi(X_1, \cdots, X_n) = 1$. Since the correlation coefficient between T and ϕ must lie in the interval from -1 to 1, it follows that

$$E(T - ET)^2 E\phi^2 \leqq ET\phi.$$

This yields the required result.

3. Proof of (2). Consider a random sample $(X_1, X_2, \cdots, X_n)$ from a normal population with mean 0 and variance σ^2. Then

$$p(x \mid \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}.$$

Now $(1/\sigma^2)\sum_{i=1}^{n} X_i^2$ has a χ^2 distribution with n degrees of freedom; hence $(1/\sigma)\sqrt{\sum_1^n X_i^2}$ has probability density

$$\frac{1}{2^{(n/2)-1}\Gamma\left(\frac{n}{2}\right)}\, v^{n-1}e^{-v^2/2}, \qquad v > 0,$$

the first moment of which is

$$a = \frac{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Consequently

$$T = \frac{1}{a}\sqrt{\sum_1^n X_i^2}$$

is an unbiased estimator σ. It is easily verified that the variance of T is given by

$$\sigma_T^2 = \sigma^2\left(\frac{n}{a^2} - 1\right).$$

Further, since

$$\frac{\partial \log p(x \mid \sigma)}{\partial \sigma} = -\frac{1}{\sigma} + \frac{x^2}{\sigma^3},$$

the lower bound for σ_T^2 given in (2) is $\sigma^2/2n$. However, since

$$\phi(X_1, X_2, \cdots, X_n) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_1^n X_i^2,$$

it is clear that T is not a linear function of ϕ; hence the bound in (3) is not attained by σ_T^2. Thus

$$\sigma^2\left[\frac{n}{a^2} - 1\right] > \frac{\sigma^2}{2n},$$

which reduces to

$$\left[\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2 < \frac{n^2}{2n+1}, \qquad n = 1, 2, \cdots. \tag{4}$$

For $n = 2k$, the inequality (4) may be written as

$$\pi < \left[\frac{(2k)!!}{(2k-1)!!}\right]^2 \frac{4}{4k+1}, \qquad k = 1, 2, \cdots,$$

while for $n = 2k+1$ it may be written as

$$\frac{4k+3}{(2k+1)^2}\left[\frac{(2k!!)}{(2k-1)!!}\right]^2 < \pi, \qquad k = 1, 2, \cdots.$$

Combining these two results yields the inequalities in (2).

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A MINIMUM SOLUTION OF A DIOPHANTINE EQUATION

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Let a, b, and c be given integers, a and b relatively prime, and $ab \neq 0$. The traditional procedure in solving the diophantine equation,

$$ax + by = c, \tag{1}$$

namely, determining a solution of

$$a\xi + b\eta = 1 \tag{2}$$

and multiplying it by c, generally leads to excessively large numerical values for x and y and consequently to an unnecessarily cumbersome form for the general solution. For example, in the case of the equation,

$$94x + 53y = 77, \tag{3}$$

which is satisfied by $x = -2$, $y = 5$, it yields the solution, $x = 1694$, $y = -3003$.

On the other hand, there is a familiar elementary method of solving (1) independently of (2) which does not have this defect. We recall this method by applying it to the example (3). Solving for y, we obtain $y = (77 - 94x)/53 = 1 - 2x + (24 + 12x)/53$. Then, since x and y are integers, so is $w = (24 + 12x)/53$; and w satisfies another diophantine equation, $53w - 12x = 24$, of the form (1) but with smaller coefficients than (3). Repeating the process by solving the latter equation for x, we arrive at a homogeneous equation, $12v - 5w = 0$, which has the obvious solution $w = v = 0$. Then we find from the preceding equation that $x = -2$ and hence $y = 5$. This we call the *definitely least solution* in the sense that $|x|$ and $|y|$ both attain their least possible values. It is the purpose of this note to show that with a slight modification, which incidentally shortens the computation, this method always leads to the definitely least solution when it exists and, when it does not, to a solution in which, with the notation arranged so that $|a| \geqq |b|$, $|x|$ assumes its minimum value. This we call a *minimum solution with respect to* x. We also give conditions for the existence of the definitely least solution.

It is not necessary to write out and solve explicitly the successive equations as was done in the example. The process can be reduced to an algorithm. In this form it is a generalization to arbitrary c of a highly efficient algorithm suggested originally by Gauss for the solution of (2), which, as D. H. Lehmer has pointed out,* requires only half as much labor as the usual continued fraction procedure. Setting $b_0 = a$, $b_1 = b$, $c_0 = c$ for uniformity of notation, we perform successively for $i = 1, 2, \cdots, n$ the two sequences of divisions with numerically least remainders,

$$b_{i-1} = q_i b_i + b_{i+1}, \qquad |b_{i+1}| \leqq \tfrac{1}{2}|b_i|, \tag{4}$$

$$c_{i-1} = g_i b_i + c_i, \qquad |c_i| \leqq \tfrac{1}{2}|b_i|. \tag{5}$$

* A note on the linear diophantine equation, this MONTHLY, vol. 48, 1941, pp. 240–246.

The value $i=n$ at which the divisions terminate is uniquely determined by the conditions,

$$0 < |c_i| < \tfrac{1}{2}|b_i| \qquad (i = 1, 2, \cdots, n-1) \tag{6}$$

and either

$$c_n = 0, \quad b_n \neq 0, \text{ or} \tag{7a}$$

$$c_n = \pm \tfrac{1}{2}b_n \neq 0. \tag{7b}$$

That (6) and (7) will hold for some integer n follows from the fact that the sequence (4) would constitute a Euclidean algorithm for the greatest common divisor (b_0, b_1) if continued until a zero remainder b_{m+1} was reached. Then, $b_m=(b_0, b_1)=(a, b)=1$, and hence by (5) for $i=m$, $c_m=0$, so that (6) and either (7a) or (7b) are satisfied by some positive integer $n \leqq m$. This brings us to our algorithm.

THEOREM 1. *From the q_i, b_i, g_i determined in the divisions* (4) *and* (5) *let integers x_i be computed as follows: x_n and x_{n+1} are defined by either*

$$x_n = x_{n+1} = 0, \text{ or} \tag{8a}$$

$$x_n = \pm \tfrac{1}{2}b_n, \qquad x_{n+1} = \mp \left[\tfrac{1}{2}b_{n+1}\right] \tag{8b}$$

accordingly as (7a) *or* (7b) *holds respectively. The upper signs are to be used in* (8b) *if the upper sign was chosen in* (7b); *otherwise the lower signs are to be used. Then $x_{n-1}, x_{n-2}, \cdots, x_1, x_0$ are obtained successively by the recurrence relation,*

$$x_{i-1} = g_i - q_i x_i + x_{i+1}. \tag{9}$$

Then

$$x = x_1, \qquad y = x_0 \tag{10}$$

is a solution of the diophantine equation,

$$b_0 x + b_1 y = c_0, \qquad |b_0| \geqq |b_1| > 0, \ (b_0, b_1) = 1. \tag{11}$$

More generally, we show that

$$b_j x_{j+1} + b_{j+1} x_j = c_j \qquad (j = 0, 1, \cdots, n). \tag{12}$$

When (8a) applies, (12) is immediate for $j=n$. (8b) can apply only when b_n is even and hence only when b_{n+1} is odd so that $x_{n+1}=\mp\frac{1}{2}(b_{n+1}-1)$, from which we can again verify (12) for $j=n$ by direct substitution. Now to complete the proof by finite descending induction, suppose (12) holds for a value $j=i>0$. Then $x_{i+1}=(c_i-b_{i+1}x_i)/b_i$ so that from (9), (5) and (4)

$$x_{i-1} = g_i - q_i x_i + \frac{c_i - b_{i+1}x_i}{b_i} = \frac{g_i b_i + c_i - (q_i b_i + b_{i+1})x_i}{b_i} = \frac{c_{i-1} - b_{i-1}x_i}{b_i},$$

and hence (12) holds for $j=i-1$ also. Thus (12) is established for $j=n, n-1, \cdots, 0$; and the case $j=0$ is precisely our theorem.

To establish the minimum property of the solution (10) we state the following lemma, which is readily proved from the usual form of the general solution of (1).

LEMMA. *A solution (X, Y) of* (1) *is a minimum solution with respect to x (y) if and only if $|X| \leqq \frac{1}{2}|b|$ $(|Y| \leqq \frac{1}{2}|a|)$, and is the only such solution except when the equality holds. It is the definitely least solution if and only if both $|X| \leqq \frac{1}{2}|b|$ and $|Y| \leqq \frac{1}{2}|a|$.*

THEOREM 2. *Let the integers x_i be computed as in Theorem* 1 *with the stipulation that the ambiguous sign in* (7b) *is to be chosen positive if b_{n-1} and c_{n-1} have the same sign and negative if they have opposite sign. Then $x=x_1$, $y=x_0$ is the minimum solution of the diophantine equation* (11) *with respect to x.*

Proof. We first show that for $n>1$

$$|x_j| < \tfrac{1}{2}|b_j| \qquad (j = 1, 2, \cdots, n-1). \tag{13}$$

When (8a) applies, $|b_n x_{n-1}| = |b_{n-1}x_n + b_n x_{n-1}| = |c_{n-1}| < \frac{1}{2}|b_{n-1}|$ by (12), (6) and hence, since $b_n \neq 0$ by (7a), $|x_{n-1}| < \frac{1}{2}|b_{n-1}|$. When (8b) applies and the sign in (7b) is chosen as stipulated in the hypothesis, $x_n = \frac{1}{2}|b_n|$ sgn $(b_{n-1}c_{n-1})$ so that by (12)

$$\begin{aligned} |x_{n-1}| &= \left|\frac{c_{n-1} - b_{n-1}x_n}{b_n}\right| = \left|\frac{c_{n-1} - \tfrac{1}{2}|b_{n-1}b_n| \operatorname{sgn} c_{n-1}}{b_n}\right| \\ &= \left|\frac{|c_{n-1}| - \tfrac{1}{2}|b_{n-1}b_n|}{b_n}\right| = \left|\left|\frac{c_{n-1}}{b_n}\right| - \frac{1}{2}|b_{n-1}|\right|. \end{aligned} \tag{14}$$

But $|c_{n-1}/b_n| \leqq |c_{n-1}| < \frac{1}{2}|b_{n-1}|$ by (6) and hence again $|x_{n-1}| < \frac{1}{2}|b_{n-1}|$. Thus (13) is established for $j=n-1$. In the case $n=2$ the proof of (13) is already complete. For $n>2$ the proof is by descending induction; so we suppose (13) holds for any value $j=k>1$. Then, since b_k and x_k are integers, $2|x_k| \leqq |b_k| - 1$. Hence by (12) and (6)

$$\begin{aligned} |x_{k-1}| &= \left|\frac{c_{k-1} - b_{k-1}x_k}{b_k}\right| \leqq \left|\frac{c_{k-1}}{b_k}\right| + \left|\frac{b_{k-1}x_k}{b_k}\right| \\ &< \left|\frac{b_{k-1}}{2b_k}\right| + \left|\frac{b_{k-1}(|b_k| - 1)}{2b_k}\right| = \frac{1}{2}|b_{k-1}|; \end{aligned}$$

so (13) holds for $j=k-1$ also. Thus (13) is established for $j=n-1, n-2, \cdots, 1$ provided $n>1$. When $n=1$, $x_1=0$ or $x_1 = \pm\frac{1}{2}b_1$ by (8a) or (8b). Combining this with the case $j=1$ of (13), we have

$$|x_1| < \tfrac{1}{2}|b_1| \text{ when } n > 1; \qquad |x_1| \leqq \tfrac{1}{2}|b_1| \text{ when } n = 1. \tag{15}$$

Hence by the lemma (x_1, x_0) is a minimum solution of (11) with respect to x.

We note that by the lemma (x_1, x_0) is the only minimum solution of (11) with respect to x except when $x_1 = \pm\frac{1}{2}b_1$. Then there is just one other such solution, say (x_1', x_0'); and, since by the lemma $|x_1'| = \frac{1}{2}|b_1|$ also, we must have $x_1' = -x_1$. In view of (15) this case can occur only when $n=1$. Then (14) becomes $|x_0| = ||c_0/b_1| - \frac{1}{2}|b_0||$, while in like manner $|x_0'| = |c_0 - b_0x_1'|/|b_1| = |c_0 + b_0x_1|/|b_1| = |c_0/b_1| + \frac{1}{2}|b_0|$; so $|x_0'| > |x_0|$. This proves the corollary,

COROLLARY 1. *Either* (x_1, x_0) *is the only minimum solution of* (11) *with respect to* x *or there is just one other, and then* (x_1, x_0) *is the one in which* $|y|$ *is smaller.*

Since the definitely least solution of (11), if it exists, is *a fortiori* a minimum solution with respect to x and is the one in which $|y|$ is least, we then have

COROLLARY 2. (x_1, x_0) *is the definitely least solution of* (11) *whenever such a solution exists.*

The following example illustrates the operation of the algorithm with the termination (7b) in conjunction with the convention for choosing the ambiguous sign:

(16) $$20243x + 9153y = 69084$$

i	b_i	q_i	c_i	g_i	x_i
0	20243		69084		−10104
1	9153	2	5013	7	4572
2	1937	5	−798	3	−967
3	−532	−4	−266	1	−266
4	−191				96

Since $|c_3| = \frac{1}{2}|b_3|$, $n=3$. b_2 and c_2 have opposite sign; so the quotient g_3 is chosen to give a negative remainder c_3. Then, since $c_3 = +\frac{1}{2}b_3$, we use the upper signs in (8b) to obtain $x_3 = \frac{1}{2}b_3 = -266$, $x_4 = -[\frac{1}{2}b_4] = -[-191/2] = 96$. The rest of the last column is then calculated upward by means of (9) giving finally $x = x_1 = 4572$, $y = x_0 = -10104$, the definitely least solution. The traditional method yields the solution $x = 46{,}355{,}364$, $y = -102{,}520{,}656$.

We conclude by establishing conditions for the existence of a definitely least solution (d.l.s.) of (1). There is no loss of generality if we take b odd, $a>0$, $b>0$, $c \geqq 0$.

THEOREM 3. *Let* $a>0$, $b>0$, $c \geqq 0$. *There are integers* L *and* R *depending only on* a *and* b *such that the diophantine equation* (1) *has a d.l.s. for every* $c<L$, *for no* $c>R$, *and for exactly half the values of* c *in the interval* $L \leqq c \leqq R$. *There is a* d.l.s. *for one but not the other of each pair of values* $c=C$ *and* $c=C'$ *such that* $C+C' = L+R$. *In particular, there is a* d.l.s. *for* $c=R$ *but not for* $c=L$. *With the notation arranged so that* b *is odd the values of* L *and* R *are given explicitly by*

(17) *a odd:* $$L = \tfrac{1}{2}(a+b), \qquad R = ab - \tfrac{1}{2}(a+b);$$

(18) *a even:* $$L = \tfrac{1}{2}a + b, \qquad R = ab - \tfrac{1}{2}a.$$

Proof. Let a and b be odd and let (X, Y) be a solution of (1) for the value $c=C$. Then $a(-X)+b(a-Y)=ab-(aX+bY)=ab-C$; *i.e.*, $(-X, a-Y)$ is a solution of

$$ax + by = C' = L + R - C \tag{19}$$

with L and R given by (17). Now, if (X, Y) is the d.l.s. of (1) for $c=C$, by the lemma (to Theorem 2) $|X|<\frac{1}{2}b$, $|Y|<\frac{1}{2}a$. Hence also $|-X|<\frac{1}{2}b$ but $|a-Y|>\frac{1}{2}a$ so that by the lemma $(-X, a-Y)$ is the (unique) minimum solution with respect to x of (19) but not the d.l.s. However, when the d.l.s. exists, it is necessarily also a minimum solution with respect to x. Hence we conclude that (19) has no d.l.s. Conversely, suppose (1) has no d.l.s. for a value $c=C$ such that $0\leqq C\leqq L+R$ but that the solution (X, Y) is a minimum with respect to x so that $|X|<\frac{1}{2}b$. Then $Y>\frac{1}{2}a$; for otherwise, since (X, Y) is not a d.l.s., $Y<-\frac{1}{2}a$, and it would follow that $C=aX+bY<a\cdot\frac{1}{2}b-b\cdot\frac{1}{2}a=0$ contrary to hypothesis. At the same time $Y<3a/2$, since, if $Y\geqq 3a/2$, we would have $C>a(-\frac{1}{2}b)+b(3a/2)=ab=L+R$ as given by (17). Accordingly $\frac{1}{2}a<Y<3a/2$ and hence $-\frac{1}{2}a=a-3a/2<a-Y<a-\frac{1}{2}a=\frac{1}{2}a$, or $|a-Y|<\frac{1}{2}a$; and, since $|-X|<\frac{1}{2}b$, the solution $(-X, a-Y)$ of (19) is a d.l.s. For a even we readily verify that, if (X, Y) is a solution of (1) for $c=C$, $(-X, a+1-Y)$ is a solution of (19) with L and R given by (18). We can then proceed as before to show that, if $0\leqq C\leqq L+R$, one but not both of the equations (1) (with $c=C$) and (19) has a d.l.s. Next, if (1) has the d.l.s., (X, Y) and a and b are odd, $|X|\leqq\frac{1}{2}(b-1)$, $|Y|\leqq\frac{1}{2}(a-1)$, and $c\leqq a|X|+b|Y|\leqq a\cdot\frac{1}{2}(b-1)+b\cdot\frac{1}{2}(a-1)=R$ as given by (17). On the other hand, if a is even, $|Y|\leqq\frac{1}{2}a$ and then $c\leqq a\cdot\frac{1}{2}(b-1)+b\cdot\frac{1}{2}a=R$ as given by (18). In either case (1) can have no d.l.s. for $c>R$, but for $c=R$ it does have the d.l.s., $x=\frac{1}{2}(b-1)$, $y=\frac{1}{2}(a-1)(a$ odd), $y=\frac{1}{2}a(a$ even). Finally, since, as we have just seen, there is no d.l.s. for any value $c=C$ such that $R<C\leqq L+R$, there is a d.l.s. for every value $c=C'=L+R-C$, where $0\leqq C'<L$. However, because there is a d.l.s. for $c=C=R$, there is none for $c=C'=L$.

Thus, referring to the example (3), $94x+53y=c$ has a d.l.s. for every integer $c<100$ but for no $c>4935$, there being a d.l.s. for $c=4935$ but not for $c=100$. Since from the example (16) $20243x+9153y=c$ has a d.l.s. for $c=69084$, it has none for $c=20243\cdot 9153-69084=185{,}215{,}095$. There is a d.l.s. for every integer $c<14698$ but for no $c>185{,}269{,}481$.

It is natural to ask how our results would be affected if in the divisions (4) and (5) we used least positive remainders instead of least absolute remainders. The following is easily shown:

THEOREM 4. *Let $b_0>b_1>0$. Replace the inequalities in* (4) *and* (5) *by*

$$0 \leqq b_{i+1} < b_i, \qquad 0 \leqq c_i < b_i$$

respectively, terminating the sequence of divisions with (7a). *Determine the integers x_i by* (8a) *and* (9). *Then $x=x_1$, $y=x_0$ is a solution of* (11), *and $|x_1|<b_1$.*

The equation $11x+7y=6$ has the solution $(-2, 4)$ while in this case $x_1=5$, $x_0=-7$; *i.e.*, (x_1, x_0) is not necessarily a minimum solution of (11) with respect to x. However, since (11) has only two solutions with $|x|<b_1$, it follows from Theorem 4 that there is at most one solution such that $|x|<|x_1|$.

A SINGULAR INTEGRAL EQUATION CONTAINING A PARAMETER

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This note concerns the inversion of the singular equation

$$(\pi i)^{-1}\int_L \cos\lambda(t-t_0)g(t)dt/(t-t_0) = f(t_0), \qquad t_0 \in L, \tag{1}$$

where L (consisting of a finite number of smooth non-intersecting contours of finite length $L_0, L_1, \cdots, L_n$ with L_0 containing all the rest) is the boundary of a plane connected region, the functions $f(t)$ and $g(t)$ are subject to the Hölder condition on L, and λ is a parameter.

The well-known solution of (1) for $\lambda=0$, *i.e.*,

$$g(t_0) = (\pi i)^{-1}\int_L f(t)dt/(t-t_0), \tag{2}$$

indicates the possibility of a general solution of the form

$$g(t_0) = (\pi i)^{-1}\int_L G[\lambda(t-t_0)]f(t)dt/(t-t_0), \tag{3}$$

where $G[\lambda(t-t_0)]$ is of Hölder type with respect to t, t_0, and $G(0)=1$.

Thus, replacing $g(t)$ in (1) by means of (3) and applying the Poincaré-Bertrand transformation formula [1, p. 56]

$$\int_L \frac{dt}{t-t_0}\int_L \frac{K(t,\xi)}{\xi-t}d\xi = -\pi^2K(t_0,t_0) + \int_L d\xi\int_L \frac{K(t,\xi)}{(t-t_0)(\xi-t)}dt, \tag{4}$$

it is seen that the function $G(t)$ must satisfy the following conditions:

$$\int_L f(\xi)d\xi\int_L \frac{\cos\lambda(t-t_0)G[\lambda(\xi-t)]}{(t-t_0)(\xi-t)}dt = 0, \qquad G(0)=1. \tag{5}$$

But an expansion of $[(t-t_0)(\xi-t)]^{-1}$ in partial fractions immediately reveals $G(t)=\cos t$. For

$$\int_L \frac{\cos\lambda(t-t_0)G[\lambda(\xi-t)]}{(t-t_0)(\xi-t)}\,dt$$

$$= \frac{1}{\xi-t_0}\int_L \left\{\frac{1}{t-t_0}-\frac{1}{t-\xi}\right\}\cos\lambda(t-t_0)G[\lambda(\xi-t)]\,dt \tag{6}$$

$$= \frac{\pi i}{\xi-t_0}\{G[\lambda(\xi-t_0)]-\cos\lambda(\xi-t_0)\}$$

provided $G[\lambda(\xi-Z)]$ is analytic as a function of Z, where Z belongs to the region bounded by L. Therefore $G(t)=\cos t$ satisfies (5), and hence the relations

$$(\pi i)^{-1}\int_L \cos\lambda(t-t_0)g(t)dt/(t-t_0) = f(t_0) \quad \text{and} \tag{7}$$

$$(\pi i)^{-1}\int_L \cos\lambda(t-t_0)f(t)dt/(t-t_0) = g(t_0) \quad \text{are reciprocal.} \tag{8}$$

Further, it is easily seen that similar relations hold for other special functions $K[\lambda(t-t_0)]$ with $K(0)=1$, *e.g.*, $\cosh\lambda(t-t_0)$.

Reference

1. N. I. Muskhelishvili, Singular Integral Equations, Translation from the Russian edited by J. R. M. Radok, P. Noordhoff, N. V. Groningen-Holland, 1953.

THE ROW-SUMS OF THE INVERSE MATRIX, II

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It was remarked in I (this MONTHLY, vol. 58, 1951, pp. 614–615) that an obvious fact about finite matrices failed to hold for infinite matrices, namely, if the row-sums of a matrix A are s those of its inverse are $1/s$. Let us state this as (*); *if* $BA=I$, $\sum_k a_{nk}=s$ *for each* n; *then* $\sum_k b_{nk}=1/s$ *for each* n. *Proof* (with $s=1$): $B1=B(A1)=(BA)1=1$, where 1 is the column vector of ones.

Clearly (*) holds for infinite matrices if B is row-finite, or if $\|A\|<\infty$, $\|B\|<\infty$, where $\|A\|=\sup_n \sum_k |a_{nk}|$. That neither of these hypotheses can be dropped follows from an example in I.

In the finite case, $s=0$ implies that B does not exist. We give an example to show that this is false in the infinite case. Specifically, *there exists a matrix* A (which turns out to be row-finite) *with row-sums zero, and a matrix* B *with row-sums one, such that* $BA=I$. (Example 1, below.) As a variant of (*) we state: *if* B *is row-finite and* 1-1 (*i.e.* $Bx=0$ *implies* $x=0$), *if* $BA=I$ *and* $\sum_k b_{nk}=s$, *then* $\sum_k a_{nk}=1/s$ (and $s\neq 0$).

Proof (with $s=1$): $B(1-A1)=1-B(A1)=1-(BA)1=0$.

We may replace the hypothesis of row-finiteness by a boundedness assump-

tion as in (*). *Neither "row-finite" nor "1-1" may be omitted.* That "1-1" may not be omitted follows from the last paragraph but one of I. That "row-finite" may not be omitted follows from Example 1, below, in which it may be checked that B is 1-1 (simpler examples are available without this property).

Finally, Example 1 also verifies the conjecture made in I that (*) need not hold for infinite matrices if it is assumed only that $\sum_k b_{nk}$ exists for each n.

These failures result of course from the non-associative character of matrix multiplication. (See Vermes, Non-associative rings of infinite matrices, *Math. Rev.*, vol. 14, 1953, p. 7.)

Example 1.

$$B = \begin{Vmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \cdots \\ 3/4 & 0 & 1/8 & 1/16 & 1/32 \cdots \\ 7/8 & 0 & 0 & 1/16 & 1/32 \cdots \\ 15/16 & 0 & 0 & 0 & 1/32 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

$$A = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \cdots \\ 4 & -4 & 0 & 0 & 0 \cdots \\ 0 & 8 & -8 & 0 & 0 \cdots \\ 0 & 0 & 16 & -16 & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

Reference

A. Wilansky and K. Zeller: Inverses of matrices and matrix transformations, Proc. Amer. Math. Soc., vol. 6, 1955, pp. 414–420.

FUNCTIONS WITH PREASSIGNED DERIVATIVES

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The existence of a function of one real variable whose derivatives of all orders have preassigned values at a point has been shown in a simple way by A. Rosenthal.* In this note we give an alternative proof, and make the extension that if the initial values for the derivatives are C^∞ functions of a number of real parameters, the desired function can be chosen to be a C^∞ function of the parameters. This proposition has for a simple corollary the corresponding proposition for functions of several real variables whose partial derivatives are to have preassigned values at a point.

* A. Rosenthal. On functions with infinitely many derivatives, Proc. Am. Math. Soc., vol. 4, pp. 600–602, 1953.

THEOREM 1. *Let* $\{a_k(u)\}$, $k=0, 1, 2, \cdots$ *be a sequence of* C^∞ *functions of the m-dimensional real variable* $u=(u_1, \cdots, u_m)$, *each defined for* $u \in U$, *an open set in* R^m. *Then there is a* C^∞ *function* $f(x, u)$ *defined for* $x \in R^1$, *and* $u \in U$, *such that* $\partial^k f(0, u)/\partial x^k = a_k(u)$, $k=0, 1, 2, \cdots$.

Proof. The obvious formal solution is:

$$f(x, u) \approx \sum_{\mu=0}^{\infty} a_\mu(u) x^\mu/\mu!. \tag{1}$$

Let $\phi(x)$ be a function of x satisfying:

(i) $\phi(x)$ is a C^∞ function defined for $x \in R^1$

(ii) $\phi(x)=0$ for $|x| \geqq 1$

(iii) $\phi(0)=1$; $\phi^{(k)}(0)=0$ for $k=1, 2, \cdots$.

Such a function is readily shown to exist. We could take for example:

$$\phi(x) = \begin{cases} \exp\left[-(1-|x|)^{-1} \exp\left(-|x|^{-1}\right)\right], & 0 < |x| < 1 \\ 0, & |x| \geqq 1 \\ 1, & x = 0. \end{cases}$$

We will now modify the formal solution (1) to take the form:

$$f(x, u) = \sum_{\mu=0}^{\infty} a_\mu(u) \frac{x^\mu}{\mu!} \phi(x/\epsilon_\mu). \tag{2}$$

In view of the property (iii) of the function ϕ, it is evident that (2) is again a formal solution to our problem. We shall show that it is possible to choose positive numbers ϵ_μ, so that (2), and the result of differentiating (2) any number of times formally with respect to x and a combination of the variables u, converges uniformly on every compact subset of $R^1 \times U$. By well-known theorems of analysis, this implies that f has the properties asserted in the theorem.

Let us consider the contribution of the μth term of (2) to a formal derivative, p times with respect to a combination of the u variables and q times with respect to x. This will be the product of two factors:

(a) The corresponding pth derivative of $a_\mu(u)$

(b) The qth derivative of $(x^\mu/\mu!)\phi(x/\epsilon_\mu)$.

To find an appropriate bound for the factor (a), let $\{C_k\}$, $k=1, 2, \cdots$ be an expanding sequence of compact subsets of U whose interiors cover U. Then $a_k(u)$ and each of its derivatives is bounded on C_k. Let N_k be a common upper bound on the magnitude of $a_k(u)$, and of its derivatives of orders $\leqq k-1$. Then the factor (a) will be no greater than N_μ, if $\mu \geqq p+1$, $\mu \geqq k$, and $u \in C_k$, since $\{C_k\}$ is an expanding sequence of sets.

To bound (b), let M_k be a common upper bound for the magnitudes of $\phi(x)$, and of its derivatives of orders $\leqq k-1$. The qth derivative of $(x^\mu/\mu!)\phi(x/\epsilon_\mu)$ is the sum of 2^q terms, each of which is the product of an rth derivative of $(x^\mu/\mu!)$ and an sth derivative of $\phi(x/\epsilon_\mu)$, where $r+s=q$. Since $\phi(x/\epsilon_\mu)$ and its derivatives

vanish for $|x| > \epsilon_\mu$, we may assume $|x| \leqq \epsilon_\mu$, and a bound for any one of the terms will be $(\epsilon_\mu)^{\mu-r} \cdot \epsilon_\mu^{-s} M_\mu = \epsilon_\mu^{\mu-q} M_\mu$, the factor ϵ_μ^{-s} arising from the chain rule for differentiating $\phi(x/\epsilon_\mu)$. This bound is valid, provided $\mu \geqq q+1$. Multiplying by the number of terms gives $2^q \epsilon_\mu^{\mu-q} M_\mu$ as a bound for (b), when $\mu \geqq q+1$.

Multiplying the bounds for (a) and (b) together, we obtain $2^q \epsilon_\mu^{\mu-q} M_\mu N_\mu$ as a bound for the contribution of the μth term of (2) to the formal derivative we are considering, provided $u \in C_k$, and $\mu \geqq \sup(p+1, q+1, k)$. Now, if ϵ_μ is chosen so small that $\epsilon_\mu M_\mu N_\mu \leqq 1$, the bound may be replaced by $2^q \epsilon_\mu^{\mu-q-1}$; and if $0 < \epsilon_\mu \leqq 1/2$, the bound may be replaced by $(1/2)^{\mu-2q-1}$, since the exponent of ϵ_μ in the previous bound is non-negative, for $\mu \geqq q+1$. Since the new bound is the general term in a convergent series of positive constants, we have established uniform convergence of the result of differentiating (2) term by term any number of times, for $u \in C_k$ which in turn shows that the function $f(x, u)$ defined by (2) satisfies the conditions of the theorem.

We can extend this result immediately to the case of several real variables:

COROLLARY. *Let $a_{k_1 \cdots k_n}(u)$, $k_1, \cdots, k_n = 0, 1, 2, \cdots$ be a collection of C^∞ functions defined for $u \in U$, as in the theorem. Then there is a C^∞ function $f(x, u)$, where $x = (x_1, \cdots, x_n)$, defined for $x \in R^n$, and $u \in U$, such that for every choice of indices, $k_1, \cdots, k_n$, we have:*

$$\frac{\partial^k f(0, u)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = a_{k_1 \cdots k_n}(u), \qquad k = k_1 + \cdots + k_n.$$

Proof. We use induction on n, the theorem being the case $n = 1$. For $n > 1$, let $x' = (x_1, \cdots, x_{n-1})$; and by the induction hypothesis, there exists, for each integer k_n, a C^∞ function $f_{k_n}(x', u)$, satisfying:

$$\frac{\partial^k f_{k_n}(0, u)}{\partial x_1^{k_1} \cdots \partial x_{n-1}^{k_{n-1}}} = a_{k_1 \cdots k_n}(u), \qquad k = k_1 + \cdots + k_{n-1}.$$

Letting $x'' = x_n$, there is now, by the theorem for the case $n = 1$ a C^∞ function $f(x', x'', u)$, such that for each value of k_n:

$$\frac{\partial^{k_n} f(x', 0, u)}{\partial x''^{k_n}} = f_{k_n}(x' u).$$

It is immediately verified that under the identification $(x', x'') = (x_1, \cdots, x_n)$, the function f has the desired properties.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large numbers of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE CUBIC EQUATION AND THE MANNHEIM SLIDE RULE

H. A. ARNOLD, University of California, Davis

The graphical discussion of Klamkin in [1], if slightly modified, leads to an unusually simple slide rule solution of the cubic, in which, except for a preliminary cube root, the only adjustment made is with the cursor (hairline). The cubic, with real coefficients,

$$z^3 + a_1z^2 + a_2z + a_3 = 0, \qquad a_3 \neq 0, \tag{1}$$

can be reduced by the substitution $z = w - a_1/3$ to the form

$$w^3 + aw + b = 0, \qquad b \neq 0, \tag{2}$$

where

$$\begin{aligned} a &= (3a_2 - a_1^2)/3 \\ b &= (2a_1^3 - 9a_1a_2 + 27a_3)/27. \end{aligned} \tag{3}$$

Setting $w = b^{1/3}x$, equation (2) reduces to

$$x^3 - Sx + 1 = 0, \qquad S = -(a/b)b^{1/3}, \tag{4}$$

$$x^2 + 1/x = S. \tag{4'}$$

$|b|^{1/3}$ is found with or without the K-scale using the slide-rule instruction.

Using the standard relations between the roots and coefficients of a cubic, see [2, p. 64], equation (4) has exactly one negative root. It has no positive (hence two conjugate imaginary) roots, two coincident or two distinct positive roots according as the discriminant $\Delta = -4a^3 - 27b^2$ is <0, $=0$, >0; that is, as S is less than, equal to, or greater than $3/4^{1/3} = 1.890$. See [2, pp. 95–97].

Close the slide rule, with the A, C' and D-scales visible and their ends aligned. (If the tongue is upside down, the C-scale serves as a C'-scale). The hairline (in any position) now lies over: some number x on the D-scale; $1/x$ on the C'-scale; x^2 on the A-scale. This remark used with equation (4') justifies Steps I and II, and fixes the position of the decimal point on each scale.

Step I. The negative root, $-v$. For all S. Set the hairline so that

$$A\text{-reading } \textit{minus } C'\text{-reading} = S. \tag{5}$$

The negative of the D-reading is the root $-v$. If S is a negative number, multiply equation (5) by -1.

Step II. The positive roots. $S \geqq 3/4^{1/3} = 1.890$. Set the hairline so that

(6) $$A\text{-reading } \textit{plus } C'\text{-reading} = S.$$

The D-reading is a positive root. Two such settings, or one, or no such setting is possible. The position of the decimal point may be different on a given scale for each setting.

After Step I, one may divide equation (4) by $x-(-v)$ and solve the resulting quadratic for the remaining roots x_2, x_3

(7) $$x^2 - vx + 1/v = 0$$

(8) $$x_2,\ x_3 = \tfrac{1}{2}(v \pm \sqrt{v^2 - 4/v}) \text{ if } v \geqq 4^{1/3} = 1.587+,$$
$$x_2,\ x_3 = \tfrac{1}{2}(v \pm i\sqrt{-v^2 + 4/v}) \text{ if } v < 4^{1/3}.$$

This yields any imaginary roots and checks Step II for real roots. As a final check, the sum of the roots of (4′) is zero.

An example is equation (4), where $S=3.5$. In Step I, the readings for A, C', D are 4, .5, 2 respectively, so $x_1 = -2$; in Step II the two sets of respective readings are 2.91, 0.586, *1.706* and 0.086, 3.41, *0.293*. Consequently, $x_2 = 1.706$, and $x_3 = 0.293$.

References

1. M. S. Klamkin, A geometric determination of the nature of the roots of the cubic, biquadratic and quintic equations, this MONTHLY, vol. 61, 1954, pp. 340–342.

2. W. V. Lovitt, Elementary Theory of Equations. Prentice-Hall, New York, 1950.

PREDETERMINED DETERMINANT VALUES USING FIBONACCI NUMBERS

J. H. MEANS, Huston-Tillotson College

In plane analytic geometry the condition that three lines intersect in a common point is that the determinant

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

If the instructor wishes to obtain a number of such lines he may do so by using the Fibonacci numbers. The well-known Fibonacci series is 1, 2, 3, 5, 8, $\cdots$, the law of formation being that any term after the second is the sum of the two preceding terms. Let the terms be denoted by U_1, U_2, U_3, $\cdots$ with U_0 being defined as unity, then $U_{n+2} = U_{n+1} + U_n$. Then we have

$$\begin{vmatrix} U_n & U_{n+1} & U_{n+2} \\ U_{n+1} & U_{n+2} & U_{n+3} \\ U_{n+2} & U_{n+3} & U_{n+4} \end{vmatrix} = 0, \qquad \text{for } n = 0, 1, 2, 3, \cdots$$

Example:

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 0.$$

The three equations of the lines which intersect in a common point are

$$\begin{aligned} x + y + 2 &= 0 \\ x + 2y + 3 &= 0 \\ 2x + 3y + 5 &= 0. \end{aligned}$$

The instructor may use the following scheme to arrive at a similar result.

$$\begin{vmatrix} U_n & U_{n+2} & U_{n+4} \\ U_{n+1} & U_{n+3} & U_{n+5} \\ U_{n+2} & U_{n+4} & U_{n+6} \end{vmatrix} = 0.$$

Example:

$$\begin{vmatrix} 3 & 8 & 21 \\ 5 & 13 & 34 \\ 8 & 21 & 55 \end{vmatrix} = 0.$$

In plane analytic geometry where the area of a triangle is given by a third order determinant, a predetermined area may be handily obtained like this:

$$\begin{vmatrix} NU_n & U_{n+1} & 1 \\ NU_{n+1} & U_{n+2} & 1 \\ NU_{n+2} & U_{n+3} & 1 \end{vmatrix} = (-1)^{n+1}N, \qquad \text{where } N \text{ is any number.}$$

Example (with $N=4$):

$$\begin{vmatrix} 12 & 5 & 1 \\ 20 & 8 & 1 \\ 32 & 13 & 1 \end{vmatrix} = (-1)^4(4).$$

If the coordinates of the vertices of a triangle are (12, 5), (20, 8), and (32, 13) the area of the triangle is 2.

with period independent of the initial value x_0 (for $x_0 \neq k_1, k_2$).

If (2) is an elliptic projectivity with ω not a rational multiple of π, then $\{x_n\}$ (for a fixed real x_0) is everywhere dense.

Three further observations may be made. First, if in the sequence $\{x_n(x_0)\}$ determined by an initial value $x_0(\neq k_1, k_2)$, there exist some two terms that are equal

$$x_N(x_0) = x_{N+M}(x_0), \tag{13}$$

then $x_n(x_0)$ is a periodic function of n, for all x_0—for (13) implies that τ^M leaves fixed x_0 as well as the elements fixed under τ and hence τ^M is the identity. Second, if we consider the sequences $\{x_{-n}\}$ $(n>0)$, we note that the set of cluster points of $\{x_{-n}\}$ is of the same type as that of $\{x_n\}$, and that $\{x_{-n}\}$ converges if and only if $\{x_n\}$ converges. If τ is parabolic, both sequences converge to the same value, but if τ is hyperbolic and not an involution, $\lim x_{-n} = k_2$, where k_2 is the fixed element $\neq k_1$. Finally, we note that our geometric analysis of the sequence (1) can be made, without significant modification, when the variable and the parameters in (1) are complex. In complex projective space, a projectivity τ is either parabolic or nonparabolic. If τ is parabolic, it follows from (7a) that $\{x_n\}$ converges to the unique double element of τ, determined by (5). If τ is nonparabolic, we see from (7a) that we must distinguish between $|k| \neq 1$ and $|k| = 1$. If $|k| \neq 1$, we can suppose that $|k| > 1$, so that $y_n \to \infty$. Then $\{x_n\}$ converges to the solution k_1 of (5) given by (9) provided that the ambiguity in the determination of R in (9) is resolved by choosing for R the root that makes k, in (10), have absolute value greater than one. If, on the other hand, $|k| = 1$, we have two possibilities: either y_n is a periodic function of n or the values of y_n are everywhere dense on the circle $|y| = |y_0|$. But in the original x coordinate system, the circle $|y| = |y_0|$ is clearly the circle through x_0 orthogonal to the pencil of circles containing the two solutions of (5).

FINDING THE CARTESIAN EQUATION OF A LOCUS

E. B. Leach, Case Institute of Technology

Here is a method of finding an equation in rectangular coordinates of a plane locus for which we are given an equation in polar coordinates. It may be applied to a locus having a polar equation whose sides are polynomials in r and the sines and cosines of rational multiples of θ. The idea is to find an equation of the locus in r^2, $r\cos\theta$, and $r\sin\theta$, so that the Cartesian equation can be found simply by identifying these with x^2+y^2, x, and y, respectively.

Let $g(r, \theta) = 0$ be the given equation. If we replace θ by $\theta+\pi$ and r by $-r$, we get another equation of the same locus, $g_1(r, \theta) = 0$. On iterating the process we get a sequence of equations, each determining the same locus:

$$g(r, \theta) = 0, \quad g_1(r, \theta) = 0, \cdots, g_k(r, \theta) = 0,$$

k being chosen so that the transformation applied to $g_k(r, \theta)$ will give either $g(r, \theta)$, or $-g(r, \theta)$. We then form the product

$$G(r, \theta) = g(r, \theta) \cdot g_1(r, \theta) \cdot \cdots \cdot g_k(r, \theta).$$

The locus $G(r, \theta) = 0$ is evidently equivalent to the original given locus, and further, is not altered except for change of sign if we replace θ by $\theta + \pi$ and r by $-r$. It then appears that the coefficients of powers of r in $G(r, \theta)$ are periodic of period 2π, and so can be written in terms of $\sin \theta$ and $\cos \theta$. If G is then multiplied by a suitable power of r, the result can be expressed in terms of r^2, $r \cos \theta$ and $r \sin \theta$, and the identification described above can be made.

Example 1. $r = 1 + \cos \theta$. The two distinct forms given by the transformation are:

$$r - 1 - \cos \theta = 0$$

$$-r - 1 + \cos \theta = 0.$$

On multiplying these together and simplifying, we obtain:

$$\sin^2 \theta + 2r \cos \theta - r^2 = 0.$$

This is then multiplied by r^2 and we immediately see the Cartesian form of the equation:

$$y^2 + 2x(x^2 + y^2) - (x^2 + y^2)^2 = 0.$$

Example 2. $r = \sin(\theta/3)$. The different polar forms of the locus are:

$$r - \sin \frac{\theta}{3} = 0$$

$$-r - \sin \frac{\theta + \pi}{3} = 0$$

$$r - \sin \frac{\theta + 2\pi}{3} = 0.$$

Multiplying these together gives the complicated expression:

$$r^3 + r^2\left(-\sin\frac{\theta}{3} + \sin\frac{\theta+\pi}{3} - \sin\frac{\theta+2\pi}{3}\right) + r\left(-\sin\frac{\theta+\pi}{3}\sin\frac{\theta+2\pi}{3} + \sin\frac{\theta}{3}\sin\frac{\theta+2\pi}{3} - \sin\frac{\theta}{3}\sin\frac{\theta+\pi}{3}\right) + \sin\frac{\theta}{3}\sin\frac{\theta+\pi}{3}\sin\frac{\theta+2\pi}{3} = 0.$$

This can be simplified with the aid of various trigonometrical formulas to:

$$r^3 - \tfrac{3}{4}r + \tfrac{1}{4}\sin\theta = 0.$$

Then the Cartesian equation is found as in the first example to be:

$$(x^2 + y^2)^2 - \tfrac{3}{4}(x^2 + y^2) + \tfrac{1}{4}y = 0.$$

In the first example, it is seen that the main advantage of this method is that it avoids rationalization of an algebraic expression. The second example would be extremely difficult to carry out without the aid of a special device.

It should be noted that when multiplication by a power of r is required, the origin may be added to the original locus. In the above examples, the origin was already part of the locus, and so no alteration occurred. But the equation $r \sin\theta \cos^2\theta - 1 = 0$ gives a simple case where the origin appears in the Cartesian equation of the locus, but not in the polar equation.

THE EUCLIDEAN ALGORITHM

PHILIP FRANKLIN, Massachusetts Institute of Technology

Several elementary texts on number theory, such as the recent one by Harriet Griffin (1954) and the older one by R. D. Carmichael (1914), include the following as an exercise.

Prove that the number of divisions required to find the greatest common divisor of two positive integers written in the scale of 10 by means of the Euclidean algorithm does not exceed five times the number of digits in the smaller integer.

Both authors recognized this as a hard problem by its position at the end of a set, and Carmichael added a star. Since its difficulty for students is confirmed by experience in the classroom, a short solution may be of interest to teachers or students of number theory.

We first observe that a typical step of Euclid's algorithm at any stage uses the division transformation $a = qb + r$ to replace the pair a, b with $a > b$ by b, r with $b > r$. If $r < (5/8)b$, this single step reduces the smaller number in the ratio $r/b < 5/8$.

Next suppose that $r \geqq (5/8)b$. Then the following step of the algorithm uses $b = q_1 r + r_1$ with $q_1 = 1$ and $r_1 = b - r < (3/8)b$. This leads to the pair of numbers r, r_1 with $r > r_1$. And the smaller number has been reduced in the ratio $r_1/b \leqq 3/8$ for two steps. This is an average ratio of reduction less than 5/8 per step, since $3/8 < (5/8)^2$.

For five steps, the average ratio of reduction is less than $(5/8)^5 = 10^5/2^{20} = (1/10)(1000/1024)^2 < 1/10$. Hence for $5n$ steps the ratio will be less than 10^{-n}. Let n be the number of digits in the smaller integer of the original pair. Then the smaller integer initially had a value less than 10^n. Thus it would be reduced to less than unity if $5n$ steps were required. But the process stops when the remainder used as the new smaller number is zero. This proves that at most $5n$ steps will suffice.

The algorithm may be modified by using positive or negative remainders. In this case for a typical step we use $a = qb + r$ with $|r| \leqq b/2$ to replace the pair a, b with $a > b$ by $b, |r|$ with $b > |r|$. If $|r| < (3/7)b$, the first step reduces the smaller number in the ratio $|r|/b < 3/7$.

Next suppose that $|r| \geqq (3/7)b$. Then the following step of the algorithm uses $b = q_1|r| + r_1$, with $q_1 = 2$. Since $|r| \leqq b/2$, $r_1 \geqq 0$. And $r_1 = b - 2|r| < b - (6/7)b = b/7$. This leads to the pair of numbers $|r|$, r_1. And the smaller number has been reduced in the ratio $r_1/b \leqq 1/7$ for two steps. This is an average ratio of reduction less than $3/7$ per step, since $1/7 < (3/7)^2$.

For three steps, the average ratio of reduction is less than $(3/7)^3 = 27/343 < 1/10$. Hence for $3n$ steps the ratio will be less than 10^{-n}. This shows that if n is the number of digits in the smaller integer of the original pair, for the modified algorithm $3n$ steps will suffice.

The result for the Euclidean algorithm was found by G. Lamé in 1844. He used properties of the Fibonacci series in the proof. The result for the modified algorithm was found by Lionnet in 1857. Other references and details are given in Dickson's *History of the Theory of Numbers*, vol. I, pages 332 and 394.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1236. *Proposed by Hazel E. Evans, University of Pittsburgh*

For $a > b$ and $N < ab$ find the maximum value of N for which the equation

$$ax + by = N$$

has a solution in non-negative integers.

E 1237. *Proposed by Viktors Linis, University of Ottawa*

Let E be an ellipse, r_1 and r_2 focal radii, α the angle between the focal radii, and ds the element of arc. Evaluate the integrals

$$\int_E ds/(r_1 r_2)^{1/2} \quad \text{and} \quad \int_E (\cos \alpha/2)ds.$$

E 1238. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Determine integral values of $n>0$ such that 3^n, 3^{n+1}, 3^{n+2} all have the same number of digits in their denary expansions.

E 1239. *Proposed by Josef Langr, Prague, Czechoslovakia*

Let $Q' \equiv A'B'C'D'$ be the quadrangle formed by the orthocenters A', B', C', D' of triangles BCD, CDA, DAB, ABC of a given convex quadrangle $Q \equiv ABCD$. Show that: (1) the vertices of Q and Q' lie on a common equilateral hyperbola, (2) Q and Q' have equal areas.

E 1240. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Find six-piece dissections of a regular dodecagon into a square and into a Greek cross.

SOLUTIONS

Diophantine Consideration of the Optic Formula

E 1206 [1956, 186]. *Proposed by W. E. Briggs, University of Colorado*

If n is a given positive integer, how many integral solutions does

$$1/n = 1/x + 1/y$$

have with x and y positive and unequal?

Solution by N. J. Fine, University of Pennsylvania. Put $x=n+r$, $y=n+s$, with $r \neq s$. The given equation reduces to $n^2=rs$. Thus the required number of solutions is $d(n^2)-1$, where $d(N)$ is the number of divisors of N.

Also solved by J. L. Alperin, Leon Bankoff, G. E. Bardwell, H. W. Becker, D. M. Bloom, B. J. Boyer, A. L. Brown, D. H. Browne, Leonard Carlitz, W. B. Carver, George Cherlin, A. E. Danese, J. E. D'Atri and Stephen Lichtenbaum (jointly), G. W. Day, Monte Dernham, David DeVol, F. J. Duarte, Hazel E. Evans, Michael Goldberg, A. J. Goldman, D. S. Greenstein, Cornelius Groenewoud, E. H. Grossman, Nathaniel Grossmann, Virginia S. Hanly, B. A. Hausmann, R. E. Heaton, J. M. Howell, A. R. Hyde, F. I. John, M. S. Klamkin, Sam Kravitz, Aaron Lieberman, D. C. B. Marsh, Erich Michalup, J. B. Muskat, C. S. Ogilvy, Hyman Orlin, C. F. Pinzka, M. A. Rashid, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, C. M. Sandwick, Sr., Erwin Schmid, J. P. Scholz, John Selfridge, G. J. Simmons, Malcolm Smith, Art Steger, G. B. Thomas, G. W. Walker, Chih-yi Wang, R. M. Warten, William Zlot, and the proposer.

Becker called attention to L. E. Dickson, *History of the Theory of Numbers*, vol. II, pp. 688–691. Thomas expressed interest in the related equation

$$1/n^2 = 1/x^2 + 1/y^2.$$

A Locus Problem

E 1207 [1956, 186]. *Proposed by R. D. Gordon, California State Polytechnic College*

Given two mutually perpendicular lines l_1 and l_2 in a plane, and a point Q in the plane located equally distant from l_1 and l_2. Determine the locus of a point P in the plane if its distance from Q equals the sum of its distances from l_1 and l_2.

Solution by N. G. Gunderson, University of Rochester. Choose axes and units so that l_1 is the y-axis, l_2 is the x-axis, and Q is at (1, 1). Then $P:(x, y)$ is on the locus if and only if

$$|x| + |y| = [(x-1)^2 + (y-1)^2]^{1/2},$$

or, squaring, if and only if $|xy| = -x-y+1$, and so, if and only if it is a point on the hyperbola $(x+1)(y+1)=2$ for which $xy \geqq 0$ or else a point on the lines $(x-1)(y-1)=0$ for which $xy \leqq 0$.

Also solved by A. N. Aheart, J. L. Alperin, W. A. Al-Salam, Robert Bart, Julian Braun, D. H. Browne, W. B. Carver, A. E. Danese, J. E. Darraugh, J. E. D'Atri, A. G. Davis, Hüseyin Demir, Hazel E. Evans, David Freedman, Herta T. Freitag, Michael Goldberg, A. J. Goldman, Cornelius Groenewoud, E. H. Grossman, J. H. Holdsworth, R. A. Hoodes, R. H. D. Hou, A. R. Hyde, I. M. Isaacs, F. I. John, Virginia L. Johnson, M. S. Klamkin, D. C. B. Marsh, Beckham Martin, Morris Morduchow, C. S. Ogilvy, Hyman Orlin, D. J. Peterson, M. A. Rashid, B. E. Rhoades, Azriel Rosenfeld, J. W. Ross, Erwin Schmid, Sister M. Stephanie, Malcolm Smith, Chih-yi Wang, G. W. Walker, David Zeitlin, and the proposer.

This problem illustrates the pitfalls of "squaring out," and for this reason has been given by the Proposer to his classes as a *prize problem*. Many of the above solutions were only partially correct.

Covering a Closed Interval

E 1208 [1956, 186]. *Proposed by Gretchen Geller, Student, University of California, Davis, Calif.*

Can a closed interval I be covered by an infinite set J of nondegenerate closed intervals in such a manner that no finite subset of J covers I?

Solution by J. L. Brown, Jr., Pennsylvania State University. Let I be the closed interval $[-1, 1]$. Then the infinite sequence of nondegenerate closed intervals

$$[-1, 0], [1/2, 1,], [1/4, 1], \cdots, [1/2^n, 1], \cdots$$

has the required property.

Also solved by Peter Crawley, A. E. Danese, N. J. Fine, A. J. Goldman, R. L. Helmbold, P. G. Hodge, Jr., F. A. Homann, L. E. Isenecker, F. I. John,

B. E. Rhoades, L. A. Ringenberg, J. W. Ross, Malcolm Smith, David Zeitlin, and the proposer. Late solution by M. S. Klamkin.

This problem occurs, without solution, as prob. *n*, p. 32, in M. E. Monroe, *Introduction to Measure and Integration.*

A Cross Ratio Associated with Any Triangle

E 1209 [1956, 186]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F, and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC.

I. *Solution by W. B. Carver, Cornell University.* This is obviously a metrically special case of a more general projective theorem. The incircle may be replaced by *any conic* tangent to the sides at D, E, F, with the conic cutting the lines BE and CF at E' and F' respectively. By one of the limiting cases of Brianchon's theorem the lines AD, BE, CF meet in a point G. We set up a homogeneous coordinate system with A, B, C, G as the points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) respectively. It then follows readily that D, E, F are the points (0, 1, 1), (1, 0, 1), (1, 1, 0); the conic has the equation

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0;$$

E', F' are the points (1, 4, 1), (1, 1, 4); the lines through D have the equations $kx+y-z=0$ with $k=1, -1, -3, 3$ for DE, DF, DE', DF' respectively; and the required anharmonic ratio is therefore

$$(1 + 3)(-1 - 3)/(1 - 3)(-1 + 3) = 4.$$

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* By a central projection, triangle ABC and its incircle (I) can be transformed into an equilateral triangle and its incircle. The anharmonic ratio $D(E, F, E', F')$ is invariant under this transformation and consequently is constant for all triangles. It is easy to show that $D(E, F, E', F') = 4$.

Also solved by N. A. Court, P. W. M. John, D. C. B. Marsh, O. J. Ramler, Roscoe Woods, and the proposer.

A Dissection of a Pair of Equilateral Triangles

E 1210 [1956, 186]. *Proposed by Michael Goldberg, Washington, D. C.*

Given two equilateral triangles of edges a and b. Show how to dissect them by straight cuts into a total of no more than six pieces which can be assembled into another equilateral triangle. When the ratio of the larger to the smaller satisfies $a/b \geqq \sqrt{3}$, then five pieces suffice. When $a/b = 4/3$, four pieces suffice.

Solution by H. Lindgren, Patent Office, Canberra, Australia. The maximum number of pieces required is not six but five.

When $a/b = 4/3$, the triangle of side b is dissected as in Figure 1 into a trape-

zoid, and the other triangle is placed on AB. This is the special case for which four pieces suffice.

When $0<a/b<4/3$, the triangle of side b is dissected as in Figure 2 into a trapezoid whose top edge is equal to a, and again the other triangle is placed on AB. We have

$$AC = (a^2 + b^2)^{1/2} - a,$$

and the cuts are determined as follows:

$$DE = AC, \quad EF = FG, \quad GH = AC, \quad HK = KL.$$

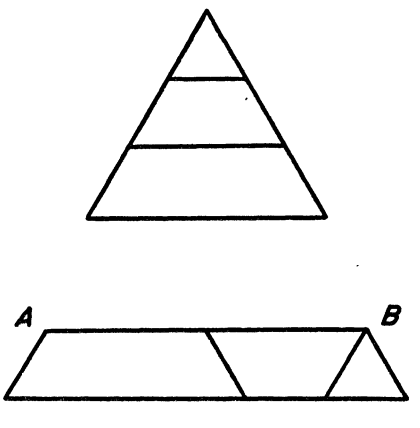


FIG. 1

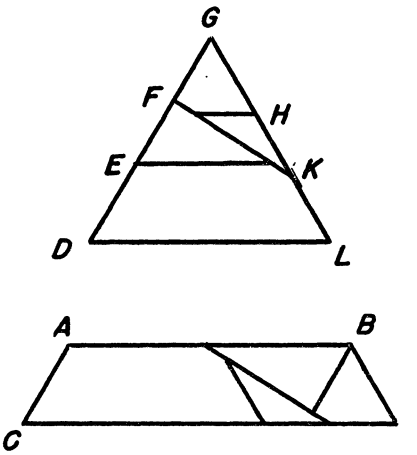


FIG. 2

This is the general case for which five pieces suffice. It is possible if F is not lower than H and K not higher than E. (The relative heights of E and H are immaterial.) Therefore it is possible if

$$DE \geqq (DG)/3,$$

that is, if

$$\begin{aligned}(a^2 + b^2)^{1/2} - a &\geqq b/3,\\ a^2 + b^2 &\geqq a^2 + 2ab/3 + b^2/9,\\ 4b &\geqq 3a.\end{aligned}$$

The edges of the triangles have been named so that this condition is satisfied.

Also solved by C. F. Pinzka and the proposer.

By an alternative method, called *superposition of strips*, Lindgren obtained other dissections. In the dissections of Pinzka and the Proposer, only the larger triangle is cut. Pinzka's dissection requires only five pieces when $a/b>4/3$. W. B. Carver pointed out that four pieces are sufficient when $a/b=\sqrt{3}$; one has merely to bisect each triangle by a cut along an altitude.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4708. *Proposed by H. A. Bender, University of Rhode Island*

Prove the following relationship:

$$\int_0^1 e^{c(1-x^2)}dx = \sum_{n=0}^{\infty} \frac{2^n c^n}{3\cdot 5\cdot\cdots\cdot(2n+1)}.$$

4709. *Proposed by R. W. Marsh, Arlington, Va. and A. M. Gleason, Harvard University*

Let $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ be an irreducible polynomial with coefficients in the finite field of q elements. Suppose that the roots of f are primitive; that is, they generate the multiplicative group of their extension field. Prove that

$$F(x) = x^{q^n-1} + a_{n-1}x^{q^{n-1}-1} + \cdots + a_0$$

is irreducible.

4710. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a complete quadrangle inscribed in a circle (O) one pair of opposite sides are isotomic lines with respect to a triangle inscribed in (O), then the remaining pairs of opposite sides are also isotomic lines with respect to the same triangle.

4711. *Proposed by Morgan Ward, California Institute of Technology*

If $\pi(n)$ is the number of prime numbers not greater than n, show that

$$\pi(n) < \frac{5}{4}\,\frac{n}{\log n}$$

for every composite number n.

4712. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that if $x_i \geqq 0$ $(i=1, \cdots, n)$ and $\sum_{i=1}^n 1/(1+x_i) \leqq 1$, then $\sum_{i=1}^n 2^{-x_i} \leqq 1$.

SOLUTIONS

Power Series Conditionally Convergent on the Unit Circle

4648 [1955, 496]. *Proposed by W. S. Loud, University of Minnesota*

A power series, $\sum_{n=0}^{\infty} a_n z^n$, has radius of convergence unity. It is known to converge at every point of its circle of convergence. Does it necessarily converge absolutely on the circle of convergence?

Editorial Note. It seems to be well known that a negative answer is called for. The following references are supplied by our contributors.

G. H. Hardy has shown that if we let $(1-z)^{-i}=\sum b_n z^n$, then the series $\sum(b_n/\log n)z^n$ converges conditionally and also uniformly on the circle of convergence. (See Dienes, *The Taylor Series*, pp. 464–465.)

Actually, problem 4449 [1952, 650] is equivalent to the present problem. There the series of Pringsheim is cited:

$$\sum_{n=1}^{\infty}(-1)^{[\sqrt{n}]}z^n/n.$$

It is convergent at every point of the unit circle. See also Pringsheim, *Vorlesungen über Funktionenlehre*, V. II, part 1, pp. 250–252; Knopp, *Theory and Application of Infinite Series*, p. 454; Knopp, *Problem Book in the Theory of Functions*, p. 31.

Another example is given by F. Herzog and G. Piranian, *Duke Mathematical Journal* (1949), p. 532. For schlicht Taylor series which converge uniformly but not absolutely on the unit circle, see P. Erdös, F. Herzog and G. Piranian, *Pacific Journal of Mathematics* (1951) pp. 75–82; also D. Gaier, *Math. Zeitschrift* (1953), pp. 349–350.

Power Series with Uniformly Bounded Partial Sums

4649 [1955, 496]. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Stratford, Conn.*

Does the fact that a power series has uniformly bounded partial sums on the circle $|z|=1$ necessarily imply that it converges everywhere on $|z|=1$?

Editorial Note. The conclusion is not necessary. References sent in by readers include: Herzog and Piranian, *Duke Mathematical Journal*, 20 (1953) pp. 45–50; Erdös, Herzog and Piranian, *Math. Scand.*, 2 (1954), pp. 262–264.

Change of Sign in a Determinant

4650 [1955, 496]. *Proposed by J. L. Massera, Instituto de Matematica Y Estadistica, Montevideo, Uruguay*

Let $p_1, \cdots, p_k$ be any fixed indices, $0 \leqq p_1 < p_2 < \cdots < p_k < n$ and let $D(x_1, \cdots, x_k)$ be the nth order determinant whose elements a_{ij} are defined by: $a_{ij}=x_h$ if $i+j=p_h+1$, $h=1, \cdots, k$; $a_{ij}=-x_h$ if $i+j=n+p_h+1$, $h=1, \cdots, k$; $a_{ij}=0$ otherwise. Show that D, considered as a function of the real variables

$x_1, \cdots, x_k$, may change its sign only in the neighborhood of those values which satisfy one of the relations $x_1 \pm x_2 \pm \cdots \pm x_k = 0$.

Solution by the Proposer. D vanishes if and only if

$$x_1 z^{p_1} + x_2 z^{p_2} + \cdots + x_k z^{p_k} = 0, \tag{1}$$

where z is an nth root of -1. This follows either directly from the factorization of D given by R. F. Scott, Note on a determinant theorem of Mr. Glaisher's, *Quarterly Journal of Mathematics*, 17 (1880) pp. 129–132, or by writing down the Euler resultant of the system formed by (1) (considered as an equation of degree $n-1$) and $z^n+1=0$ and then making suitable elementary transformations in this resultant which show that it is equal to $\pm D$.

Let $q_h = p_h - p_i$; the x_h being real, we will have

$$\begin{aligned} x_1 + x_2 z^{q_2} + \cdots + x_k z^{q_k} &= 0, \\ x_1 + x_2 \bar{z}^{q_2} + \cdots + x_k \bar{z}^{q_k} &= 0. \end{aligned} \tag{2}$$

Suppose the point $(x_1, \cdots, x_k)$ in k-dimensional real space satisfies (2) where z is an nth root of -1. If the rank of the matrix

$$\begin{pmatrix} 1 & z^{q_2} \cdots z^{q_k} \\ 1 & \bar{z}^{q_2} \cdots \bar{z}^{q_k} \end{pmatrix}$$

is 2, the set of points which satisfy (2) is a linear variety of dimension $k-2$; and since this variety does not separate the k-dimensional space, in the neighborhood of such points D cannot change its sign. If the rank is 1, then $z^{q_h} = \bar{z}^{q_h}$ which implies $z^{q_h} = \pm 1$ and one of the relations $x_1 \pm x_2 \pm \cdots \pm x_k = 0$ is satisfied.

Equilateral Triangles on a Parallelogram

4651 [1955, 496]. *Proposed by the late Joseph Rosenbaum*

On the sides of a parallelogram $A_1A_2A_3A_4$, equilateral triangles $A_iA_{i+1}B_i$ are constructed exteriorly. Then equilateral triangles $B_iB_{i+1}C_i$ are constructed interiorly to $B_1B_2B_3B_4$. Prove that $C_1C_2C_3C_4$ coincides with $A_1A_2A_3A_4$.

Generalize to the case where the original parallelogram is replaced by a polygon of n sides.

Solution by Joseph Langr, Prague, Czechoslovakia. Let the sides of the given parallelogram be $A_1A_2 = A_3A_4 = a$, $A_2A_3 = A_4A_1 = b$, and let angle $A_4A_1A_2 = \theta$. The three triangles $A_1B_1B_4$, $A_4A_3B_4$, $A_2B_1A_3$ are congruent, having $A_1B_1 = A_3A_4 = A_2B_1 = a$, $A_1B_4 = A_4B_4 = A_2A_3 = b$, and angles $B_4A_1B_1 = B_4A_4A_3 = B_1A_2A_3 = \theta + 120°$. Thus $B_1B_4 = B_4A_3 = B_1A_3$ and $A_3 \equiv C_4$. In the same manner we can show that A_1, A_2, A_4 coincide with C_2, C_3, C_1.

The situation is not essentially different if the triangles $A_iA_{i+1}B_i$ are constructed interiorly with respect to parallelogram $A_1A_2A_3A_4$, and the proof goes through unaltered. One must note, however, that it depends upon the shape of the original parallelogram whether triangles $B_iB_{i+1}C_i$ must be constructed ex-

teriorly or interiorly (with respect to $B_1B_2B_3B_4$) in order to guarantee the coincidence of A_i with C_{i+1}.

The same argument shows that, if we start with $A_1A_2 \cdots A_n$ and generate $C_1C_2 \cdots C_n$ by the above construction, each C_i is the fourth vertex of the parallelogram two of whose sides are adjacent sides of the given n-gon.

Also solved by Michael Goldberg, and R. L. Helmbold.

Editorial Note. The generalization intended by the proposer is the following. *Let $A_1A_2 \cdots A_n$ be a projection of a regular n-gon, and on the sides A_iA_{i+1} as bases construct exteriorly isosceles triangles $A_iA_{i+1}B_i$, where the vertex angle B_i is such that*

$$\sin \frac{B_i}{2} = \frac{\sqrt{2}}{2} \sin \frac{\pi}{n};$$

and on the sides B_iB_{i+1} as bases construct interiorly isosceles triangles $B_iB_{i+1}C_i$ similar to $A_iA_{i+1}B_i$; then when n is even the n-gons $C_1C_2 \cdots C_n$ and $A_1A_2 \cdots A_n$ are coincident, and when n is odd the two n-gons are congruent with corresponding sides parallel.

A Theorem on Measure

4652 [1955, 496]. *Proposed by Albert Wilansky, Lehigh University*

Let f be a Lebesgue integrable function on an interval. Let t_n be the measure of the set of points x such that $|f(x)| > n$. Prove that $\sum t_n < \infty$.

Solution by S. I. Goldberg, Wayne University, Detroit. This is an immediate consequence of the following well known result given as a problem in Titchmarsh, *The Theory of Functions*, 2nd ed., p. 343: *If $f(x)$ is a measurable function over an interval and u_n is the measure of the set of points x such that $n-1 \leqq f(x) < n$, the necessary and sufficient condition that $f(x)$ be integrable is that $\sum_{n=-\infty}^{\infty} |n| u_n$ be convergent.*

To see this, note that $\sum t_n = \sum |n| u_n$.

Also solved by P. T. Bateman, Joshua Barlaz, G. U. Brauer and Chih-yi Wang, W. M. Faucett, Harley Flanders, A. J. Goldman, C. D. Gorman, D. S. Greenstein, P. R. Halmos, J. Horváth, B. C. Kenny, William Kruskal, A. E. Livingston, Neill McShane, C. M. Pearcy, H. G. Tucker, and the Proposer.

Editorial Note. The result is given as an exercise (accompanied by a strong hint of the solution) in Halmos, *Measure Theory* (New York, 1950), p. 115. The exercise points out that the condition of problem 4652 is sufficient as well as necessary, and also, for complete precision, requires the interval to be bounded.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All new material for this section should be sent to the incoming editor, Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

Fundamental Concepts of Geometry. By B. E. Meserve. Addison-Wesley Publishing Company, Cambridge, Massachusetts, 1955. ix+321 pages. $7.50.

In this work one finds material which should be valuable for anyone teaching mathematics, especially geometry, as it shows the relationship of several types of geometries to one another.

The aims of this book, according to the author, are to help the reader

1. Discover how euclidean plane geometry is related to, and often a special case of, many geometries,
2. Obtain a practical understanding of "proof,"
3. Obtain the concept of a geometry as a logical system based upon postulates and undefined elements, and,
4. Appreciate the historical evolution of our geometrical concepts and the relation of euclidean geometry to the space in which we live.

The first chapter of this text is devoted to a study of various sets of postulates, some of their consequences, and the associated logic. Then in Chapter Two synthetic projective geometry is developed from the postulational viewpoint. Later coordinate systems are introduced, and some of the basic properties of analytic projective geometry are developed. The relation of this geometry to the synthetic projective geometry already considered is then discussed. Affine geometry is next considered as a special case of analytic projective geometry, and euclidean geometry is developed as a special case of affine geometry. The last three chapters of the book are devoted to a historical sketch of the development of geometry, a brief discussion of some non-euclidean geometries, and a descriptive approach to topology.

Klein's definition of a geometry, "The study of the invariants of a configuration under a group of transformations" is frequently mentioned, and the emphasis is on the study of the transformations used in the various geometries and the invariants that exist under these transformations. Many of the basic properties of the various geometries are developed, but no attempt is made to study completely any one of the geometries under consideration. However, enough of each geometry is developed so that the restrictions which need to be imposed on it to obtain another geometry are evident.

From the pedagogical standpoint, it should be noted that the proofs of some of the theorems needed in the development of the text are left as exercises for

the student. Also, there is a paucity of illustrative examples, which might make it difficult for an embryonic mathematician to see what is wanted or how to proceed in some of the exercises. As far as mathematical techniques are concerned, only those of College Algebra are needed, but a greater mathematical background is needed for an appreciation of the material presented.

R. G. SANGER
Kansas State College

Analytic Geometry and Calculus. By T. S. Peterson. New York, Harpers Mathematics Series. Harper and Brothers, Publishers, 1955. 456 pages. $6.50.

Over-all picture of text. This text covers the analytic-geometry topics, the calculus topics, and the introduction-to-differential-equations found in traditional, approved analytic-geometry and calculus texts. Formulas and curves of more elementary mathematics together with tables, mathematical symbols, and the Greek alphabet are conveniently located. The order of topic-presentation and the pattern of exposition (theory, then illustrations, examples and solutions) are conventional.

Concerning the author's objectives. As to the design, purpose, and intended rigor, Peterson writes, "This book is designed to serve as an introductory course in Analytic Geometry and Calculus for students who have some familiarity with algebra and trigonometry," and "It is the purpose of the book to present Analytic-Geometry and Calculus not only as a powerful tool that can be used in applied fields but also as an important branch of mathematical analysis. To serve this latter purpose, all definitions, theorems, and general procedures are as accurately presented as the demands of clarity and simplicity permit."

Peterson achieves the objective of the first quotation by writing in a simple and concise manner. In fact Peterson writes so concisely that one is likely on first reading to conclude that the book is too abridged. However, on a second reading one realizes that there is still much material in this text. In fact, any student who thoroughly masters all the material in Peterson's text will be definitely superior to the average run-of-the-mill calculus student.

In the second quotation, the decision to develop rigor compatible with clarity and simplicity seems wise in view of the deficiency in self-discipline prevalent in today's students in many classes.

Concerning controversial definitions. Now by way of discussing a few topics that are of interest to writers and teachers, I present this paragraph.

In defining *function*, some authors emphasize that a function is the *relationship, law or correspondence between two variables*, or is a *set of number-pairs.* Peterson writes as follows, "If two variables x and y are related so that, for each x in a range R of real numbers, we obtain one or more real values for y, then y is said to be a real function of the real variable x defined over the range R." Possibly the presentation of limits on page 73 could be improved by elaboration. Peterson does not emphasize a distinction between integration and antidifferentiation. He does not distinguish between iterated and multiple integrals. In

writing today there are new ways of defining *versus* traditional ways of defining. Both can be criticized. Peterson in most cases adheres to the traditional.

Comments and conclusions. More youths are kept in school today, and hence more non-students are found in our classes. Also the multitudinous distractions in the form of extra-curricular activities have caused some of our good students to expect more of the text and the professor. Hence great moderation must be exerted in the extent of rigor, ramifications, and details when writing some texts. The reviewer feels that Peterson's text is very appropriate for many of today's students.

E. M. PEASE
University of Rhode Island

Analytic Geometry, By N. H. McCoy, and R. E. Johnson. New York, Rinehart and Company, 1955. 301+xiv pages. $4.00.

Analytic Geometry, New Second Edition, By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1955. 310+ix pages. $3.75.

Each instructor has his own idea concerning the teaching of analytic geometry, and from the great wealth of textbooks, he is sure to find one to suit his purposes. As most students study analytic geometry as preparation for the calculus, the instructor is guided in his selection as to the purposes of his students, the basic ideas, and the presentation of the methods for graphing. A textbook then stands or falls according to the preferences of the instructor.

McCoy and Johnson in the preface state: "The emphasis throughout the book is on an understanding of the basic principles of analytic geometry." The text contains the traditional material, but in an order which does not always force the best understanding of the basic principles. In the discussion of conic sections, only the simplest standard forms are discussed very fully. Two chapters later, the standard form is again introduced after the translation of axes has been discussed. There seems to be a tendency throughout the textbook to encourage the students to substitute in memorized relations derived from proven theorems. For example, the center and the radius of the circle $x^2+y^2+Dx+Ey+F=0$ are found from the relations $(-D/2, -E/2)$ and $\frac{1}{2}\sqrt{D^2+E^2-4F}$ rather than the completion of the squares in each individual case.

The discussion of the general properties of curves in Chapter 3, entitled "Equations and Graphs", concerns itself with the intercepts, extent, and symmetry, but only of the conic sections. It is felt that the rapid sketching of curves is a most important contribution of analytic geometry to the student's future work in mathematics and should not be confined to only the simplest of curves. The authors seem however to give only "lip service" to these properties, since throughout the text extensive tables of values are compiled (some entries to three decimal places) and used in the sketching of algebraic and transcendental curves. These tables of values have a place in introducing rectangular coordinates in algebra courses but are not desirable in analytic geometry.

The format of the text is excellent in general, with large, clear type and excellent illustrations especially in the section devoted to solid analytic geometry. Some of the earlier illustrations, *i.e.*, in the chapter on polar coordinates, are too extensive and could gain from judicious elimination of excess information.

The preface of the textbook by Middlemiss sets forth his plan and arrangement, and states in part: "The original edition of this text was written with the primary aim of giving students a better preparation for calculus and the sciences. To this end, less than the usual amount of space was devoted to the conic sections—and more than the average amount was devoted to the polynomials and rational fractional functions and to the graphs and properties of the exponential, logarithmic, and trigonometric functions. . . . All these characteristics of the original text have been retained in this new edition."

A brief introduction compiled of essential definitions and formulas from algebra and trigonometry include the evaluation of determinants by minors, the definitions of trigonometric functions, and the fundamental relations between the trigonometric functions.

In Chapter 4, the way is paved easily to the general discussion of curves through the polynomials, while the more formalized discussion of intercepts, symmetry, extent, and asymptotes, is introduced in Chapter 5 using rational fractional functions. The commendable practice of orderly discussion of curves in the numerous examples throughout the text should greatly aid those students who have hitherto had only a vague notion of what was required in rapid sketching.

The introduction of transformation of coordinates before the discussion of the circle and the other conic sections permits a fuller discussion of the general standard forms together with the simpler forms.

A fine feature of the text is a more complete discussion of trigonometric, exponential, and logarithmic curves than is ordinarily found in analytic geometry texts. The essential properties of lines, planes, and surfaces required in the calculus have been included in the section on solid analytic geometry.

Students using the text by Middlemiss will have a clearer understanding of the nature of graphs and equations and a more solid foundation for future application and study. The less important aspects of the curves have been relegated to the problem sets, which are numerous and varied.

The format of the text is excellent, with clear-cut and ample illustrations.

E. F. MYERS
University of Pittsburgh

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

MATHEMATICS DIVISION OF ASEE

The Mathematics Division of the American Society for Engineering Education met on June 25–29, 1956, at Iowa State College. Four well-attended sessions were held, including a joint conference with the Iowa Section of the Mathematical Association of America. The following new officers of the Division were elected at the annual business meeting: Chairman, Professor W. G. Warnock, Rensselaer Polytechnic Institute; Secretary, Professor W. E. Restemeyer, University of Cincinnati; Director, Professor G. B. Thomas, Massachusetts Institute of Technology; Director, Professor C. O. Oakley, Haverford College; Director and Representative to General Council of ASEE, Dr. R. S. Burington, Department of Navy. The next annual meeting of the Mathematics Division of the ASEE will be held in June, 1957, at Cornell University. For further information write to Professor W. E. Restemeyer, University of Cincinnati.

MATHEMATICS TEACHING ESSAY CONTEST

Kappa Mu Epsilon and the Science Teaching Improvement Program of the American Association for the Advancement of Science are cooperating in the sponsorship of an essay contest on "Opportunities in Teaching Mathematics in Secondary Schools." Satisfactory essays will be published in *The Pentagon*. First prize in the contest will be $50. There will be second and third prizes of $25 and $15, respectively.

The Mathematics Teaching Essay Contest is planned to increase interest in the teaching of mathematics at the secondary-school level, and to encourage all students in mathematics to consider the advantages of a career in secondary-school mathematics teaching. It is also hoped that the preparation, as well as the reading, of the essays may attract good students with an aptitude for and interest in mathematics to enter the teaching profession. The importance of the ability to express oneself in writing, particularly on the part of teachers, should also be emphasized by such an essay contest.

Essays submitted in the contest should reach Professor Carl V. Fronabarger, Southwest Missouri State College, Springfield, Missouri, no later than April 1, 1957. They must be not more than 1,000 words in length, and should be typed double-spaced on a good grade of paper. Four copies should be submitted by each contestant. The content of the essay should be as specific as possible, and point out the advantages of preparation for the teaching of mathematics at the secondary-school level. The essay may consider one or more of the special facets

of the profession of mathematics teaching, or cover the general area as completely as the length of the essay will permit. Undergraduate and graduate students in mathematics are eligible to enter the contest. The essays will be judged on accuracy and objectivity of the data presented, the degree to which the essay appears to be convincing in the case presented for mathematics teaching, and composition and neatness.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1956 Preliminary Actuarial Examination are as follows:

First Prize of $200

Pratt, Richard L.	Washington University

Additional Prizes of $100 each

Brillinger, David R.	University of Toronto
Earle, Clifford J., Jr.	Swarthmore College
Kaplan, Stanley	Cornell University
Mosher, Robert E.	Kenyon College
Riehm, Carl R.	University of Toronto
Rubin, Jerrold	Columbia University
Schweitzer, Paul A.	Holy Cross College
Soderquist, George D.	Drake University

The Society of Actuaries has authorized a similar set of nine prizes for the 1957 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination.
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. General Mathematics Examination.
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. Special Mathematics Examination.
(Finite differences, probability and statistics.)

The 1957 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 15, 1957. Applications should be submitted before April 1, 1957, to the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

PERSONAL ITEMS

Fresno State College announces the following: Dr. S. J. Bryant, formerly a teaching assistant at the University of Missouri, Dr. V. E. Howes, and Dr. T. C. Kipps of the University of Santa Clara have been appointed to instructorships; Assistant Professor G. D. Alkire has been promoted to an associate professorship.

Purdue University reports: Associate Professor R. A. Oesterle of Eastern Oregon College of Education has been appointed Assistant Professor of Mathematics and Education; Mr. G. D. Miller, previously principal of Shadeland High School, Indiana, and Dr. Judah Rosenblatt have been appointed to instructorships; Dr. Rosenblatt is also a consultant in the Statistical Laboratory.

At San Diego State College: Dr. C. V. Holmes of Northrop Aircraft and Assistant Professor Margaret Willerding of Harris Teachers College have been appointed to assistant professorships; Mr. Roger Shaw of the University of California has been appointed to an instructorship; Associate Professor A. R. Harvey has been promoted to a professorship; Dr. L. J. Warren has been promoted to an assistant professorship.

Southwest Texas State Teachers College announces: Mr. W. C. Akin, Mrs. Saphrona B. Krause, and Mr. A. W. Spear have been appointed to instructorships.

Mr. D. S. Adorno of the University of Utah has been appointed to an instructorship at Iowa State College.

Mr. F. T. Baker, previously a student at Yale University, has a position as a programmer with the International Business Machines Corporation, New York, New York.

Mr. J. M. Behr, formerly a technical writer for Librascope, Glendale, California, has a position as a computer applications engineer with the Royal Precision Corporation.

Assistant Professor Dean Clifton Benson of South Dakota School of Mines and Technology has been appointed to an assistant professorship at Chico State College.

Mr. W. S. Bicknell of Wayne University has been appointed Assistant Professor of Commerce and Actuarial Science at the University of Wisconsin.

Mr. I. W. Boxer of California State Polytechnic College has been appointed to an instructorship at El Camino College.

Mrs. Carolina M. Brennan has been appointed to an instructorship at the University of the Philippines.

Professor A. R. Brown, Jr. of Drury College has accepted a position as chief, Office of Ballistic Computations, Air Force Armament Center, Eglin Air Force Base, Florida.

Mr. Donald Brown, previously a student at Hofstra College, is employed now as an aeronautic research intern by the National Advisory Committee on Aeronautics, Langley Field, Virginia.

Mrs. Mary H. Brown of Florida State University has been appointed to an instructorship at Perkinston Junior College.

Assistant Professor W. O. Buschman of Portland State College has been appointed to an assistant professorship at California State Polytechnic College.

Mr. C. N. Campopiano, formerly a project engineer for the Sperry Gyroscope Company, Great Neck, New York, has accepted a position as a research associate at the Micro-wave Research Institute of the Polytechnic Institute of Brooklyn.

Associate Professor F. L. Celauro of East Tennessee College has been promoted to a professorship.

Mr. E. W. Cheney, recently a teaching fellow at the University of Kansas, has a position as a senior research engineer with Consolidated-Vultee Aircraft Corporation, San Diego, California.

Assistant Professor Eckford Cohen of the University of South Carolina has been appointed to an assistant professorship at the University of Tennessee.

Assistant Professor R. M. Conkling of the University of New Hampshire has been appointed to an assistant professorship at New Mexico College of Agriculture and Mechanic Arts.

Mr. B. G. Cooper, formerly a graduate assistant at Kansas State College, is a teacher at Coronado High School, California.

Dr. F. W. Donaldson, previously a senior aerophysics engineer for Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas, is now Supervisor of the Computing Laboratory, General Electric Company, Philadelphia, Pennsylvania.

Mr. Willard Draisin, formerly a student at Brooklyn College, is employed as a staff member of Lincoln Laboratories, Lexington, Massachusetts.

Dr. Jacqueline P. Evans of Smith College has been appointed to an assistant professorship at Wellesley College.

Professor S. H. Gould of Williams College has been appointed Executive Editor of Mathematical Reviews.

Mr. E. P. Graney is employed as an analyst by the Westinghouse Electric Corporation, Kansas City, Missouri.

Dr. R. C. Gunning of the University of Chicago has been appointed a Higgins Lecturer at Princeton University.

Dr. N. A. Hall, assistant dean of the College of Engineering, New York University, has been appointed Professor and Chairman of the Department of Mechanical Engineering of Yale University.

Dr. Melvin Hausner of Brooklyn College has been appointed to an assistant professorship at Stevens Institute of Technology.

Mr. F. E. Hazlett, previously a computer applications engineer at Librascope, Glendale, California, has a position as a computer applications engineer with the Royal Precision Corporation.

Mr. Simon Hellerstein, formerly a graduate assistant at Syracuse University, has been appointed to an instructorship at the University.

Brigadier General Harris Jones, professor of mathematics and dean of the U. S. Military Academy, has retired.

Mr. Edward Keegan, previously a graduate student at Kansas State Teachers College, is employed as a mathematician by the Coast and Geodetic Survey, Washington, D. C.

Associate Professor L. G. Kelly of Clemson Agricultural College has been appointed a research scientist at Lockheed Missiles Systems Division, Mathematical and Computing Center, Palo Alto, California.

Dr. A. V. Kozak, chairman of the Department of Mathematics and engineering officer of Concord College, has been granted a leave of absence to do additional graduate study and to work as a member of the operations research staff at the Glenn L. Martin Company, Baltimore, Maryland.

Assistant Professor L. T. LaBorde of the University of the South has been appointed to an assistant professorship at the University of Cincinnati.

Captain W. T. Lee, recently a teacher at Oklahoma Military Academy, has a position as a nuclear engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. J. L. Lewis, Jr., has a position as an engineer for Westinghouse Electric Corporation, Baltimore, Maryland.

Mr. B. W. Marks, formerly a teacher at San Jacinto Junior High School, Midland, Texas, has a position as an analytical chemist at Continental Oil Company, Westlake, Louisiana.

Dr. Ceslovas Masaitis of the University of Tennessee has a position as a mathematician at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Dr. L. F. McAuley of the University of Maryland has been appointed to an instructorship at the University of Wisconsin.

Professor C. N. Mills of Augustana College has been appointed Visiting Lecturer at Florida State University.

Dr. C. V. Newsom, formerly Executive Vice-President, has been elected President of New York University. Dr. Newsom served as Editor of this MONTHLY from 1947 to 1951.

Dr. R. H. Owens, recently a mathematician at the Office of Naval Research, Washington, D. C., has been appointed to an assistant professorship at the University of New Hampshire.

Dr. Edgar Reich of the Rand Corporation has been appointed to an assistant professorship at the University of Minnesota.

Mr. L. A. Rife of Mississippi Southern College has been appointed to an assistant professorship at Clemson Agricultural College.

Dr. P. D. Ritger of Stevens Institute of Technology has been promoted to an assistant professorship.

Mr. D. H. Schmieder, formerly a graduate assistant at the University of Kentucky, is a physicist at the Army Ballistic Missile Agency, Huntsville, Alabama.

Miss Janet H. Scott (Mrs. Allsbrook), previously a graduate assistant at the University of South Carolina, is now a mathematician at the David Taylor Model Basin, Washington, D. C.

Mr. R. M. Shank, formerly a graduate student at the University of Kentucky, is teaching at Nitro High School, West Virginia.

Mr. G. J. Simmons, previously a graduate student at the University of Oklahoma, is employed as a research scientist at the Lockheed Missile Division, Van Nuys, California.

Mr. R. L. Slater, Jr., formerly an assistant engineer for Chicago Midway Laboratory, Chicago, Illinois, is now an associate laboratory director at Bjorksten Research Laboratories, Madison, Wisconsin.

Dr. Paul Slepian of Brown University is employed now as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor R. W. Sloan of the University of New Hampshire has been appointed to an assistant professorship at Carleton College.

Mr. R. A. Spong, previously a mathematician at the U. S. Navy Underwater Sound Laboratory, New London, Connecticut, has accepted a position as a scientist-mathematician in the Special Projects Division of the Research and Development Department of the Electric Boat Division, General Dynamics Corporation, Groton, Connecticut.

Mr. Vivian Spurgeon has been appointed to an assistant professorship at East Texas Baptist College.

Dr. R. D. Stalley of Fresno State College has a position as a mathematician at the U. S. Naval Ordnance Test Station, China Lake, California.

Mr. Robert Tates, recently a graduate student at Syracuse University, is now an engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Associate Professor H. E. Taylor of Florida State University is on leave during 1956–57 as a Carnegie Intern in General Education and is Visiting Associate Professor at the University of Chicago.

Mr. Jack Warga of the Electro Data Corporation, Pasadena, California, is on leave of absence on a Chaim Weizmann Memorial Fellowship at the Weizmann Institute of Science, Rehovoth, Israel.

Dr. Ti Yen of Lehigh University has been appointed to an assistant professorship at Illinois Institute of Technology.

President Emeritus F. C. Ferry of Hamilton College died on August 14, 1956. He was a charter member of the Association.

Professor Emeritus L. C. Karpinski of the University of Michigan died on January 25, 1956. He was a charter member of the Association.

Professor Emeritus William Marshall, who was formerly Head of the Department of Mathematics of Purdue University, died on July 31, 1956.

Sir Edmund T. Whittaker of Edinburgh, Scotland, died on March 24, 1956.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SEVENTH SUMMER MEETING OF THE ASSOCIATION

The thirty-seventh summer meeting of the Mathematical Association of America was held at the University of Washington, Seattle, Washington, on Monday and Tuesday, August 20 and 21, 1956, in conjunction with the summer meetings of the American Mathematical Society, the Biometric Society, the Econometric Society, and the Institute of Mathematical Statistics. There were 919 persons registered, including 303 members of the Association.

Sessions of the Association were held on Monday morning and afternoon and on Tuesday morning in the Guggenheim Auditorium of the University of Washington. President W. L. Duren and former Vice-President F. L. Griffin presided on Monday morning, Professor W. A. Golomski on Monday afternoon, and Vice-President G. B. Price and President Duren on Tuesday morning. The fifth series of Earle Raymond Hedrick Lectures was delivered by Professor J. C. Oxtoby of Bryn Mawr College. The Program Committee for the meeting consisted of C. R. Wylie, Jr., Chairman, C. L. Clark, W. A. Golomski, and F. E. Hohn.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Category and Measure"; Lecture I, by Professor J. C. Oxtoby, Bryn Mawr College.

"The Metropolitan New York Mathematical Competition," by Professor W. H. Fagerstrom, City College of New York.

"The Stanford University Competitive Examination in Mathematics," by Professor H. M. Bacon, Stanford University.

"The William Lowell Putnam Mathematical Competition," by Professor L. E. Bush, Kent State University.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Oxtoby.

"Dynamic Programming," by Dr. Richard Bellman, Rand Corporation. (By title.)

"The Practice of Mathematics," by Dr. R. E. Gaskell, Boeing Airplane Co.

"Games of Survival," by Dr. L. S. Shapley, California Institute of Technology.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Oxtoby.

Business Meeting of the Association.

"Report from the Commission on Mathematics of the College Entrance Examination Board," by Dean A. E. Meder, Rutgers University, and Professor G. B. Thomas, Massachusetts Institute of Technology.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the lounge of McKee Hall in the Women's Residence Hall, with twenty members present. Among the more important items of business transacted were the following:

The Board voted to accept the resignation of Professor Ralph Hull as Governor from the Indiana Section and elected Professor P. D. Edwards of Ball State Teachers College to the Board for the balance of the unexpired term.

The Secretary-Treasurer was instructed to prepare amendments to the By-Laws to be voted on at the annual meeting of December 1956 providing for a more specific statement of the control of the Association over its sections, and for an increase in the annual dues to five dollars and in the dues of emeritus members who receive the MONTHLY to two dollars. The Board also voted to increase the annual subscription rate of the MONTHLY to $6 effective January 1, 1957.

It was voted to hold the Forty-first Annual Meeting at the University of Cincinnati and the Hotel Sheraton-Gibson, Cincinnati, Ohio, on Friday, January 31, 1958.

Professor Leo Zippin of Queens College was invited to deliver the Hedrick Lectures at the 1957 Summer Meeting.

Approval was voted for the holding of a meeting for the organization of a New Jersey Section of the Association, at which tentative by-laws are to be adopted and officers elected.

The Board voted that the Association will sponsor an annual contest in mathematics for secondary school students in the United States and Canada, the first such contest to be held in 1958 under the supervision of a standing committee on High School Contests.

BUSINESS MEETING OF THE ASSOCIATION

At a business meeting held on Tuesday morning, announcement was made that a grant of $1,250 had been received from the National Science Foundation for the support of the work of the Committee on Mathematical Personnel and Education. It was also announced that the membership of the Association was 6,461 on August 6.

The following committee chairmen reported on the work of their respective committees: B. W. Jones for the Committee on Visiting Lecturers; W. L. Duren

for the Committee on the Undergraduate Program in Mathematics; G. B. Price for the Committee to Study the Activities of the Association; Tomlinson Fort for the Committee on Mathematical Personnel and Education; R. P. Bailey on High School Contests; and J. S. Frame for the Committee on Employment Opportunities.

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the lounge of McKee Hall. About thirty persons were present representing 22 of the 26 sections.

Among the topics discussed were the various types of contests now being conducted by the sections, the recruitment of college mathematics teachers, revision of the undergraduate mathematics curriculum, and programs of sections. Emphasis was placed on traveling lectureships and other contacts with high school teachers and students in order to make known the many professional opportunities now open to mathematicians.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday afternoon. The colloquium lectures on "Harmonic analysis and probability" were delivered by Professor Salomon Bochner of Princeton University.

Meetings of the Biometric Society, the Econometric Society, and the Instistitute of Mathematical Statistics were also held during the week of August 20.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: A. E. Livingston, Chairman; R. A. Beaumont, Z. W. Birnbaum, D. G. Chapman, D. B. Dekker, H. M. Gehman, V. L. Klee.

Registration headquarters was in the Student Union Building of the University of Washington. The Employment Register and the Book Exhibit were on display in the Union Building. Dormitory accommodations were available in the Women's Residence Hall and meals were served in the Union.

The facilities of the Officers' Club at the Sand Point Naval Air Station were available for an informal party on Tuesday evening. A salmon bake was held on Wednesday afternoon and evening at Seward Park. A boat cruise along the Seattle waterfront and around Bainbridge Island was held on Thursday.

A motion prepared by Professor Tomlinson Fort was adopted by the five mathematical organizations, expressing thanks to the members of the Department of Mathematics of the University of Washington and to the ladies of the department for their labor with hand and heart for the comfort and happiness of the visiting mathematicians.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 44 persons have been elected to membership by the Board of Governors on applications duly certified.

R. L. ADLER, A.M. (Columbia) United States Army.

E. A. ANDERSON, JR., Student, Lebanon Valley College.

SEYMOUR BACHMUTH, B.S. (Brooklyn) Brooklyn, New York.

D. J. BOEDICKER, B.S. (LeMoyne) Instrument Engr., International Business Machines Corp., Poughkeepsie, N. Y.

B. R. BUZBY, B.S. (Antioch) Grad. Student, Indiana University.

A. A. CAPORASO, M.S. (N.Y.U.) Math., International Business Machines Corp., New York, N. Y.

JULIANNE COLLINS, Student, Cardinal Stritch College.

C. M. CONDIT, Student, Idaho State College.

W. M. CUNNEA, S.M. (Chicago) Teaching Asst., University of Chicago.

F. C. DESUA, Ph.D. (Pittsburgh) Asst. Professor, University of Pittsburgh.

A. J. GRUBER, Student, Kent State University.

S. M. HENDLEY, M.A. (South Carolina) Engr., Westinghouse Electric Corp., East Pittsburgh, Pa.

P. G. HEYDA, B.S. (London) Math., deHavilland Aircraft Co., Hatfield, Herts, England.

MRS. ALICE A. HUFFMAN, Student, University of California, Riverside.

J. P. JACOB, M.A. (Columbia) Teacher, St. Luke's School, New Canaan, Conn.

W. J. KELSEY, B.A. (Hunter) Junior Actuary, State Insurance Fund, New York, N. Y.

M. J. KEMP, Student, University of Utah.

FRANK LEVIN, Ph.D. (Cincinnati) Asst. Professor, University of Kentucky.

SADIA M. MAKKY, M.S. (Queen Alia C., Iraq) Grad. Student, Indiana University.

H. W. MARTIN, B.A. (West Texas S.C.) Teacher, Clarendon Junior College, Texas.

G. E. MCCUNE, B.A. (Kansas) Pvt., United States Army, Fort Bliss, Texas.

D. S. MCMANUS, B.S. (Columbia) Asst. Professor of Engineering, Norwich University.

E. G. MCNIEL, M.S. (DePaul) Instr., DePaul University.

E. W. MEHNER, B.S. (Case I.T.) Res. Engr., National Cash Register Co., Hawthorne, Calif.

E. L. MESSERE, Student, University of Rhode Island.

C. E. MILLER, Ph.D. (Chicago) Res. Math., California Research Corp., Richmond, Calif.

G. H. PAVLAKOS, M.S. (DePaul) Asst. Professor, Elmhurst College.

GEORGE PEPPERDINE, II, Student, George Pepperdine College.

RONALD PYKE, M.S. (Washington) Res. Asst., University of Washington.

THEODOR RANOV, Dr.Ing. (Technical U., Berlin) Professor of Engineering, University of Buffalo.

L. E. REYNOLDS, M.Ed. (Alfred) Chairman of the Department of Mathematics, State University of New York, Agricultural and Technical Institute, Alfred.

W. C. ROUNDS, Math. Instr., Ford Apprentice Training School, Chicago, Ill.

R. M. SCAMURRA, B.A. (Buffalo) Junior Math., Cornell Aeronautical Lab., Buffalo, N. Y.

S. H. SCHOT, M.A. (Maryland) Res. Asst., University of Maryland.

H. M. SCOFIELD, B.Ch.E. (Cornell) Chemical Engr., Linde Air Products, Buffalo, N. Y.

DONNA J. SEAMAN, B.S. (Washington) Physicist, U. S. Naval Proving Ground, Dahlgren, Va.

PEDRO SEN, B.S.E.E. (Mapua I.T.) Instr., University of Santo Tomas, Manila, Philippines.

SEYMOUR SHERMAN, S.M. (Harvard) Staff Member, Dian Labs., New York, N. Y.

E. E. SHULT, B.A. (Southern Illinois) Res. Asst., Biological Research Lab., Southern Illinois University.

R. S. SPIRA, Student, University of California, Berkeley.

THOMAS THEOCLITUS, B.S. (Ohio U.) Dynam-

ics Engr., Bell Aircraft Corp., Niagara Falls, N. Y.

G. P. TRICOLES, A.B. (U.C.L.A.) Res. Engr., Convair Division, General Dynamics Corp., San Diego, Calif.

MRS. MARY K. TULOCK, M.A. (George Peabody) Asst. Professor, California State Polytechnic College.

L. A. WELLER, M.A. (Toronto) Physicist, Foster Wheeler Corp., New York, N. Y.

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-fourth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Marquette University, Milwaukee, Wisconsin, on May 7, 1956. Professor A. C. Moeller, Chairman, presided. There were 66 present, including 41 members.

The following officers were elected for the coming year: Chairman, Mr. C. J. Vanderlin, Jr., Wisconsin State College, Whitewater; Vice-Chairman, Mr. Joseph Kennedy, Wisconsin High School, Madison; Secretary-Treasurer, Sister Mary Felice, Mount Mary College, Milwaukee.

The following papers were presented:

1. *Probability and statistics in general education*, by Professor J. V. Talacko, Marquette University, Milwaukee.

Mathematical probability and statistics are an important part of general education. Statistics is a coherent, uniform, mathematical science based on the laws of probability. Institutions of higher learning should have in their curriculum courses in probability and statistics having two objectives: (1) to give students basic instruction in the concepts and methods of statistics, so they may be able to understand statistical reasoning and be able to apply its methods in the fields of their specialization; (2) to train professional, applied or theoretical statisticians.

The number of courses will depend on the size of the institution and on the determination of objectives.

2. *What is a surface?* by Professor L. C. Young, University of Wisconsin, introduced by Professor R. C. Wagner.

The basic notion, originating with Sophus Lie, is that of a weighted set of pairs x, $J(x)$, where x is a point and $J(x)$ determines a positively oriented plane element through x. A surface is defined as such a weighted set subject to an irreducibility condition (akin to connectedness) and a boundary condition. It is derivable from a traditional surface of finite type by an arbitrarily small modification in area. The definition is sufficiently general to ensure, for the first time, the existence of a solution to Plateau's problem without restrictions of topological type. More general weighted sets define "generalized" surfaces.

3. *Geometries*, by Mr. C. J. Vanderlin, Jr., Wisconsin State College, Whitewater.

The paper presented a brief survey of various geometries and their relationships. An outline of Artin's proof that a projective plane can be co-ordinatized over a division ring if and only if Desargue's theorem is satisfied was given. This division ring is commutative if and only if the Pappus-Pascal theorem is satisfied. Following Klein's Erlanger program, affine, unimodular affine, euclidean, metrical, and non-euclidean geometries were presented as sub-geometries of projective geometry.

4. *The Science Teaching Improvement Program of the American Association for the Advancement of Science*, by Dr. J. R. Mayor, Director of Program, Washington, D. C.

Professional scientific organizations are now more deeply concerned about science and mathematics education in the secondary schools than at any other time in this century. Scientists recognize that they have a responsibility for the improvement of secondary-school teaching which has not been adequately met. Throughout the nation there is evidence of new interest in teacher education and of a desire on the part of college and university staff members in academic departments to cooperate with secondary-school teachers and administrators in improving secondary-school programs. If this interest and spirit of cooperation can be properly coordinated and fostered, desirable modification of some goals and trends in secondary education that have been questioned by scientists could result.

5. *Special Mathematics Programs*

a. Report by Professor P. C. Rosenbloom, University of Minnesota.

Since April, 1955, special programs in mathematics have been conducted for superior high school, and even younger, students with special classes weekly, one for high school students, one for junior high, and some for 4th to 6th grade children. Attendance is purely voluntary, but has consistently averaged 40 in the first two groups and 20 in the third. This program was set up with the cooperation of H. O. Jackson, recently appointed consultant in mathematics to the Minneapolis public school system, and F. Roessel, principal of Lincoln Junior High School. The importance of having people in the school systems with primary responsibility for the gifted students was stressed.

b. Report by Mr. Joseph Kennedy, Wisconsin High School, Madison.

Mr. Kennedy described briefly an experimental twelfth grade course at Wisconsin High School. This course included a number of topics from modern mathematics; the text was *Principles of Mathematics* by Allendoerfer and Oakley. It is difficult to draw conclusions from one year's work, but it appeared that students grasped abstract ideas easily, their approach to problems became less inhibited, they displayed knowledge of important concepts, and they found the course an enjoyable experience. This teacher believes that almost any twelfth grade class would profit from a course based on this text, although some topics would have to be omitted for a less able group.

7. *A universal metric for experimental sciences, Biological time and gamma function, Elementary problems in theory and elasticity, On the geometry of* $\Omega = z^2$, by Professor Jose Gallego-Diaz, Special School of Agricultural Engineering, Madrid, Spain. (Presented by title.)

SISTER MARY FELICE, *Secretary*

ORGANIZATIONAL MEETING OF THE NEW JERSEY SECTION

At a meeting of New Jersey mathematicians to be held November 3, 1956, at Rutgers University a motion to organize a New Jersey section of the Association will be entertained.

Dean A. E. Meder, Jr., is temporary chairman; Professor C. A. Nelson is chairman of the committee on by-laws; Professor I. L. Battin is chairman of the nominating committee; and Professor D. R. Davis is chairman of the program committee.

I. L. BATTIN, *Temporary Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Spring, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS

KENTUCKY, Berea College, Berea, April 20, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College of William and Mary, Williamsburg, Virginia, December 1, 1956.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, March 23, 1957.

MINNESOTA

MISSOURI, Southeast Missouri State College, Cape Girardeau, Spring, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 27, 1957.

NEW JERSEY, Organizational Meeting, Rutgers University, November 3, 1956.

NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.

NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, Muhlenberg College, Allentown, Pennsylvania, November 24, 1956.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, Spring, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May, 1957.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9, 10 and 11 additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. The most recent issues are:

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

No. 11. *Irrational Numbers* by Ivan Niven, xii+164 pages.

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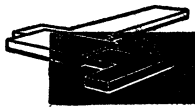
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VOLUME 63 NUMBER 9

PART II

The Conjugate Coordinate System for Plane Euclidean Geometry

WALTER B. CARVER

Number 5
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

NOVEMBER 1956

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THE CONJUGATE COORDINATE SYSTEM FOR PLANE EUCLIDEAN GEOMETRY

By

WALTER B. CARVER

Professor of Mathematics, Emeritus, Cornell University

The Fifth

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

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PREFACE

The author of this paper claims no originality for either the method or the geometric results presented. He is attempting an elementary exposition of a method of handling certain types of problems in plane Euclidean geometry. The method was used by Professor Frank Morley in a number of papers published just before and after the turn of the century, and some explanation of it is given in the second part of his book on *Inversive Geometry*. Morley passed the method on to his students at Johns Hopkins University, and some of them have made use of it in papers appearing in this MONTHLY, the *American Journal of Mathematics* and other periodicals.

The method is, in the opinion of the author, sufficiently simple for presentation to undergraduates with only the first course in calculus and an interest in mathematics as prerequisites; and the material of this paper is based largely on notes for a one-semester course which the author gave several times at Cornell University to mixed classes of undergraduate and graduate students. The paper is addressed to that level of mathematical maturity.

The intention is to present the method rather than geometric results. Most of the latter may be found in books using synthetic methods, such as Altshiller-Court's *College Geometry* or Johnson's *Modern Geometry*. Such books will furnish the interested reader with further geometric theorems the proof of which will be good exercises in the use of conjugate coordinates.

The author wishes to acknowledge the help of Professor Howard Eves, who read the manuscript and made a number of helpful suggestions.

W. B. C.

Chapter I

COMPLEX NUMBERS AND THEIR GRAPHICAL REPRESENTATION

1.1. We first call the attention of the reader to certain facts, probably already familiar to him, about complex numbers and their graphical representation in the plane.

A complex number a is a number of the form $\alpha+i\beta$, where α and β are real and $i^2=1$. The same complex number may be expressed in two other forms, thus

$$a = \alpha + i\beta = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta},*$$

where ρ and θ are real numbers, $\rho \geqq 0$, and $\cos\theta$, $\sin\theta$ mean the cosine and sine of an angle of θ radians; and we have the relations

$$(1.11)\quad \begin{cases} \rho = \sqrt{\alpha^2+\beta^2}, & \cos\theta = \dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \qquad \sin\theta = \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}, \\ \alpha = \rho\cos\theta, & \beta = \rho\sin\theta. \end{cases}$$

If a is zero, $\alpha=\beta=\rho=0$, and θ is any real number whatever. For a given complex number not zero, α, β, and ρ are uniquely fixed, but θ may have any one of an infinite set of values differing from each other by multiples of 2π. We call ρ the *absolute value* (or *modulus*) of a, and θ the *amplitude* of a, and we write

$$\rho = |a|, \qquad \theta = \operatorname{amp} a.$$

If two complex numbers,

$$a_1 = \alpha_1 + i\beta_1 = \rho_1 e^{i\theta_1}$$

and

$$a_2 = \alpha_2 + i\beta_2 = \rho_2 e^{i\theta_2},$$

are equal, we must have

$$(1.12)\qquad \alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 = \beta_2;$$

and also

$$(1.13)\qquad \begin{array}{ll} \text{either} & \rho_1 = \rho_2 = 0, \\ \text{or} & \rho_1 = \rho_2 \neq 0 \end{array}$$

with θ_1 and θ_2 equal or differing only by a multiple of 2π.

* The relation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

may be regarded as the definition of $e^{i\theta}$. See D. R. Curtiss: Analytic Functions of a Complex Variable, p. 51; and Goursat-Hedrick: Mathematical Analysis, vol. II, part I, p. 25.

1.2. A one-to-one correspondence is established between complex numbers and the points of a plane as follows. Let a polar coordinate system be superposed on a rectangular Cartesian system with the polar axis coinciding with the posi-

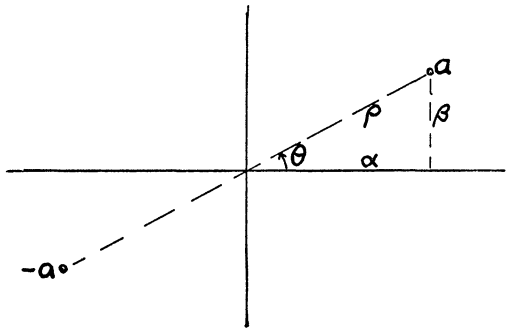


FIG. 1

tive end of the x-axis. Then the complex number a corresponds to the point whose Cartesian coordinates are (α, β) and polar coordinates are (ρ, θ), θ in radians. We may then speak of the *point* a, meaning the point corresponding to

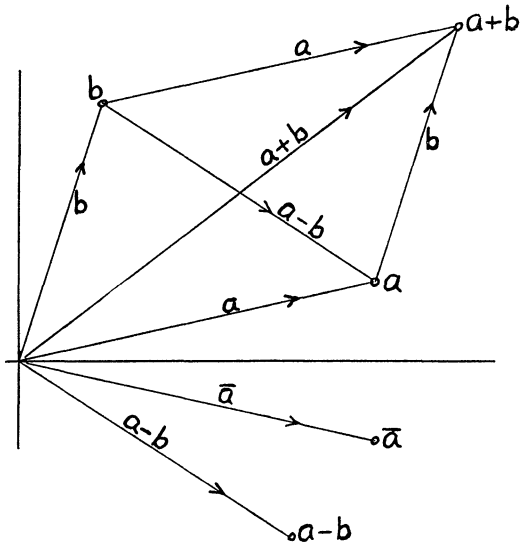


FIG. 2

the complex number a in the above sense. The points a and $-a$ are symmetric with respect to the origin.

We shall also make use of a correspondence between complex numbers and *vectors* in the plane. The complex number a corresponds to the vector having

the length and direction of the vector from the origin to the point a. (We use *vector* in the sense of a *free vector*, the vector being uniquely determined by its length and direction without regard to the position of its initial point.) Because of this correspondence we may speak of the *vector* a. The usual graphical addition and subtraction of vectors will correspond to the algebraic addition and subtraction of the complex numbers. Frequent use will be made of the fact that the vector from the *point* b to the *point* a is the vector $(a-b)$.

1.3. The notation $\bar{a}$ will be used to indicate the complex number *conjugate* to a, *i.e.*,

$$\text{if} \qquad a = \alpha + i\beta = \rho e^{i\theta},$$
$$\text{then} \quad \bar{a} = \alpha - i\beta = \rho e^{-i\theta}.$$

The point $\bar{a}$ is the reflection of the point a in the x-axis. The sum and the product of two conjugate complex numbers, $a+\bar{a}$ and $a\bar{a}$, are always real, and $a\bar{a} \geqq 0$. The real numbers α, β, ρ and θ may be expressed in terms of a and $\bar{a}$ as follows:

$$\alpha = \frac{1}{2}(a + \bar{a}), \qquad \beta = \frac{1}{2i}(a - \bar{a}),$$
$$\rho = \sqrt{a\bar{a}}, \qquad \theta = -i \log \frac{a}{\sqrt{a\bar{a}}}. \tag{1.31}$$

If $a=\bar{a}$, $\beta=0$ and a is real.

If $R(z)$ is a rational function of z, $\overline{R}(z)$ will indicate the function obtained by replacing each coefficient in $R(z)$ by its conjugate. Then the conjugate of $R(a)$ is $\overline{R}(\bar{a})$.

1.4. If k is real and not zero, the direction of the vector ka is the same as, or opposite to, that of the vector a according as k is positive or negative; and the length of the vector ka is $|k|$ times that of the vector a.

If the complex number b is multiplied by the complex number $a=\rho e^{i\theta}$, the vector ab corresponding to the product is obtained graphically by turning the vector b through an angle θ and stretching (or contracting) it to ρ times its original length. If $\rho=1$, so that $a=e^{i\theta}$, the vector ab is obtained by merely turning the vector b without changing its length. For this reason a complex number of the form $e^{i\theta}$, with absolute value unity, will be called a *turn*, and will usually be represented by the letter t. A turn corresponds to a point on the unit circle or to a vector of unit length. Since $\sqrt{t\bar{t}}=1$, $\bar{t}=1/t$, *i.e.*, the conjugate of a turn is its reciprocal; and, conversely, if $\bar{a}=1/a$, then a and $\bar{a}$ are turns. The product of two or more turns is a turn, and if t is a turn then t^n is a turn for any real n. The quotient of any number by its conjugate is a turn, $a/\bar{a}=t$, and amp $t=2$ amp a.

1.5. If $t=e^{i\theta}=\cos\theta+i\sin\theta$ and m is real, then

$$t^m = (e^{i\theta})^m = e^{i(m\theta)} = \cos m\theta + i\sin m\theta.* \tag{1.51}$$

Putting $m=1/n$, n an integer greater than 1, we have

$$t^{1/n} = (e^{i\theta})^{1/n} = e^{i\theta/n},$$

and for a particular value of θ this gives *one* of the nth roots of t. But if we write, as we may,

$$t = e^{i(\theta+2\pi s)}, \qquad s \text{ any integer},$$

we have

$$t^{1/n} = e^{i(\theta+2\pi s)/n},$$

and for $s=0, 1, 2, \cdots, n-1$, this gives the n *different* nth roots of t, all of them turns. The graphical representation of these n roots is interesting. The turn t corresponds to a point on the unit circle with amplitude θ, $0<\theta\leqq 2\pi$. The point on the unit circle with amplitude θ/n will correspond to one of the nth roots of t.

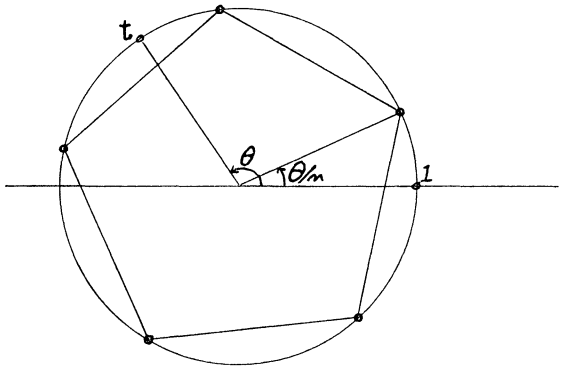


FIG. 3

With this point as one vertex, a regular polygon of n sides may be inscribed in the circle, and the vertices of this polygon will correspond to the n different nth roots of t.

If $t=1$ we have the important case of the nth roots of unity, roots of the equation $z^n=1$, represented by the n vertices of a regular n-gon with one vertex at the point 1. If $\epsilon=e^{2\pi i/n}$, the n different nth roots of unity are

$$\epsilon, \epsilon^2, \epsilon^3, \cdots, \epsilon^{n-1}, \epsilon^n, \qquad (\epsilon^n = 1)$$

---

* The trigonometric relation

$$(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$$

for m a positive integer is known as De Moivre's theorem, and is found in many textbooks in College Algebra and Trigonometry.

and the sum of these n roots is zero. Certain relations among them are quite obvious either algebraically or graphically. For instance

$$\epsilon^{k+n} = \epsilon^k$$

$$\frac{1}{\epsilon^k} = \overline{\epsilon^k} = \epsilon^{n-k}.$$

If n is even, $\epsilon^{n/2} = -1$, and $-\epsilon^k = \epsilon^{k+n/2}$.

We shall make considerable use of the three cube roots of unity,

$$1, \qquad \omega = e^{2\pi i/3}, \qquad \omega^2 = e^{4\pi i/3}$$

which satisfy the relations

$$(1.52) \qquad \begin{cases} 1 + \omega + \omega^2 = 0, \\ \overline{\omega} = \dfrac{1}{\omega} = \omega^2, \quad \overline{\omega^2} = \dfrac{1}{\omega^2} = \omega. \end{cases}$$

The n different nth roots (n a positive integer) of any complex number $a = \rho e^{i\theta}$ are, similarly,

$$\sqrt[n]{\rho}\, e^{i(\theta+2\pi s)/n}, \qquad s = 0, 1, 2, \cdots, n-1,$$

where $\sqrt[n]{\rho}$ means the positive nth root of the positive number ρ; and these n roots correspond to the vertices of a regular n-gon inscribed in a circle of radius $\sqrt[n]{\rho}$ with center at the origin.

1.6. To define the (natural) logarithm of a complex number we agree that when $z = e^u$ then $u = \log z$. Then since

$$t = e^{i(\theta+2\pi s)}, \qquad s \text{ any integer},$$

we have $\log t = i(\theta + 2\pi s)$. Thus, for a fixed t, $\log t$ has infinitely many values differing from each other by multiples of $2\pi i$, and $2\pi i s$, for s any integer, is a logarithm of 1.

1.7. We give some simple examples to show how methods using complex numbers and vectors may be useful in handling geometric problems in the plane.

Two points A and B (Figure 4) are given by the complex numbers a and b; and we wish to find the complex number c corresponding to the point C on the segment AB which divides this segment in the given ratio $m:n$, m and n *positive* numbers.

The vectors from A to C and from C to B correspond respectively to the numbers $c-a$ and $b-c$. Then since these vectors have the same direction (see section 1.4), we have

$$c - a = \frac{m}{n}(b - c)$$

$$nc - na = mb - mc$$

$$(m + n)c = mb + na$$

and

$$c = \frac{mb + na}{m + n}. \tag{1.71}$$

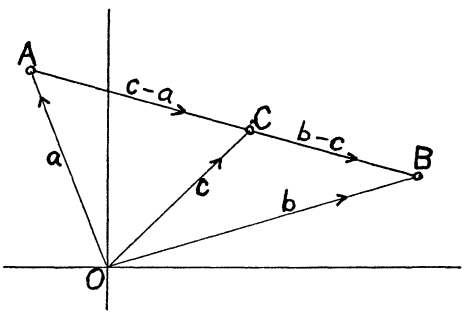


FIG. 4

As a particular case, the point

$$c = \frac{a + b}{2} \tag{1.72}$$

is the *mid-point* of the segment AB.

Using 1.72 and 1.71 one proves readily that if points a, b, c are the vertices of a triangle the point $(a+b+c)/3$ is the *centroid* of the triangle (center of gravity, point of intersection of the medians).

As another example, let us find the condition on the complex numbers a, b, c such that the triangle with vertices at the three corresponding points may be an equilateral triangle with the vertices a, b, c in counterclockwise order.

It is evidently necessary and sufficient that the vector $a-c$ shall be equal to the vector $c-b$ turned through an angle $2\pi/3$; that is

$$a - c = e^{2\pi i/3}(c - b),$$

or

$$a - c = \omega(c - b),$$

and we have

$$a + \omega b - (1 + \omega)c = 0,$$

or (see 1.52)

$$a + \omega b + \omega^2 c = 0, \tag{1.73}$$

as the required condition.

Similarly,

$$a + \omega^2 b + \omega c = 0 \tag{1.74}$$

is the condition that the points a, b, c should be the vertices of an equilateral triangle in clockwise order.

The reader may use 1.73 to prove

THEOREM 1.75. *If on the sides of any triangle, equilateral triangles are constructed outwardly, the centers of the equilateral triangles will be the vertices of an equilateral triangle.*

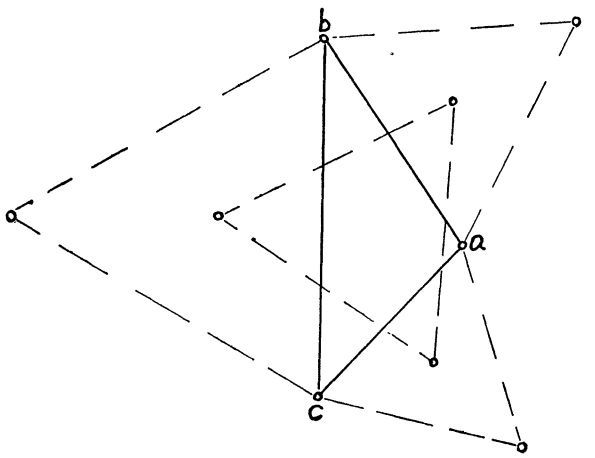


FIG. 5

Chapter II

THE CONJUGATE COORDINATE SYSTEM

2.1. The geometry we shall study will be the *real* Euclidean geometry in the Euclidean plane. A rectangular Cartesian coordinate system is first set up, the Cartesian coordinates of any point in the plane being a pair of *real* numbers, (X, Y). The *conjugate* (or *circular*) *coordinates*, (x, y), of any point are then defined by the transformation

$$\begin{cases} x = X + iY \\ y = X - iY \end{cases}, \tag{2.11}$$

and we have the inverse transformation

$$\begin{cases} X = \dfrac{1}{2}(x + y) \\ Y = \dfrac{1}{2i}(x - y) \end{cases}. \tag{2.12}$$

Since X and Y are always real, x and y are always conjugate complex numbers. The origin is the point whose coordinates are (0, 0) in both systems. Either x or y alone is sufficient to fix the point (x, y), the coordinate x being the complex number corresponding to the point in the sense explained in the preceding chapter; and we may designate a point as the point (x, y) or simply as the point x.

The distance between the points $(a, \bar{a})$ and $(b, \bar{b})$ is the length of the vector $a-b$, and is therefore (see 1.31)

$$d = \sqrt{(a - b)(\bar{a} - \bar{b})}. \tag{2.13}$$

Also the coordinates of the point which divides the segment from $(a, \bar{a})$ to $(b, \bar{b})$ in the ratio $m:n$, m and n positive, are (see 1.71)

$$\left(\frac{na + mb}{m + n}, \frac{n\bar{a} + m\bar{b}}{m + n}\right). \tag{2.14}$$

2.2. We shall be interested in certain algebraic curves, curves which, in the Cartesian coordinate system, have equations of the form

$$P(X, Y) = 0, \tag{2.21}$$

where $P(X, Y)$ is a proper polynomial* in X and Y with *real* coefficients. From the form of the transformations 2.11 and 2.12 it is evident that such a curve will have an equation in the conjugate coordinate system,

$$p(x, y) = 0, \tag{2.22}$$

where $p(x, y)$ is a proper polynomial of the same degree as $P(X, Y)$ but with

* We use the phrase "proper polynomial" in the sense of a polynomial *of degree greater than zero*, so that a constant (which is a polynomial of degree zero) is not a *proper* polynomial.

complex coefficients. But we will show that a proper polynomial equation in x and y does not correspond to a curve unless it satisfies certain conditions.

We assume for the moment that the polynomial $p(x, y)$ is irreducible, *i.e.*, that it is not the product of any two proper polynomials. If the equation $p(x, y) = 0$ represents a curve it must be satisfied by infinitely many *conjugate* pairs of values x_1, y_1, and if $p(x_1, y_1) = 0$ then also $\bar{p}(y_1, x_1) = 0$. Thus the two equations $p(x, y) = 0$ and $\bar{p}(y, x) = 0$ would have infinitely many common solutions, and the polynomials $p(x, y)$ and $\bar{p}(y, x)$ would have a common proper polynomial factor.* But since $p(x, y)$, and therefore also $\bar{p}(y, x)$, is irreducible, it follows that these polynomials are identical except for a possible constant factor, *i.e.*,

$$\bar{p}(y, x) \equiv \lambda p(x, y), \qquad \lambda \text{ a constant} \neq 0. \tag{2.23}$$

When this condition is satisfied we say that $p(x, y)$ is a *self-conjugate polynomial* and that $p(x, y) = 0$ is a *self-conjugate equation*. We have then shown that:

If $p(x, y)$ is an irreducible† polynomial, a necessary‡ condition that $p(x, y) = 0$ be the equation of a curve is that $p(x, y)$ be self-conjugate.

2.3. The equation of a straight line in conjugate coordinates will obviously be linear, $ax + by + c = 0$; and since it must also be self-conjugate we must have

$$\bar{a}y + \bar{b}x + \bar{c} \equiv \lambda(ax + by + c),$$

or

$$\bar{a} = \lambda b, \qquad \bar{b} = \lambda a, \qquad \bar{c} = \lambda c, \qquad \lambda \neq 0. \tag{2.31}$$

Since a and b cannot both be zero, these conditions imply that *neither a nor b* can be zero. Also, since $\bar{a}/\bar{b} = b/a$ and $\bar{c}/\bar{a} = c/b$, it follows that a/b is a turn and that c/a and c/b are conjugates.

Conversely, it is easily seen (by transforming to Cartesian coordinates) that the equation $ax + by + c = 0$ always corresponds to a straight line when conditions 2.31 are satisfied.

The equations $x = 0$ and $y = 0$ are not self-conjugate and hence do not represent lines, and we do not speak of an x-axis or y-axis. Instead we make use of the *axis of reals* with the equation $x - y = 0$, and the *axis of pure imaginaries* with the equation $x + y = 0$, and these are respectively the X-axis and Y-axis of the Cartesian system.

---

* Bôcher, Introduction to Higher Algebra, p. 211.

† If $p(x, y)$ is reducible it is not necessary that it be self-conjugate but only that one (or more) of its factors be self-conjugate. Thus $2ix^2 - xy + iy^2 \equiv (2ix + y)(x + iy)$ is not self-conjugate but the factor $x + iy$ is, and $x + iy = 0$ is the equation of a line every point of which satisfies the equation $2ix^2 - xy + iy^2 = 0$.

‡ The condition is not sufficient except for polynomials of the first degree. Thus

$$2xy - ix + iy + 4 = 0$$

is a self-conjugate equation, but it is not satisfied by any pair of *conjugate* numbers x, y, and hence does not represent a curve.

Chapter III

THE STRAIGHT LINE

3.1. A different approach to the straight line will illustrate methods to be used later in the treatment of other curves.

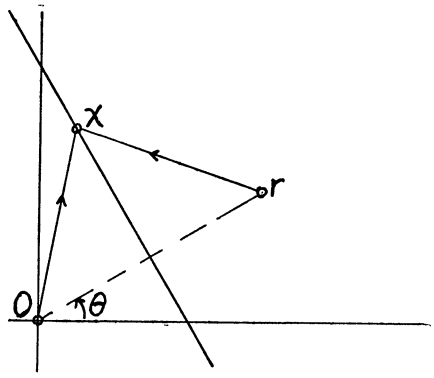


FIG. 6

By the *reflex point* of a line we shall mean the reflection of the origin in the line. If the line goes through the origin, the reflex point is the origin itself. Let r be the reflex point of a line not through the origin, $r=\rho e^{i\theta}$, $\rho \neq 0$, and let x be a variable point on the line. Then the vectors x and $x-r$ are of the same length, and we have

$$x - r = tx,$$

where t is a turn running through all turn values (except the value 1) as the point x runs along the line. Solving for x we have

$$x = \frac{r}{1-t}. \tag{3.11}$$

Such an equation, expressing the x coordinate of a variable point on a curve as a function of a variable *turn* t, will be called a *map equation* of the curve. If x is equal to $r/(1-t)$, then the conjugate of x must be equal to the conjugate of $r/(1-t)$; in other words, the map equation 3.11 implies the conjugate equation

$$y = \frac{\bar{r}}{1-\frac{1}{t}} = \frac{t\bar{r}}{t-1}, \tag{3.12}$$

and 3.11 and 3.12 together give us a pair of parametric equations of the line.

Eliminating the parameter t from the two equations, we obtain a self-conjugate equation of the line,

$$\bar{r}x + ry = r\bar{r}.$$

Since $r \neq 0$, we may divide by r, obtaining

$$\frac{\bar{r}}{r}x + y = \bar{r}.$$

Now $\bar{r}/r(=e^{-2i\theta})$ is a fixed turn τ, and $\bar{r}=\tau r$, and hence a self-conjugate equation of the line is obtained in the standard form

$$\tau\bar{x} + y = \bar{\tau}r. \tag{3.13}$$

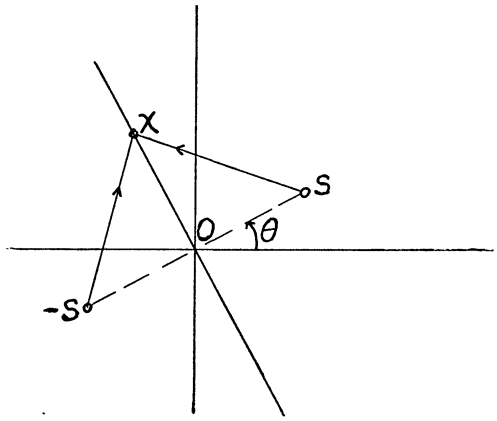


FIG. 7

If the line passes through the origin, $r=0$, and certain steps in the above procedure would not be valid. In this case let s be any point on the perpendicular to the line at the origin, $s=\rho e^{i\theta}$, $\rho \neq 0$. We then have

$$x - s = t(x + s),$$

or

$$x = \frac{s(1+t)}{1-t}, \tag{3.14}$$

as a map equation of the line. This implies the conjugate equation

$$y = \frac{\bar{s}\left(1+\frac{1}{t}\right)}{1-\frac{1}{t}} = \frac{\bar{s}(t+1)}{t-1},$$

and eliminating t as before we have the self-conjugate equation

$$\bar{s}x + sy = 0.$$

Letting $\bar{s}/s = \tau$, this becomes

$$\tau x + y = 0,$$

which agrees with 3.13 when $r = 0$. Hence equation 3.13 is a general standard form for any straight line, whether or not it goes through the origin.

3.2. The turn τ is called the *clinant* of the line, and it gives the orientation* of the line in somewhat the same way as the *slope* gives the orientation of a line in the Cartesian system. It will be helpful to the reader later if he will make himself thoroughly familiar with the relation between the clinant τ of a line and the orientation of the line. The simplest way to see this relationship is to picture it graphically. The following figure shows for certain values of the turn τ segments of lines having the clinant τ. It may be noted that as the turn τ runs

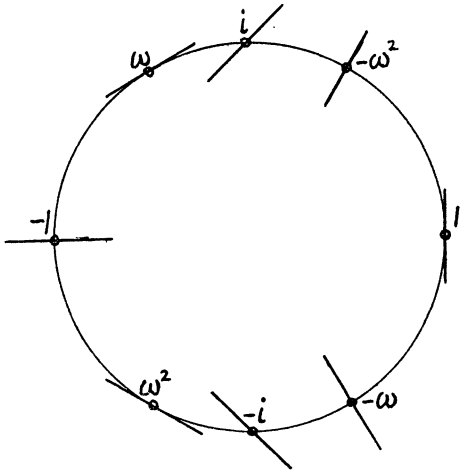


FIG. 8

completely around the unit circle counter-clockwise, the line segment with clinant τ swings around clockwise through 180°.

We now express this relationship analytically. Let ϕ be an angle through which the axis of reals may be turned to bring it into parallelism with the line $\tau x + y = \tau r$. There are infinitely many such angles ϕ, but always just one, ϕ_0 such that $0 \leqq \phi_0 < \pi$; and all other values of ϕ differ from ϕ_0 by multiples of π.

* We use the word *orientation* rather than *direction*, preferring to reserve the latter word for the thing indicated by a pointed finger or an arrowhead on a line. Thus "north" indicates a direction and "south" a different direction. A line, or line segment, without an arrowhead on it has orientation but not direction; and two parallel lines or segments have the same orientation but may have different directions.

This angle ϕ_0 is often called the *inclination* of the line. Since

$$\frac{1}{\tau} = e^{2i\theta},$$

we have

$$2i\theta = \log\left(\frac{1}{\tau}\right) = -\log(\tau),$$

and

$$\theta = -\frac{1}{2i}\log(\tau).$$

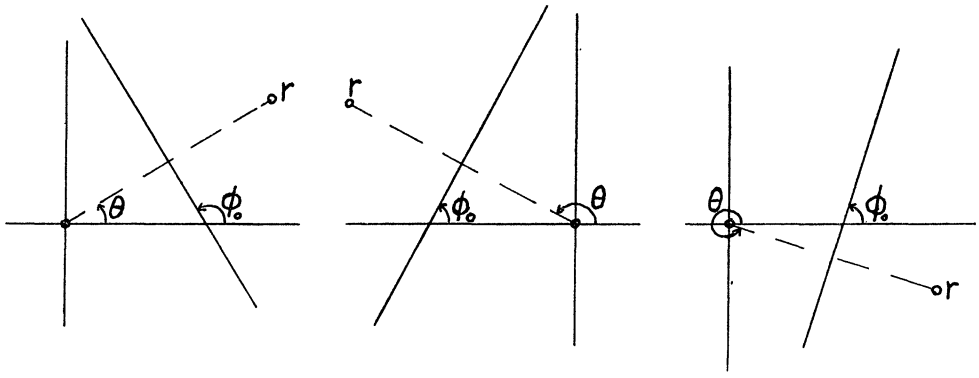


FIG. 9

Now when

$$0 \leqq \theta < \frac{\pi}{2}, \quad \phi_0 = \theta + \frac{\pi}{2};$$

when

$$\frac{\pi}{2} \leqq \theta < \frac{3\pi}{2}, \quad \phi_0 = \theta - \frac{\pi}{2} \quad \text{or} \quad \phi_0 + \pi = \theta + \frac{\pi}{2};$$

when

$$\frac{3\pi}{2} \leqq \theta < 2\pi, \quad \phi_0 = \theta - \frac{3\pi}{2} \quad \text{or} \quad \phi_0 + 2\pi = \theta + \frac{\pi}{2};$$

and hence there is always a value of ϕ such that

$$\phi = \theta + \frac{\pi}{2}.$$

Then

$$\phi = \frac{\pi}{2} - \frac{1}{2i}\log(\tau)$$

$$= -\frac{1}{2i}[-\pi i + \log(\tau)]$$

$$= \frac{i}{2}[\log(-1) + \log(\tau)]$$

and hence

$$\phi = \frac{i}{2}\log(-\tau). \tag{3.21}$$

Since $\log(-\tau)$ has infinitely many values differing by multiples of $2\pi i$ (see section 1.6), the formula 3.21 gives the infinitely many values of ϕ differing from each other by multiples of π, and one of these values is the inclination ϕ_0. This is similar to the situation in Cartesian coordinates where, the slope of the line being m, we have

$$\phi = \tan^{-1} m.$$

The reader may be interested in verifying the following relations between the slope m of a line in the Cartesian system (m real) and the clinant τ of the same line in the conjugate coordinate system (τ a turn):

$$m = \frac{i(\tau+1)}{\tau-1}, \qquad \tau = \frac{m+i}{m-i},$$
$$\tan^{-1} m = \frac{i}{2}\log(-\tau). \tag{3.22}$$

3.3 By the directed angle ϕ_{12} *from* a line l_1 *to* a line l_2 we shall mean any angle through which the line l_1 may be turned to bring it into parallelism with the line l_2. There are infinitely many values of ϕ_{12} differing from each other by multiples of π. From 3.21 we see that the angle *from* a line with clinant τ_1 *to* a line with clinant τ_2 is

$$\phi_{12} = \frac{i}{2}\log(-\tau_2) - \frac{i}{2}\log(-\tau_1)$$

or

$$\phi_{12} = \frac{i}{2}\log\left(\frac{\tau_2}{\tau_1}\right). \tag{3.31}$$

Evidently two lines are parallel if $\tau_1 = \tau_2$. If the lines are perpendicular we have

$$\phi_{12} = \frac{\pi}{2} + \pi s, \qquad s \text{ any integer},$$

$$\frac{i}{2} \log\left(\frac{\tau_2}{\tau_1}\right) = \frac{\pi}{2} + \pi s,$$

$$\log\left(\frac{\tau_2}{\tau_1}\right) = -\pi i - 2\pi s i,$$

$$\frac{\tau_2}{\tau_1} = e^{i(-\pi - 2\pi s)} = -1,$$

$$\tau_2 = -\tau_1.$$

Hence two lines are perpendicular when the clinant of one is the negative of the clinant of the other. This is a relation we shall use frequently.

The reader may verify that if the self-conjugate equations of two non-parallel lines are solved simultaneously, the resulting solutions for x and y will always be conjugates. If either or both of two self-conjugate equations is of degree higher than one the solutions for x and y may or may not be conjugates.

If τ_1 and τ_2 are the clinants of two intersecting lines, and τ_3 is the clinant of either of the bisectors of the angles between the lines, we must have

$$\frac{\tau_1}{\tau_3} = \frac{\tau_3}{\tau_2}.$$

Hence the clinants of the two bisectors must be

$$\tau_3 = \pm \sqrt{\tau_1 \tau_2}. \tag{3.32}$$

3.4. Because of the linear form of the equation of a line, $ax+by+c=0$, it follows at once that the line through the two points (x_1, y_1) and (x_2, y_2) has the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \tag{3.41}$$

or

$$(y_1 - y_2)x - (x_1 - x_2)y + x_1 y_2 - x_2 y_1 = 0,$$

or

$$-\frac{y_1 - y_2}{x_1 - x_2} x + y = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}.$$

Hence we see that the clinant of the line through the points (x_1, y_1) and (x_2, y_2) is

$$\tau = -\frac{y_1 - y_2}{x_1 - x_2}. \tag{3.42}$$

Also it is obvious at once that the equation of the line having the clinant τ and passing through the point (x_1, y_1) is

$$\tau x + y = \tau x_1 + y_1. \tag{3.43}$$

3.5. Let $\tau x + y = \tau r$ be the equation of a given line, and (x_1, y_1), (x_2, y_2) two points satisfying the condition

$$\tau x_1 + y_2 = \tau r. \tag{3.51}$$

This implies also the conjugate relation

$$\frac{y_1}{\tau} + x_2 = \frac{\bar{r}}{\tau} = r$$

or

$$\tau x_2 + y_1 = \tau r. \tag{3.52}$$

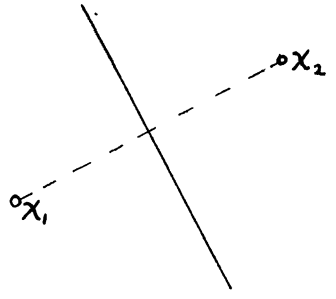


FIG. 10

Adding 3.51 and 3.52 and dividing by 2, we have

$$\tau\left(\frac{x_1 + x_2}{2}\right) + \left(\frac{y_1 + y_2}{2}\right) = \tau r,$$

which shows that the mid-point of the segment from x_1 to x_2 lies on the given line. If we subtract 3.52 from 3.51, we have

$$\tau(x_1 - x_2) + (y_2 - y_1) = 0$$

or

$$\tau = \frac{y_1 - y_2}{x_1 - x_2},$$

which shows that the given line is perpendicular to the segment from x_1 to x_2. It follows that each of the two points is the reflection of the other in the given line. We shall say that two such points are *inverse points with respect to the line*, and that each point is the inverse of the other with respect to the line.

If the equation of the line is in the form

$$ax + by + c = 0,$$

the condition that the points (x_1, y_1) and (x_2, y_2) be inverse points with respect to the line is

$$ax_1 + by_2 + c = 0, \tag{3.53}$$

or the equivalent condition

$$ax_2 + by_1 + c = 0.$$

If we use only the first coordinate to designate each point, it follows that the inverse of the point x_1 with respect to the line $ax+by+c=0$ (or $\tau x+y=\tau r$) is

$$x_2 = \frac{-b\bar{x}_1 - c}{a} = r - \frac{\bar{x}_1}{\tau}. \tag{3.54}$$

The distance between the point (x_1, y_1) and its reflection in the line $ax+by+c=0$ is

$$\sqrt{\left(\frac{-by_1 - c}{a} - x_1\right)\left(\frac{-ax_1 - c}{b} - y_1\right)} = \sqrt{\frac{(ax_1 + by_1 + c)^2}{ab}}$$

$$= \pm \frac{ax_1 + by_1 + c}{\sqrt{ab}}.$$

Hence the square of the distance from the point (x_1, y_1) to the line $ax+by+c=0$ is

$$d^2 = \frac{(ax_1 + by_1 + c)^2}{4ab}. \tag{3.55}$$

Chapter IV

THE CIRCLE

4.1. Let k be any complex number and g any complex number other than zero, and consider the circle whose center is the point k and whose radius r is

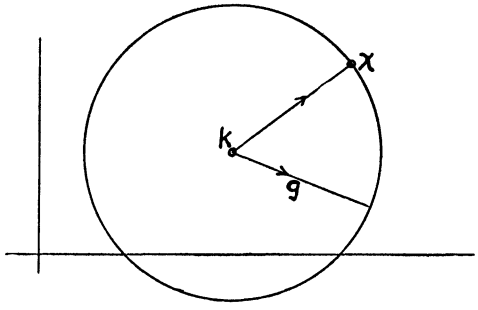


FIG. 11

the length of the vector g, so that $r^2 = g\bar{g}$. If x is any point on the circle, the vectors $x-k$ and g are of the same length and

$$x - k = gt,$$

where t is a turn varying with x. Hence

$$x = k + gt \tag{4.11}$$

is a map equation of the circle, the point x running around the circle as t runs through all turn values.

Equation 4.11 implies the conjugate equation

$$y = \bar{k} + \frac{\bar{g}}{t},$$

and, eliminating t, we obtain the self-conjugate equation of the circle

$$(x - k)(y - \bar{k}) = g\bar{g} = r^2. \tag{4.12}$$

Equation 4.12 is of the general form

$$axy + bx + cy + d = 0, \tag{4.13}$$

and we may ask under what conditions will an equation of this form represent a circle. Evidently we must have $a \neq 0$, for otherwise the equation would be linear and could only represent a line. Then in order that the equation be self-conjugate (see 2.23) it is necessary that

$$a = \lambda\bar{a}, \qquad b = \lambda\bar{c}, \qquad c = \lambda\bar{b}, \qquad d = \lambda\bar{d},$$

from which it follows that d/a must be real and b/a and c/a must be conjugates. Then the equation 4.13 may be written

$$\left(x+\frac{c}{a}\right)\left(y+\frac{b}{a}\right)=\frac{bc}{a^2}-\frac{d}{a}, \tag{4.14}$$

and $bc/a^2-d/a$ is real. Now comparing this with equation 4.12, we see that it represents a circle if $bc/a^2-d/a$ is positive. Hence, under the conditions

$$\begin{cases} a \neq 0, \\ d/a \text{ is real}, \\ b/a \text{ and } c/a \text{ are conjugates}, \\ (bc-ad)/a^2 > 0, \end{cases} \tag{4.15}$$

equation 4.13 represents a circle with center at $-c/a$ and radius $\sqrt{(bc-ad)/a^2}$.

If x_1, x_2, x_3 are any three points not on a line, the equation

$$\begin{vmatrix} xy & x & y & 1 \\ x_1y_1 & x_1 & y_1 & 1 \\ x_2y_2 & x_2 & y_2 & 1 \\ x_3y_3 & x_3 & y_3 & 1 \end{vmatrix} = 0 \tag{4.16}$$

is obviously satisfied by the coordinates of each of the three points, and is of the form 4.13 with coefficients satisfying conditions 4.15. It is therefore the equation of the circle through the three points.*

4.2. Two points are *inverse points with respect to a circle* (a proper circle, $r \neq 0$) if they lie on a line with the center, both on the same side of the center, and the product of their distances from the center is equal to the square of the radius. From this definition certain geometric facts are immediately obvious:

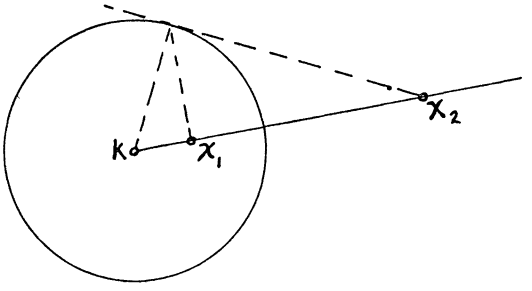


FIG. 12

if one of the points is inside the circle the other is outside; if one of the points is given, anywhere in the plane except at the center of the circle, the other is uniquely determined and may be easily constructed; and a point coincides with its inverse point when and only when it is on the circle.

* If the three points are on a line the equation is linear (the coefficient of xy vanishes) and is the equation of the line through the three points.

We now show that two points (x_1, y_1) and (x_2, y_2) are inverse points with respect to the circle

$$(x - k)(y - \bar{k}) = r^2$$

if and only if

$$(x_1 - k)(y_2 - \bar{k}) = r^2. \tag{4.21}$$

Condition 4.21 implies the conjugate condition

$$(x_2 - k)(y_1 - \bar{k}) = r^2. \tag{4.22}$$

From 4.21

$$x_1 - k = \frac{r^2}{y_2 - \bar{k}}$$

or

$$x_1 - k = \frac{r^2}{(y_2 - \bar{k})(x_2 - k)}(x_2 - k).$$

Since

$$\frac{r^2}{(y_2 - \bar{k})(x_2 - k)}$$

is real and positive, it follows that the points x_1 and x_2 are on a line with the center k and on the same side of k. Now multiplying 4.21 and 4.22 we have

$$(x_1 - k)(y_1 - \bar{k})(x_2 - k)(y_2 - \bar{k}) = r^4.$$

But $(x_1-k)(y_1-\bar{k})$ and $(x_2-k)(y_2-\bar{k})$ are respectively the squares of the distances from k to x_1 and x_2, and hence the product of these distances is r^2. This completes the proof that 4.21 or 4.22 is the condition that x_1 and x_2 be inverse points with respect to the circle.

The points (x_1, y_1) and (x_2, y_2) will be inverse points with respect to a circle whose equation is in the form

$$axy + bx + cy + d = 0$$

if and only if

$$ax_1y_2 + bx_1 + cy_2 + d = 0, \tag{4.23}$$

or (an equivalent condition) if and only if

$$ax_2y_1 + bx_2 + cy_1 + d = 0. \tag{4.24}$$

Using only the first coordinate to designate each point, the inverse of the point x_1 with respect to the circle

$$axy + bx + cy + d = 0$$

is

$$x_2 = \frac{-c\bar{x}_1 - d}{a\bar{x}_1 + b}. \tag{4.25}$$

Note that if we drop the restriction $a \neq 0$, the equation

$$axy + bx + cy + d = 0$$

includes both straight lines and circles; and when $a = 0$, 4.23 and 4.25 above reduce to 3.53 and 3.54 giving inverse points with respect to a line.

4.3. Let P_1 and P_2 be inverse points with respect to any circle with center C. Then the line l_1 through P_2 and perpendicular to the line CP_1P_2 is called the *polar line of the point P_1 with respect to the circle;* and similarly the line l_2 through P_1 and perpendicular to CP_1P_2 is the polar line of P_2. Thus, with respect to a

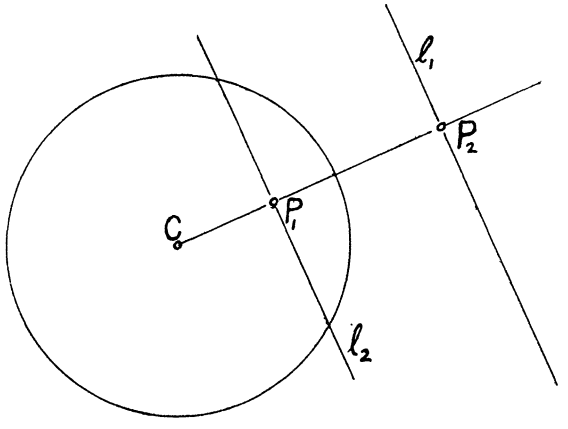


FIG. 13

fixed circle, every point P in the plane (except the center of the circle) has a unique polar line l. Also every line l in the plane (not through the center of the circle) is the polar line of a unique point P, and we say that P is the *polar point of the line l.*

The following geometric properties of these polar points and lines are quite obvious or very easy to prove.

(a) The polar line of P cuts the circle or not according as P is outside or inside the circle. If P is on the circle, the polar line is the tangent to the circle at P.

(b) The polar line of P passes through P if and only if P lies on the circle.

(c) If the line l cuts the circle at A and B, the tangents to the circle at these points intersect at the polar point of l.

(d) If the point B lies on the polar line of A, then A lies on the polar line of B.

The reader may show that the polar line of the point a with respect to the circle $(x-k)(y-\bar{k})=r^2$ has the self-conjugate equation

$$(\bar{a}-\bar{k})x+(a-k)y=k\bar{a}+\bar{k}a-2k\bar{k}+2r^2,$$

and that the equation of the tangent to the circle whose map equation is $x=k+gt$ at the point given by $t=t_1$ is

$$\bar{g}x+g\bar{t}_1^2y=k\bar{g}+\bar{k}gt_1^2+2g\bar{g}t_1.$$

With respect to a given circle a triangle is said to be *self-polar* (or *self-conjugate*) if each side of the triangle is the polar line of the opposite vertex. Each altitude of such a self-polar triangle must evidently pass through the center of the circle, and hence the center of the circle is the orthocenter of any self-polar

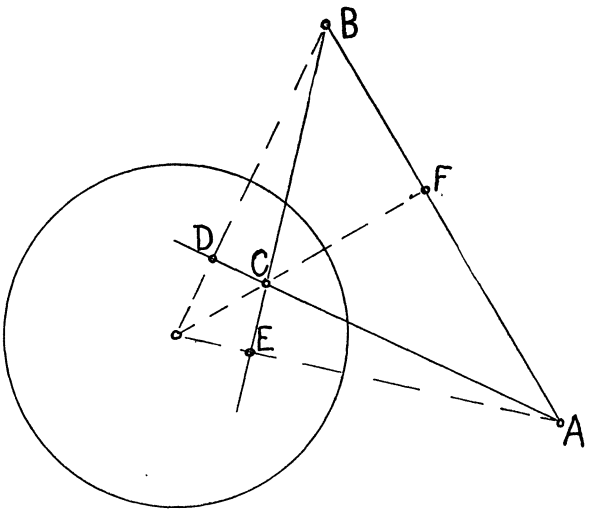


FIG. 14

triangle. Moreover, if the altitudes from the vertices A, B, and C respectively meet the opposite sides at the points D, E, and F, then A, D and B, E and C, F are pairs of inverse points with respect to the circle.

Given the circle, a self-polar triangle may be easily constructed. One vertex may be chosen arbitrarily, and a second vertex anywhere on the polar line of the first vertex; the third vertex is then uniquely determined.

Given a triangle, is there a circle with respect to which the triangle is self-polar? This question will be discussed in a later chapter (see section 7.7).

4.4. Suppose the equation of any circle is in the form

$$(x-k)(y-\bar{k})-r^2=0 \quad \text{or} \quad xy+bx+cy+d=0,$$

the coefficient of the xy term being 1. Then the value of the left member of

the equation when we substitute for x and y the coordinates (x_1, y_1) of any point in the plane is called the *power of the point* (x_1, y_1) *with respect to the circle.* Since $(x_1-k)(y_1-\bar{k})$ is the square of the distance of the point from the center of the circle, we see that when the point is outside the circle the power is positive and

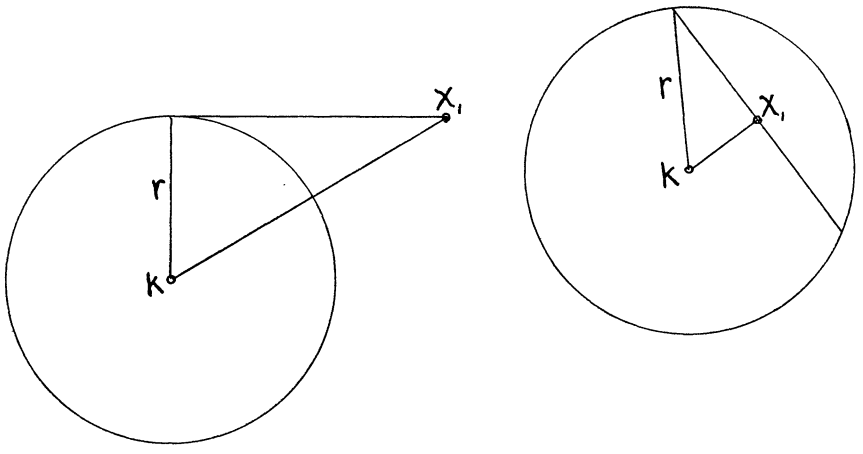


FIG. 15

is the square of the length of the tangent from the point to the circle, and when the point is inside the circle the power is the negative of the square of half the length of the minimum chord through the point.

Consider the two circles, not concentric,

$$xy - \bar{k}_1 x - k_1 y + k_1\bar{k}_1 - r_1^2 = 0 \tag{4.41}$$

and

$$xy - \bar{k}_2 x - k_2 y + k_2\bar{k}_2 - r_2^2 = 0, \qquad k_1 \neq k_2, \tag{4.42}$$

and the line

$$(\bar{k}_2 - \bar{k}_1)x + (k_2 - k_1)y + k_1\bar{k}_1 - k_2\bar{k}_2 + r_2^2 - r_1^2 = 0, \tag{4.43}$$

obtained by eliminating the xy term between the equations of the two circles. This line is called the *radical axis* of the two circles, and we shall show that there is a simple geometric relationship between this radical axis and the two circles.

Since we have the identity (in x and y)

$$\begin{aligned}(xy - \bar{k}_1 x - k_1 y + k_1\bar{k}_1 - r_1^2) - (xy - \bar{k}_2 x - k_2 y + k_2\bar{k}_2 - r_2^2)\\ \equiv \{(\bar{k}_2 - \bar{k}_1)x + (k_2 - k_1)y + k_1\bar{k}_1 - k_2\bar{k}_2 + r_2^2 - r_1^2\},\end{aligned} \tag{4.44}$$

we see that if the two circles intersect in two points the radical axis will go through these two points, and if the radical axis cuts one of the circles in two

points it must cut the other circle in the same two points. The clinant of the radical axis is $(\bar{k}_2-\bar{k}_1)/(k_2-k_1)$, and this is the negative of the clinant of the line through the centers k_1 and k_2 (see 3.42), which shows that the radical axis of two circles is perpendicular to the line of centers. It follows that if the two circles are tangent to each other, the radical axis is the common tangent line. The identity 4.44 also shows that if a point is on the radical axis its powers with respect to the two circles are equal; and this means that from any point of the radical axis lying outside the two circles the lengths of the tangents to the two circles are equal. This last property is the distinguishing geometric property of the radical axis in all cases and furnishes a method of constructing the radical axis when the two circles have no points of intersection.

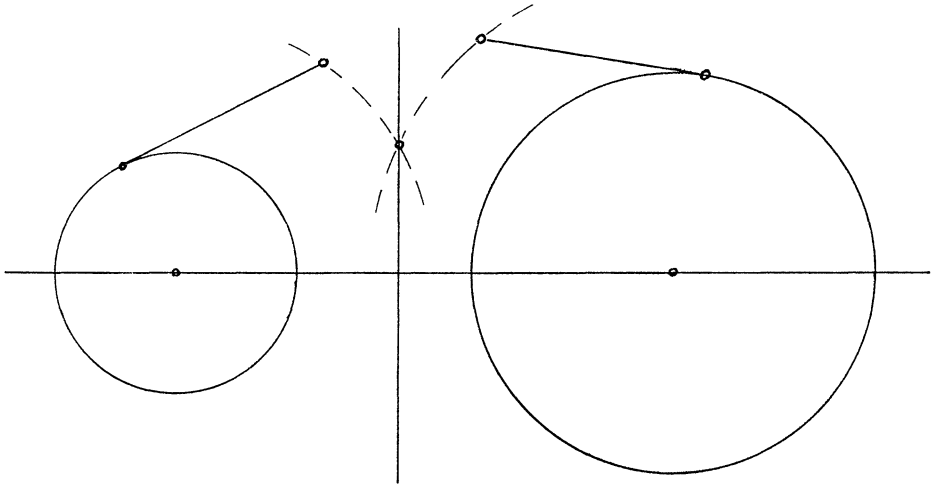


FIG. 16

4.5. Consider again the two circles 4.41 and 4.42 of the preceding section. If we multiply 4.41 by λ_1 and 4.42 by λ_2 and then add, λ_1 and λ_2 being *real* constants such that $\lambda_1+\lambda_2\neq 0$*, we obtain the equation of a third circle

$$(\lambda_1+\lambda_2)xy-(\lambda_1\bar{k}_1+\lambda_2\bar{k}_2)x-(\lambda_1 k_1+\lambda_2 k_2)y+\lambda_1 k_1\bar{k}_1+\lambda_2 k_2\bar{k}_2-\lambda_1 r_1^2-\lambda_2 r_2^2=0. \tag{4.51}$$

For varying values of λ_1 and λ_2 this equation represents an infinite set of circles which we call a *pencil of circles* (also sometimes called a coaxial set of circles); and the geometric property of such a pencil of circles is that any two circles of the set have the same radical axis as any other two circles of the set. To show this let the circle 4.51 for some fixed λ_1 and λ_2, and the circle

* We take λ_1 and λ_2 to be real in order to make equation 4.51 self-conjugate, and the restriction $\lambda_1+\lambda_2\neq 0$ is to insure that equation 4.51 shall represent a circle rather than a line.

$$(4.52)\qquad (\mu_1+\mu_2)xy-(\mu_1\bar{k}_1+\mu_2\bar{k}_2)x-(\mu_1k_1+\mu_2k_2)y + \mu_1k_1\bar{k}_1+\mu_2k_2\bar{k}_2-\mu_1r_1^2-\mu_2r_2^2=0$$

be any two different circles of the pencil (we must have $\lambda_1\mu_2-\lambda_2\mu_1\neq 0$ in order that these circles should not be the same). We find the radical axis of these two circles by eliminating the xy term, and obtain, after taking out the factor $\lambda_1\mu_2-\lambda_2\mu_1$,

$$(4.53)\qquad (\bar{k}_2-\bar{k}_1)x+(k_2-k_1)y+k_1\bar{k}_1-k_2\bar{k}_2+r_2^2-r_1^2=0.$$

Since this equation is quite independent of λ_1, λ_2 and μ_1, μ_2, it represents the radical axis of any two circles of the pencil; and this line may therefore be called the *radical axis of the pencil.* If the radical axis cuts one circle of the pencil in two points, it cuts every circle of the pencil in the same two points; if it is tangent to one circle of the pencil, it is tangent to every circle at the same point; and if it has no points of intersection with some one circle of the pencil, it has no intersections with any circle of the pencil, and no two circles of the pencil cut each other. There are three types of pencil as shown in the accompanying figures. In all cases, tangents from a point of the radical axis to all the circles of the pencil are of equal length.

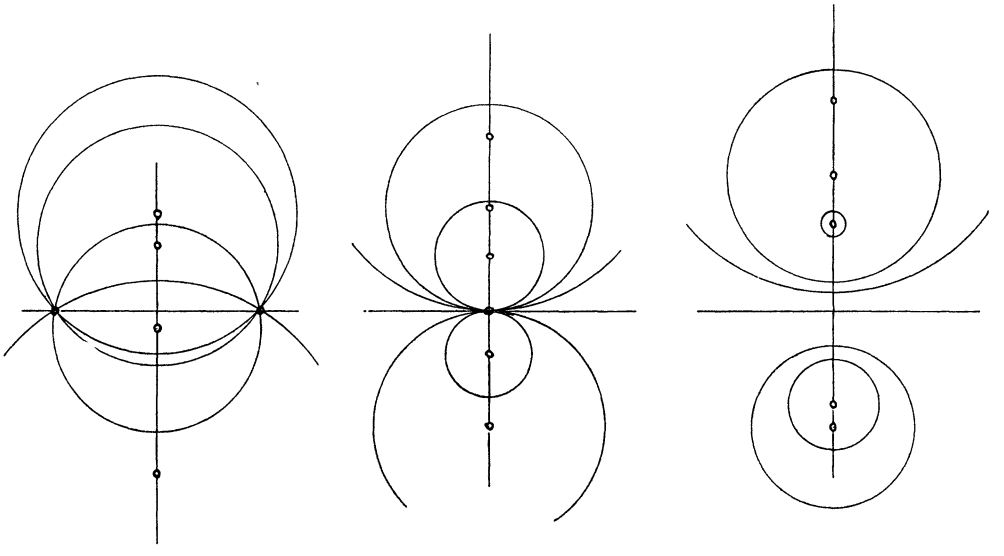


FIG. 17

Chapter V

MAP EQUATIONS AND THE DERIVATIVE $D_t x$

5.1. The map equations 3.11, 3.14, and 4.11 of the straight line and circle are examples of a kind of representation which can be used advantageously in the study of certain other rational (unicursal) curves.

Suppose the complex variable x is equated to a rational function of a variable turn t,

$$x = R(t) = \frac{P(t)}{Q(t)}, \tag{5.11}$$

where $P(t)$ and $Q(t)$ are relatively prime polynomials. As the point t runs around the unit circle the point x describes a rational curve. Equation 5.11 implies the conjugate equation

$$y = \overline{R}\left(\frac{1}{t}\right), \tag{5.12}$$

and equations 5.11 and 5.12 together are parametric equations of the curve. There may be turn values of t for which $Q(t)=0$, and such values of t would not give points of the curve (as, for example, $t=1$ in 3.11 and 3.14). Except for such values of t, $R(t)$ and $\overline{R}(1/t)$ are analytic* functions of t. Eliminating t algebraically from 5.11 and 5.12 we obtain the equation

$$f(x, y) = 0, \tag{5.13}$$

and it can be shown that this equation will always be self-conjugate. It will be satisfied by all the points obtained by giving t turn values in 5.11 and 5.12, but may sometimes be satisfied also by other points not given by the parametric equations (see section 9.2).

5.2. If in the map equation $x=R(t)$ the turn t is replaced by any rational function of t, $f(t)$, which takes on all turn values and only turn values as t takes on all turn values, the curve corresponding to the map equation is not changed. Thus it makes no change in the curve to replace t by t^n, $1/t^n$, $t_1 t^n$ (where t_1 is a fixed turn), *etc.*

If b is a complex number not zero, $b=\rho e^{i\theta}$, $\rho \neq 0$, the curve with the map equation $x=bR(t)$ has a simple geometric relation to the curve $x=R(t)$. The geometric effect of the multiplication by b is to rotate the curve through an angle θ about the origin, and to expand (or contract) it about the origin to ρ times its original size. If b is a turn, $\rho=1$, the curve is merely rotated without change of size or shape. If a complex number a is added to $R(t)$, the effect is to translate the curve without change of size or shape, the direction and distance

* See Curtiss, Analytic Functions of a Complex Variable, pp. 43–51.

of the translation being the direction and length of the vector a. Thus the curve

$$x = a + bR(t) \tag{5.21}$$

is similar to the curve $x=R(t)$, the one being obtained from the other by the operations of rotation, expansion, and translation. If any rigid motion is applied to the curve $x=R(t)$, a map equation of the curve in the new position will be

$$x = a + t_1 R(t), \tag{5.22}$$

where t_1 is a turn.

5.3. The map equations 3.11 and 3.14 for a line and 4.11 for a circle are all included in the general form

$$x = \frac{a + bt}{c + dt}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0, \tag{5.31}$$

and we now show that this more general map equation always represents either a line or a circle.

Equation 5.31 implies the conjugate equation

$$y = \frac{\bar{a}t + \bar{b}}{\bar{c}t + \bar{d}},$$

and eliminating t we have the self-conjugate equation

$$(d\bar{d} - c\bar{c})xy + (\bar{a}c - \bar{b}d)x + (a\bar{c} - b\bar{d})y + (b\bar{b} - a\bar{a}) = 0. \tag{5.32}$$

If $d\bar{d}-c\bar{c}=0$, this equation is linear and satisfies conditions 2.31, and hence represents a line. With $d\bar{d}-c\bar{c}=0$, both c and d would be zero if either one were zero, and this would contradict the condition $ad-bc\neq 0$. Hence *neither* c nor d can be zero in this case and the condition $d\bar{d}-c\bar{c}=0$ is equivalent to

$$\frac{c}{d} = \frac{\bar{d}}{\bar{c}},$$

which means that c/d is a turn.

If $d\bar{d}-c\bar{c}\neq 0$, equation 5.31 may be put in the form

$$\left(x - \frac{b\bar{d} - a\bar{c}}{d\bar{d} - c\bar{c}}\right)\left(y - \frac{\bar{b}d - \bar{a}c}{d\bar{d} - c\bar{c}}\right) = \frac{(bc - ad)(\bar{b}\bar{c} - \bar{a}\bar{d})}{(d\bar{d} - c\bar{c})^2};$$

and since $bc-ad\neq 0$, $(bc-ad)(\bar{b}\bar{c}-\bar{a}\bar{d})>0$, and we have the equation of a circle with center at

$$\left(\frac{b\bar{d} - a\bar{c}}{d\bar{d} - c\bar{c}}, \frac{\bar{b}d - \bar{a}c}{d\bar{d} - c\bar{c}}\right)$$

and radius

$$r = \sqrt{\frac{(bc - ad)(\bar{b}\bar{c} - \bar{a}\bar{d})}{(d\bar{d} - c\bar{c})^2}}.$$

We have then shown that the map equation

$$x = \frac{a + bt}{c + dt}, \quad \text{with} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

represents a line or a circle according as c/d is or is not a turn.

5.4. Suppose that the map equation of a curve is $x = R(t) = P(t)/Q(t)$ and that t_1 and t_2 are two distinct values of the turn t, with $Q(t_1) \neq 0$, and hence $Q(t_2) \neq 0$ when t_2 is sufficiently close to t_1; and let (x_1, y_1) and (x_2, y_2) be the two corresponding points on the curve. The clinant of the secant line through these two points is (see 3.42)

$$\tau = -\frac{y_1 - y_2}{x_1 - x_2} = -\frac{\dfrac{y_1 - y_2}{t_1 - t_2}}{\dfrac{x_1 - x_2}{t_1 - t_2}}.$$

If now t_1 is fixed and t_2 is made to approach t_1 *taking on only turn values*, the clinant of the secant will approach the clinant of the tangent to the curve at the point (x_1, y_1), and the fractions $(y_1 - y_2)/(t_1 - t_2)$ and $(x_1 - x_2)/(t_1 - t_2)$ will approach limits, the values of the formal derivatives* $D_t y$ and $D_t x$ for $t = t_1$. Thus, provided $D_t x \neq 0$ when $t = t_1$, the clinant of the tangent to the curve at the point (x_1, y_1) is

$$\tau = -\left.\frac{D_t y}{D_t x}\right]_{t=t_1}. \tag{5.41}$$

If either $D_t y$ or $D_t x$ is zero for $t = t_1$, the other is also. For with $x = R(t)$ we may write $D_t x = R'(t)$. Then $y = \overline{R}(1/t)$ and

$$D_t y = \overline{R}'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) = \overline{R'}\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right),$$

and since $-1/t^2$ cannot vanish, $D_t y$ vanishes when and only when $\overline{R'}(1/t) = 0$. But $\overline{R'}(1/t)$ is the conjugate of $R'(t) = D_t x$; and hence $D_t y$ vanishes when and only when $D_t x$ vanishes. This will occur only at certain singular points of the

* By the "formal derivatives" $D_t y$ and $D_t x$ we mean the functions of t obtained by applying to the functions $\overline{R}(1/t)$ and $R(t)$ the formulas used in differentiating rational functions of a *real* variable.

curve (as we shall see later), and at such points the formula 5.41 is meaningless. Because of the relation

$$D_t y = -\overline{D_t x}/t^2,$$

formula 5.41 may be written

$$\tau = \left.\frac{\overline{D_t x}}{t^2 D_t x}\right]_{t=t_1}, \tag{5.42}$$

a useful form which enables us to obtain the clinant of the tangent from the map equation 5.11 without writing the conjugate equation 5.12.

5.5. Formula 5.41 and the obvious analogy with slope in the Cartesian system suggest that the clinant of the tangent to the curve might be given by the negative of the formal derivative $D_x y$ obtained from the self-conjugate equation 5.13 by either solving for y in terms of x or using the implicit function method. This procedure would (with slight restrictions) be sound, but it will be simpler for our purposes to use 5.42. The reader may, however, be interested in checking the results against each other in the simple case of the circle, finding the clinant of the tangent by (1) using 5.42 with the map equation 4.11, (2) solving the self-conjugate equation 4.14 for y in terms of x and using $-D_x y$ as a function of x, and (3) using $-D_x y$ as a function of x and y obtained from 4.14 by the implicit function method.

Chapter VI

SYMMETRIC FUNCTIONS OF TURNS

6.1. Let n turns, $t_1, t_2, \cdots, t_n$, be the roots of the equation

$$t^n - s_1 t^{n-1} + s_2 t^{n-2} + \cdots + (-1)^{n-1} s_{n-1} t + (-1)^n s_n = 0. \tag{6.11}$$

Then the coefficient s_1 is the sum of the n turns, s_2 is the sum of the products of the turns taken two at a time, and in general s_i is the sum of the products of the turns taken i at a time. Since s_n is the product of all the turns, it is itself a turn, and hence

$$\bar{s}_n = \frac{1}{s_n}. \tag{6.12}$$

Also we have

$$\bar{s}_1 = \frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} = \frac{s_{n-1}}{s_n},$$

and in general

$$\bar{s}_i = \frac{s_{n-i}}{s_n}, \qquad i = 1, 2, \cdots, n-1. \tag{6.13}$$

These s_i are called the *elementary symmetric functions** of the turns, and any rational symmetric function of the turns can be expressed as a rational function of the s_i. Thus

$$(t_2 + t_3)(t_3 + t_1)(t_1 + t_2) = s_1 s_2 - s_3. \tag{6.14}$$

Since the turns t_i are the roots of the equation 6.11 we have *for each* t_i the identity

$$t_i^n - s_1 t_i^{n-1} + s_2 t_i^{n-2} \cdots + (-1)^n s_n = 0, \tag{6.15}$$

and this identity will be used frequently in the later chapters. Note also for future use that for any three t_i, $s_1 s_2 / s_3$ and $s_3/(s_1 s_2 - s_3)$ are real; and for three distinct t_i, $s_1 s_2 - 9 s_3 \neq 0$.

6.2. Let $\sum_1$ $\sum_2$, $\sum_3$, $\sum_4$ be the elementary symmetric functions for four turns t_1, t_2, t_3, t_4, and s_1, s_2, s_3 the elementary symmetric functions for three of the four turns when one of them, say t_k, is omitted. It is readily verified that

* Bôcher, Higher Algebra, p. 243. Dickson, New First Course in the Theory of Equations, pp. 17–18 and p. 177.

$$(6.21)\qquad \begin{cases} \Sigma_1 = s_1 + t_k \\ \Sigma_2 = s_2 + s_1 t_k \\ \Sigma_3 = s_3 + s_2 t_k \\ \Sigma_4 = \quad s_3 t_k. \end{cases}$$

To express the s_i in terms of the $\sum_i$ and t_k we may obtain the expression for s_1 from the first of the equations 6.21 and then s_2 and s_3 successively from the second and third equations; or we may obtain s_3 from the fourth equation and then s_2 and s_1 from the third and second equations. The results are

$$(6.22)\qquad \begin{cases} s_1 = \Sigma_1 - t_k & = \dfrac{\Sigma_2 t_k^2 - \Sigma_3 t_k + \Sigma_4}{t_k^3} \\ s_2 = \Sigma_2 - \Sigma_1 t_k + t_k^2 & = \dfrac{\Sigma_3 t_k - \Sigma_4}{t_k^2} \\ s_3 = \Sigma_3 - \Sigma_2 t_k + \Sigma_1 t_k^2 - t_k^3 & = \dfrac{\Sigma_4}{t_k}. \end{cases}$$

That the two different expressions for each s_i are equal may be seen at once from the identity 6.15.

More generally, if $\sum_1, \sum_2, \cdots, \sum_n$ are the symmetric functions for n turns, and $s_1, s_2, \cdots, s_{n-1}$ for the $n-1$ turns when t_k is omitted, then

$$\begin{aligned} \Sigma_1 &= s_1 + t_k \\ \Sigma_i &= s_i + s_{i-1} t_k, \qquad i = 2, 3, \cdots, n-1, \\ \Sigma_n &= \quad s_{n-1} t_k; \end{aligned}$$

and the s_i may be expressed in terms of the $\sum_i$ and t_k by expressions similar to those in 6.22.

6.3. If a rational function of n turns is symmetrical in $n-1$ of the turns (but not in all n) it may be expressed in many ways as a rational function of the odd turn and the elementary symmetric functions of the n turns. In the following chapters we shall make frequent use of this fact. For example,

$$t_2 t_3 (t_3 + t_1)(t_1 + t_2) = \frac{s_3(s_1 s_2 - s_3)}{t_1(s_1 - t_1)} = s_1 s_3 + \frac{s_3^2}{t_1^2}$$

and

$$\frac{t_2 t_3}{t_2 + t_3} + \frac{t_3 t_1}{t_3 + t_1} = \frac{s_2^2 - s_3^2/t_3^2}{s_1 s_2 - s_3}$$

where the s_i are for the *three* turns t_1, t_2, t_3.

Chapter VII

SOME GEOMETRY OF THE TRIANGLE

7.1. Given any triangle we may fix our coordinate system so that the circumcircle of the triangle will be the unit circle. Then the vertices of the triangle will correspond to three turns, t_1, t_2, t_3, which are roots of an equation

$$t^3 - s_1 t^2 + s_2 t - s_3 = 0,$$

and the circumcenter of the triangle will be the point 0. The centroid of the triangle (see section 1.7) will be the point $(t_1+t_2+t_3)/3$ or $s_1/3$.

To find the orthocenter of the triangle (the point of intersection of the three perpendiculars from the vertices to the opposite sides) we note that the clinant of the side t_2t_3 is (see 3.42)

$$-\frac{\dfrac{1}{t_2}-\dfrac{1}{t_3}}{t_2-t_3}=\frac{1}{t_2t_3},$$

and hence the equation of this side is (see 3.43)

$$\frac{x}{t_2t_3}+y=\frac{1}{t_2}+\frac{1}{t_3}. \tag{7.11}$$

Then the equation of the perpendicular from the vertex t_1 to the side t_2t_3 is

$$-\frac{x}{t_2t_3}+y=-\frac{t_1}{t_2t_3}+\frac{1}{t_1}. \tag{7.12}$$

Similarly the equation of the perpendicular from t_2 to the side t_1t_3 is

$$-\frac{x}{t_1t_3}+y=-\frac{t_2}{t_1t_3}+\frac{1}{t_2}.$$

Eliminating y between these two equations and solving for x we have $x=t_1+t_2+t_3$, or

$$x = s_1, \tag{7.13}$$

and hence the orthocenter of the triangle is the point s_1. Although we found this point as the intersection of only two of the perpendiculars, the symmetry of the result shows at once that the third perpendicular also goes through it.

If the triangle is equilateral we have (see 1.73 and 1.74)

$$\text{either} \quad t_2 = \omega t_1,\ t_3 = \omega^2 t_1 \quad \text{and hence} \quad t_1 + \omega t_2 + \omega^2 t_3 = 0,$$

$$\text{or} \quad t_2 = \omega^2 t_1,\ t_3 = \omega t_1 \quad \text{and hence} \quad t_1 + \omega^2 t_2 + \omega t_3 = 0.$$

In either case $s_1 = 0$, so that the circumcenter, centroid, and orthocenter coincide. Also $\bar{s}_1 = s_2/s_3 = 0$, and therefore $s_2 = 0$.

If the triangle is not equilateral, the circumcenter, centroid, and orthocenter are respectively the distinct points, 0, $s_1/3$, and s_1; and these points obviously lie on a line (see section 1.4) with the centroid between the other two and at one-third the distance from the circumcenter to the orthocenter. This line is known as the *Euler line* of the triangle. Its clinant is

$$-\frac{\bar{s}_1}{s_1} = -\frac{s_2}{s_1 s_3},$$

and its equation is

$$s_2 x - s_1 s_3 y = 0. \tag{7.14}$$

The reader will find the following theorem very easy to prove:

If P_i, $i = 1, 2, 3, 4$, are four points on a circle, and H_i is the orthocenter of the triangle whose vertices are the remaining three points when P_i is omitted, then the four line segments P_iH_i have a common mid-point, and hence the figures formed by the points P_i and H_i are congruent.

7.2. The mid-point of the side t_2t_3 of the triangle is $\frac{1}{2}(t_2+t_3)$ or $\frac{1}{2}(s_1-t_1)$. If now we write the map equation

$$x = \frac{s_1}{2} - \frac{1}{2}t, \tag{7.21}$$

it corresponds (see 4.11) to the circle with center at $s_1/2$ and radius $\frac{1}{2}$. Moreover, as the variable turn t takes the values t_1, t_2, t_3, the circle goes through the mid-points of the three sides. This circle, called the *nine-point circle* for reasons which will appear shortly, is half as large as the circumcircle (whose radius was taken as the unit length) and has its center $s_1/2$, the *nine-point center*, on the Euler line midway between the circumcenter and orthocenter.

We next show that this nine-point circle (Figure 18) also passes through the three feet of the altitudes from each vertex to the opposite side. The foot of the altitude from the vertex t_1 to the side t_2t_3 is found by solving the equations 7.11 and 7.12 simultaneously. Eliminating y between these two equations and solving for x, we have

$$x = \frac{1}{2}\left(t_1 + t_2 + t_3 - \frac{t_2t_3}{t_1}\right),$$

and hence the foot of the altitude from t_1 to the side t_2t_3 is the point

$$x = \frac{s_1}{2} - \frac{s_3}{2t_1^2}. \tag{7.22}$$

Now we write the map equation

$$x = \frac{s_1}{2} - \frac{s_3}{2}t, \tag{7.23}$$

which corresponds to the circle with center at $s_1/2$ and radius $\left|-s_3/2\right| = \frac{1}{2}$. This is then the same nine-point circle, and as t takes the *turn* values $1/t_1^2$, $1/t_2^2$, and $1/t_3^2$ it evidently goes through the three feet of the altitudes from t_1, t_2, and t_3 respectively.

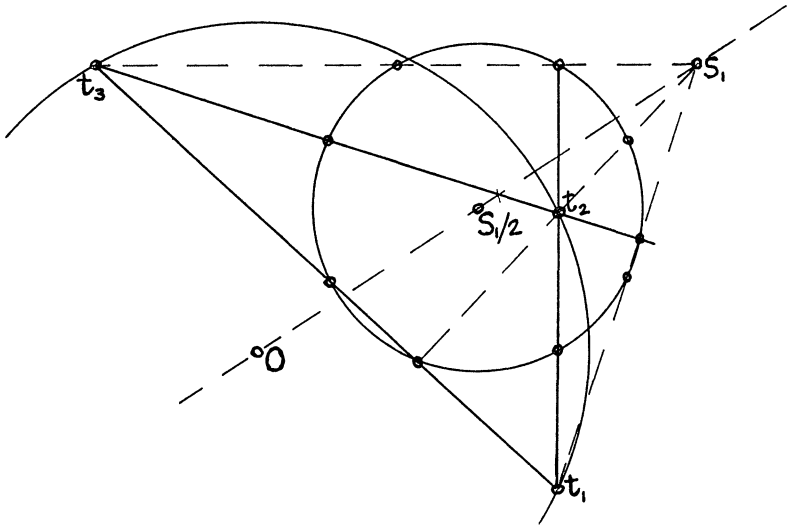


FIG. 18

We leave as an exercise for the reader to prove, *using the same method we have used twice above*, that the nine-point circle also goes through the three points mid-way between the orthocenter and each vertex of the triangle. We see then that this circle goes through the nine points which account for its name. It is obvious that when the triangle is isosceles, equilateral, or a right triangle, certain of these points will coincide.

Eliminating the parameter t between the map equation 7.21 and the conjugate equation

$$y = \frac{s_2}{2s_3} - \frac{1}{2t},$$

we have

$$\left(x - \frac{s_1}{2}\right)\left(y - \frac{s_2}{2s_3}\right) = \frac{1}{4} \tag{7.24}$$

as the self-conjugate equation of the nine-point circle.

It is easily seen geometrically that the four points consisting of the vertices of a triangle (not a right triangle) and its orthocenter t_1, t_2, t_3, s_1, make up an orthocentric set of points with the symmetric properties that each one of the four points is the orthocenter of the triangle whose vertices are the other three, that the line through any two of the points is perpendicular to the line through the other two, and that the nine-point circle for the triangle $t_1t_2t_3$ is also the nine-point circle for the triangles $s_1t_2t_3$, $t_1s_1t_3$, and $t_1t_2s_1$, and it may therefore be called the nine-point circle *for the orthocentric set.*

The proofs of the two following theorems will be good exercises for the reader.

Let P_i, $i=1, 2, 3, 4$, be an orthocentric set of points, C_i the circle through the three points other than P_i, and O_i the center of the circle C_i; then

(a) *the four circles C_i are all the same size;*

(b) *the four line segments P_iO_i have a common mid-point;*

(c) *the four points $P_iP_jO_kO_l$ ($i, j, k, l=1, 2, 3, 4$ in any order) are the vertices of a parallelogram.*

Let P_i, $i=1, 2, 3$, be the vertices of a triangle, D_i the foot of the altitude from P_i to the opposite side, and M_i the mid-point of this side; then the tangent to the circumcircle at P_i, the tangent to the nine-point circle at M_i, and the line D_jD_k, ($i, j, k=1, 2, 3$ in any order) are all parallel.

7.3 If a triangle with vertices at the points t_1, t_2, t_3 is a right triangle one of the turns must be the negative of another, that is, one of the quantities t_2+t_3, t_3+t_1, t_1+t_2 must be zero. But (see 6.14)

$$(t_2 + t_3)(t_3 + t_1)(t_1 + t_2) = s_1s_2 - s_3, \tag{7.31}$$

and therefore the condition for a right triangle is $s_1s_2-s_3=0$. If $t_2+t_3=0$, $s_1=t_1$, or (as is quite obvious) the orthocenter of a right triangle coincides with the vertex of the right angle.

In the case of an acute triangle (all three angles acute) the orthocenter lies inside the triangle and the distance from the circumcenter to the orthocenter is less than the radius of the circumcircle, that is, $|s_1|<1$ and $|s_1|^2<1$. But $|s_1|^2=s_1\bar{s}_1=s_1s_2/s_3$, and hence for an acute triangle $s_1s_2/s_3<1$ or $(s_1s_2-s_3)/s_3<0$.

Similarly for an obtuse triangle (one angle obtuse) we have $(s_1s_2-s_3)/s_3>0$.

Therefore triangle $t_1t_2t_3$ is

$$\begin{cases} \text{acute when} & (s_1s_2 - s_3)/s_3 < 0 \\ \text{right when} & (s_1s_2 - s_3)/s_3 = 0 \\ \text{obtuse when} & (s_1s_2 - s_3)/s_3 > 0. \end{cases} \tag{7.32}$$

7.4. Let us assume now that the triangle $t_1t_2t_3$ is not equilateral and is not a right triangle, so that $s_1\neq 0$, $s_2\neq 0$, and $s_1s_2-s_3\neq 0$. Then the tangents to the circumcircle at the vertices form a new triangle $P_1P_2P_3$, the vertex P_1 being the

intersection of the tangents at t_2 and t_3, *etc.* This triangle $P_1P_2P_3$ is called the *tangential triangle* of the triangle $t_1t_2t_3$.

Given any triangle $P_1P_2P_3$ there are four different circles each of which is tangent to the three lines P_2P_3, P_3P_1, P_1P_2. One of these circles lying inside the triangle is called the *incircle* (or inscribed circle) and the other three are called *excircles* (or escribed circles), and the centers of these circles are known as the *incenter* and the *excenters* of the triangle $P_1P_2P_3$. Starting, as in the preceding paragraph, with the triangle $t_1t_2t_3$, its circumcircle will be the incircle or one of the excircles of the tangential triangle $P_1P_2P_3$, depending on whether the triangle $t_1t_2t_3$ is acute or obtuse (in Figure 19 it is one of the excircles).

The tangent at t_2 has the reflex point $r_2 = 2t_2$, and its clinant is $\bar{r}_2/r_2 = 1/t_2^2$. Hence its equation is

$$\frac{x}{t_2^2} + y = \frac{2}{t_2}. \tag{7.41}$$

Similarly the tangent at t_3 has the equation

$$\frac{x}{t_3^2} + y = \frac{2}{t_3},$$

and solving these two equations for x we find that the point of intersection P_1 is

$$x = \frac{2t_2t_3}{t_2 + t_3}.$$

This may be written

$$x = \frac{2t_2t_3(t_3 + t_1)(t_1 + t_2)}{(t_2 + t_3)(t_3 + t_1)(t_1 + t_2)} = \frac{2\{t_1t_2t_3(t_1 + t_2 + t_3) + t_2^2t_3^2\}}{(t_2 + t_3)(t_3 + t_1)(t_1 + t_2)}$$

$$= \frac{2\left(s_1s_3 + \frac{s_3^2}{t_1^2}\right)}{s_1s_2 - s_3} = \frac{2s_1s_3}{s_1s_2 - s_3} + \frac{2s_3^2}{(s_1s_2 - s_3)t_1^2}.$$

The three vertices P_i are then given by

$$x = \frac{2s_1s_3}{s_1s_2 - s_3} + \frac{2s_3^2}{(s_1s_2 - s_3)t_i^2}, \qquad i = 1, 2, 3.$$

The map equation

$$x = \left(\frac{2s_3}{s_1s_2 - s_3}\right)s_1 + \left(\frac{2s_3}{s_1s_2 - s_3}\right)t \tag{7.42}$$

represents a circle with

$$\text{center } \left(\frac{2s_3}{s_1s_2 - s_3}\right)s_1 \quad \text{and radius } \left|\frac{2s_3}{s_1s_2 - s_3}\right|,$$

and since it passes through the vertices P_1, P_2, and P_3 when t takes on the turn values s_3/t_1^2, s_3/t_2^2, and s_3/t_3^2, it follows that 7.42 is the map equation of the circumcircle of the tangential triangle $P_1P_2P_3$. Since $2s_3/(s_1s_2-s_3)$ is real (see section 6.1), the center of this circle lies on the line through O and s_1, and we have

THEOREM 7.43. *Given a triangle $t_1t_2t_3$ (not a right triangle or an equilateral triangle) and its tangential triangle $P_1P_2P_3$, the circumcenter of the triangle $P_1P_2P_3$ lies on the Euler line of the triangle $t_1t_2t_3$.*

The radius of the circumcircle of the triangle $P_1P_2P_3$ is

$$R = \left|\frac{2s_3}{s_1s_2 - s_3}\right| = \pm\frac{2s_3}{s_1s_2 - s_3},$$

the minus or plus sign to be used according as the triangle $t_1t_2t_3$ is acute or obtuse.

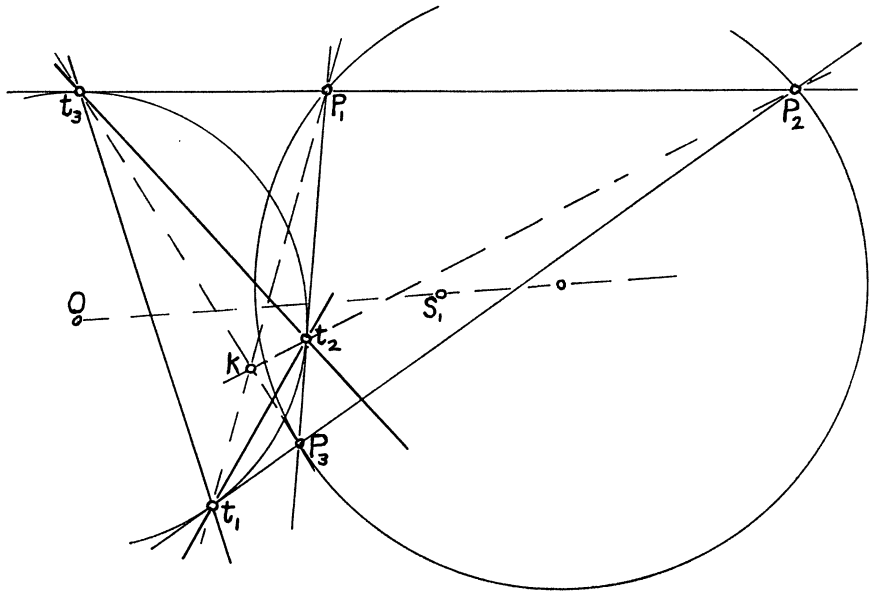


FIG. 19

The equation conjugate to 7.42 is

$$y = \frac{2s_2}{s_1s_2 - s_3} + \frac{2s_3}{(s_1s_2 - s_3)t},$$

and, eliminating t, we have

$$\left(x - \frac{2s_1s_3}{s_1s_2 - s_3}\right)\left(y - \frac{2s_2}{s_1s_2 - s_3}\right) = \frac{4s_3^2}{(s_1s_2 - s_3)^2} \tag{7.44}$$

as the self-conjugate equation of the circumcircle of the tangential triangle $P_1P_2P_3$.

7.5. Consider the line through a vertex of a triangle and the corresponding vertex of the tangential triangle. The line through the points t_1 and $2t_2t_3/(t_2+t_3)$ has the clinant

$$-\frac{\dfrac{2}{t_2+t_3} - \dfrac{1}{t_1}}{\dfrac{2t_2t_3}{t_2+t_3} - t_1} = \frac{-2t_1 + t_2 + t_3}{t_1(2t_2t_3 - t_3t_1 - t_1t_2)} = \frac{s_1 - 3t_1}{3s_3 - s_2t_1},$$

and hence its equation (see 3.43) is

$$\frac{s_1 - 3t_1}{3s_3 - s_2t_1}x + y = \frac{s_1t_1 - 3t_1^2}{3s_3 - s_2t_1} + \frac{1}{t_1}$$

or

$$\frac{s_1 - 3t_1}{3s_3 - s_2t_1}x + y = \frac{-3t_1^3 + s_1t_1^2 - s_2t_1 + 3s_3}{t_1(3s_3 - s_2t_1)}. \tag{7.51}$$

Now $t_1^3 - s_1t_1^2 + s_2t_1 - s_3 = 0$ (see 6.15) and therefore

$$-3t_1^3 + s_1t_1^2 - s_2t_1 + 3s_3 = -2s_1t_1^2 + 2s_2t_1,$$

and the equation becomes

$$(s_1 - 3t_1)x + (3s_3 - s_2t_1)y = -2s_1t_1 + 2s_2$$

or

$$(s_1x + 3s_3y - 2s_2) - t_1(3x + s_2y - 2s_1) = 0.$$

Then the equations

$$(s_1x + 3s_3y - 2s_2) - t_i(3x + s_2y - 2s_1) = 0, \qquad i = 1, 2, 3, \tag{7.52}$$

represent the three lines joining each vertex of the triangle $t_1t_2t_3$ to the corresponding vertex of the tangential triangle. These three lines will obviously be concurrent if there is a point whose coordinates satisfy the two equations*

$$s_1x + 3s_3y - 2s_2 = 0,$$
$$3x + s_2y - 2s_1 = 0.$$

* These equations are not self-conjugate and hence do not represent lines, but they are simultaneously satisfied by the conjugate coordinates of a point.

Solving these equations, we find that they are satisfied by the pair of *conjugate* values

$$x = \frac{2s_2^2 - 6s_1s_3}{s_1s_2 - 9s_3}, \qquad y = \frac{2s_1^2 - 6s_2}{s_1s_2 - 9s_3}.$$

Note that $s_1s_2 - 9s_3 \neq 0$ (see section 6.1). We have then

THEOREM 7.53. *The three lines joining the vertices of the triangle $t_1t_2t_3$ to the corresponding vertices of the tangential triangle $P_1P_2P_3$ meet in a point k,*

$$k = \frac{2s_2^2 - 6s_1s_3}{s_1s_2 - 9s_3}. \tag{7.54}$$

This point is called the *symmedian point*† of the triangle $t_1t_2t_3$.

In the foregoing discussion the case of a right triangle, $s_1s_2 - s_3 = 0$, was excluded because in that case the tangential triangle does not exist. However, two of the three lines passing through the symmedian point k do exist, and hence k is determined for a right triangle also, and is given by formula 7.54.

The line through the circumcenter and the symmedian point is called the *Brocard line* of the triangle, and its equation is

$$(3s_2 - s_1^2)x + (s_2^2 - 3s_1s_3)y = 0. \tag{7.55}$$

7.6. The mid-point of the side P_1P_2 of the tangential triangle is

$$x = \frac{1}{2}\left(\frac{2t_2t_3}{t_2 + t_3} + \frac{2t_3t_1}{t_3 + t_1}\right) = \frac{t_3\{t_2(t_3 + t_1) + t_1(t_2 + t_3)\}(t_1 + t_2)}{(t_2 + t_3)(t_3 + t_1)(t_1 + t_2)}$$

$$= \frac{(t_2t_3 + t_3t_1 + 2t_1t_2)(t_2t_3 + t_3t_1)}{(t_2 + t_3)(t_3 + t_1)(t_1 + t_2)} = \frac{(s_2 + t_1t_2)(s_2 - t_1t_2)}{s_1s_2 - s_3}$$

$$= \frac{s_2^2 - \dfrac{s_3^2}{t_3^2}}{s_1s_2 - s_3} = \frac{s_2^2}{s_1s_2 - s_3} - \frac{s_3^2}{(s_1s_2 - s_3)t_3^2}.$$

The three mid-points of the sides are therefore

$$x = \frac{s_2^2}{s_1s_2 - s_3} - \frac{s_3^2}{(s_1s_2 - s_3)t_i^2} \qquad i = 1, 2, 3.$$

Then

$$x = \frac{s_2^2}{s_1s_2 - s_3} - \frac{s_3^2}{s_1s_2 - s_3}t \tag{7.61}$$

† It is also sometimes called the *Lemoine* point or the *Grebe* point. See Johnson, Modern Geometry, p. 213.

is the map equation of the nine-point circle of the tangential triangle $P_1P_2P_3$. Note that its radius, $\left|s_3^2/(s_1s_2-s_3)\right|$, is just half the radius of the circumcircle (see 7.42) of this tangential triangle.

Writing the conjugate of 7.61,

$$y = \frac{\overset{2}{s_1}}{s_1s_2 - s_3} - \frac{1}{(s_1s_2 - s_3)t},$$

and eliminating t, we have for the self-conjugate equation of this nine-point circle

$$\left(x - \frac{\overset{2}{s_2}}{s_1s_2 - s_3}\right)\left(y - \frac{\overset{2}{s_1}}{s_1s_2 - s_3}\right) = \frac{\overset{2}{s_3}}{(s_1s_2 - s_3)^2}. \tag{7.62}$$

We now find the intersections of this circle with the circumcircle $xy=1$ of the triangle $t_1t_2t_3$. Substituting $1/x$ for y in 7.62, we have, after some reduction,

$$\overset{2}{s_1}\overset{2}{x} - 2s_1s_2x + \overset{2}{s_2} = 0.$$

If $t_1t_2t_3$ (and therefore also $P_1P_2P_3$) is equilateral, we have $s_1=s_2=0$, and the two circles coincide. Otherwise the quadratic equation has the double root $x=s_2/s_1$, which means that the nine-point circle of the tangential triangle $P_1P_2P_3$ is tangent to the circumcircle of the triangle $t_1t_2t_3$ at the point s_2/s_1.

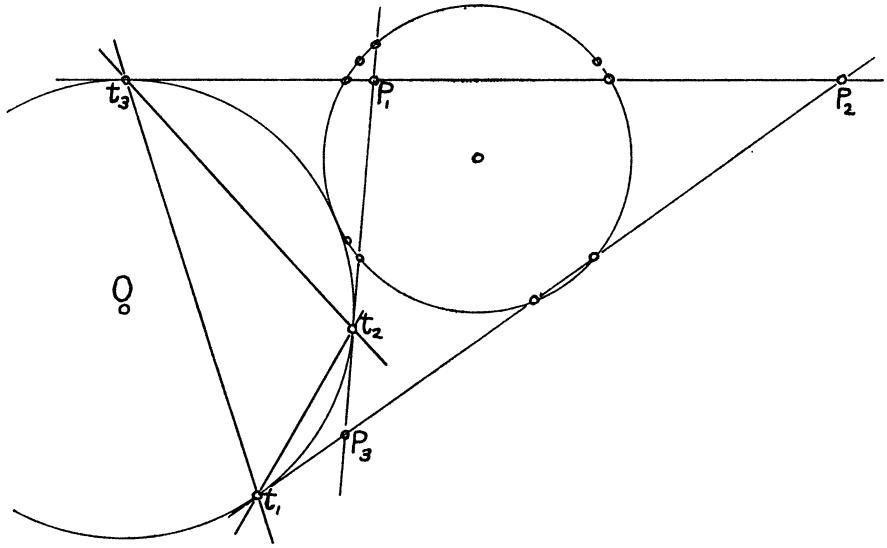


FIG. 20

We might start with any triangle $P_1P_2P_3$ (not equilateral) and take its incircle or any one of its excircles as the unit circle, with t_1, t_2, t_3 corresponding to the points of tangency of this circle with the lines P_2P_3, P_3P_1, and P_1P_2, and then

proceed with the foregoing proof. As a result we obtain what is known as *Feuerbach's theorem*:

THEOREM 7.63. *The nine-point circle of any triangle (not equilateral) is tangent to the incircle and to each of the excircles of the triangle.*

The corresponding facts for an equilateral triangle are quite obvious.

7.7. Consider now the question as to whether for a given triangle $P_1P_2P_3$ there is a circle with respect to which the triangle is self-polar (see section 4.3). If there is such a circle, its center must be the orthocenter of the triangle; and if the altitude from the vertex P_i meets the opposite side at the point D_i, each pair of inverse points, P_i, D_i, must lie on the same side of the orthocenter. This is possible only if the orthocenter is outside the triangle, and hence only an obtuse triangle can be self-polar with respect to any circle. We shall show that for any given obtuse triangle there is always a unique circle with respect to which the triangle is self-polar, and this circle may be called the *polar circle of the triangle.*

Fixing the coordinate system so that the vertices of the triangle are the points t_1, t_2, t_3 on the unit circle, the orthocenter will be the point s_1, and since the triangle is obtuse we have (see 7.32) $(s_1s_2-s_3)/s_3>0$. If there is a polar circle, its equation must be

$$(x - s_1)\left(y - \frac{s_2}{s_3}\right) = r^2, \tag{7.71}$$

where the value of r^2 is to be determined. The altitude from t_1 meets the opposite side at the point (see 7.22)

$$x = \frac{s_1}{2} - \frac{s_3}{2t_1^2},$$

and if this point and t_1 are to be inverse points with respect to the circle 7.71 we must have (see 4.21)

$$\left(\frac{s_1}{2} - \frac{s_3}{2t_1^2} - s_1\right)\left(\frac{1}{t_1} - \frac{s_2}{s_3}\right) = r^2,$$

or

$$r^2 = \frac{s_3(-s_1t_1^2 + s_2t_1 - s_3) + s_1s_2t_1^3}{2s_3t_1^3}.$$

But we have $t_1^3-s_1t_1^2+s_2t_1-s_3=0$, and hence

$$-s_1t_1^2 + s_2t_1 - s_3 = -t_1^3$$

and

$$r^2 = \frac{-s_3t_1^3 + s_1s_2t_1^3}{2s_3t_1^3} = \frac{s_1s_2 - s_3}{2s_3}.$$

Since $(s_1s_2-s_3)/s_3>0$, this value for r^2 is positive; and hence

$$(x - s_1)\left(y - \frac{s_2}{s_3}\right) = \frac{s_1s_2 - s_3}{2s_3}$$

represents a circle with center at s_1, and having the vertex t_1 and the foot of the altitude from that vertex as inverse points. The symmetrical form of the result shows that each of the other vertices together with the foot of the corresponding altitude will also be a pair of inverse points, and we have

THEOREM 7.72. *If the triangle $t_1t_2t_3$ is obtuse, so that $(s_1s_2-s_3)/s_3>0$, it is a self-polar triangle with respect to the circle*

$$(x - s_1)\left(y - \frac{s_2}{s_3}\right) = \frac{s_1s_2 - s_3}{2s_3}, \tag{7.73}$$

and this circle is the polar circle of the triangle.

This polar circle and three other circles that have been mentioned in connection with a triangle are circles of a pencil (see section 4.5). The four circles are the circumcircle, the nine-point circle 7.24, the polar circle 7.73 when it exists, and the circumcircle of the tangential triangle 7.44. We assume in this discussion that the triangle is not a right triangle and is not equilateral so that $s_1s_2-s_3\neq0$, $s_1\neq0$, $s_2\neq0$. The equations of these circles may be written respectively in the forms

$$xy - 1 = 0, \tag{7.74}$$

$$xy - \frac{s_2}{2s_3}x - \frac{s_1}{2}y + \frac{s_1s_2 - s_3}{4s_3} = 0, \tag{7.75}$$

$$xy - \frac{s_2}{s_3}x - s_1y + \frac{s_1s_2 + s_3}{2s_3} = 0, \tag{7.76}$$

$$xy - \frac{2s_2}{s_1s_2 - s_3}x - \frac{2s_1s_3}{s_1s_2 - s_3}y + \frac{4s_3}{s_1s_2 - s_2} = 0. \tag{7.77}$$

It is readily verified that if we multiply 7.74 by -1 and 7.75 by 2 and add we get 7.76; and if we multiply 7.74 by $(s_1s_2-5s_3)/(s_1s_2-s_3)$ and 7.75 by $4s_3/(s_1s_2-s_3)$ and add we get 7.77. Also, if we subtract any one of the equations from any other one, we obtain the linear equation

$$2s_2x + 2s_1s_3y - s_1s_2 - 3s_3 = 0.$$

This proves

THEOREM 7.78. *If the triangle $t_1t_2t_3$ is not a right triangle and is not equilateral, its circumcircle, nine-point circle, polar circle (when it exists), and the circumcircle of its tangential triangle all belong to a pencil whose radical axis is*

$$2s_2x + 2s_1s_3y - s_1s_2 - 3s_3 = 0.$$

The centers of these circles, as we have already seen, are on the Euler line (see 7.14, 7.21, and 7.43) $s_2x - s_1s_3y = 0$, perpendicular to the radical axis.

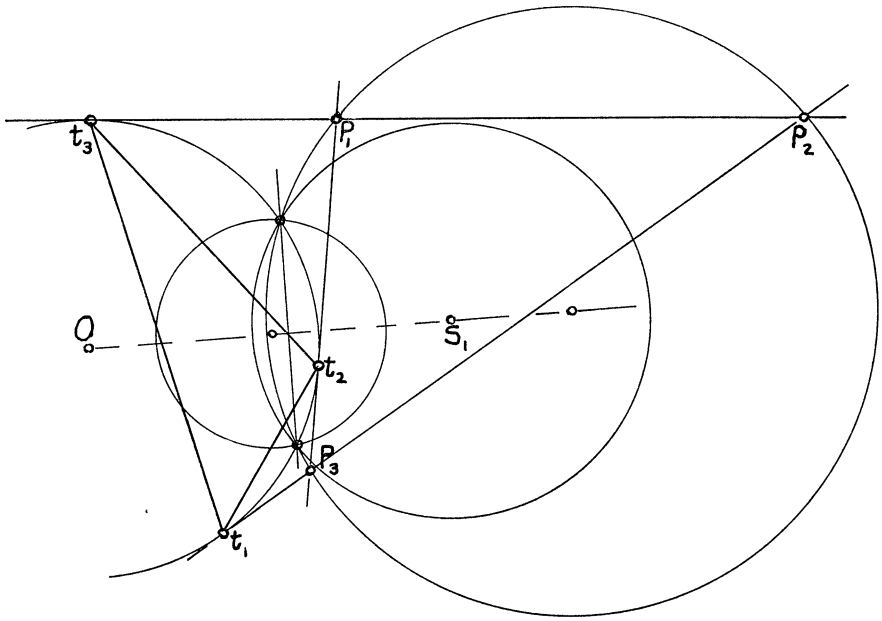


FIG. 21

The question as to whether these circles and their radical axis intersect or not can be answered by determining whether the radical axis cuts any one of the circles,* say the circle $xy - 1 = 0$. The square of the distance from the center of this circle to the radical axis (see 3.55) is

$$d^2 = \frac{(s_1s_2 + 3s_3)^2}{16s_1s_2s_3},$$

* A natural procedure to determine whether or not a line intersects a circle would be to solve the equations simultaneously and then examine the pairs of values of x and y obtained as solutions to see if they are conjugates. But these solutions for x and y would be irrational in form, and it is rather difficult to determine whether or not *irrational* expressions are conjugates. In the case of a line cutting a circle it is simpler to use the distance from the center of the circle to the line.

and hence the axis cuts the circle in two points if

$$\frac{(s_1s_2 + 3s_3)^2}{16s_1s_2s_3} - 1 < 0.$$

But

$$\frac{(s_1s_2 + 3s_3)^2}{16s_1s_2s_3} - 1 = \frac{s_1^2s_2^2 - 10s_1s_2s_2 + 9s_3^2}{16s_1s_2s_3} = \frac{(s_1s_2 - s_3)(s_1s_2 - 9s_3)}{16s_1s_2s_3}$$

$$= \frac{\left(\frac{s_1s_2}{s_3} - 1\right)\left(\frac{s_1s_2}{s_3} - 9\right)}{\frac{16s_1s_2}{s_3}}.$$

Since $s_1s_2/s_3 = s_1\bar{s}_1$, it is real and positive. Also since s_1 is the sum of three different turns, we have $|s_1| < 3$, or $|s_1|^2 = s_1s_2/s_3 < 9$. Therefore in the last fraction above the second factor in the numerator is negative, the denominator is positive, and the whole fraction will be negative when, and only when, $(s_1s_2 - s_3)/s_3 > 0$. This is the condition that the triangle should be obtuse and that the polar circle 7.73 should exist. Thus for an obtuse triangle we have *four* circles of a pencil and their radical axis intersecting in two distinct points. For an acute triangle $(s_1s_2 - s_3)/s_3 < 0$, and in this case there is no polar circle and the three circles 7.74, 7.75, 7.77 are in a non-intersecting pencil. For a right triangle, excluded from the foregoing discussion, $(s_1s_2 - s_3)/s_3 = 0$, there is no tangential triangle, circles 7.76 and 7.77 do not exist, and the circles 7.74 and 7.75 are tangent to each other and to the radical axis at the vertex of the right angle. For an equilateral triangle, also excluded from the discussion, the polar circle does not exist and the other three circles are obviously concentric.

7.8. Three circles of another pencil are related to any triangle which is not isosceles or equilateral. They are called the *circles of Apollonius* or the *Apollonian circles*. One of these circles passes through the vertex t_1 and has its center at the intersection of the side t_2t_3 and the corresponding side P_2P_3 of the tangential triangle; and the other two circles are similarly related to the other two vertices and sides of the triangle. To find the equation of the circle through the vertex t_1 we note that the equations of the lines t_2t_3 and P_2P_3 are respectively (see 7.11 and 7.41)

$$\frac{x}{t_2t_3} + y = \frac{1}{t_2} + \frac{1}{t_3} \quad \text{and} \quad \frac{x}{t_1^2} + y = \frac{2}{t_1}.$$

Eliminating y and solving for x we have

$$x = \frac{t_1(2t_2t_3 - t_3t_1 - t_1t_2)}{t_2t_3 - t_1}$$
$$= \frac{t_1(3s_3 - s_2t_1)}{s_3 - t_1^3}.$$

Then, using the relation $t_1^3 - s_1t_1^2 + s_2t_1 - s_3 = 0$, we have

$$x = \frac{t_1(3s_3 - s_2t_1)}{s_2t_1 - s_1t_1^2} = \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}.$$

The center of the circle is then the point

$$\left(\frac{3s_3 - s_2t_1}{s_2 - s_1t_1}, \frac{s_1 - 3t_1}{s_2 - s_1t_1}\right),$$

and the equation of the circle is

$$xy - \frac{s_1 - 3t_1}{s_2 - s_1t_1}x - \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}y + g = 0,$$

where g is to be determined. Since the circle goes through the vertex t_1, we must have

$$1 - \frac{s_1 - 3t_1}{s_2 - s_1t_1}t_1 - \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}\left(\frac{1}{t_1}\right) + g = 0,$$

or

$$g = \frac{(s_1 - 3t_1)t_1}{s_2 - s_1t_1} + \frac{3s_3 - s_2t_1}{t_1(s_2 - s_1t_1)} - 1$$
$$= \frac{-3t_1^3 + 2s_1t_1^2 - 2s_2t_1 + 3s_3}{t_1(s_2 - s_1t_1)}.$$

Using the identity $t_1^3 - s_1t_1^2 + s_2t_1 - s_3 = 0$, we have

$$g = \frac{-s_1t_1^2 + s_2t_1}{t_1(s_2 - s_1t_1)} = 1,$$

and the Apollonian circle corresponding to the vertex t_1 is therefore

$$xy - \frac{s_1 - 3t_1}{s_2 - s_1t_1}x - \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}y + 1 = 0. \tag{7.81}$$

The equations of the three Apollonian circles may then be written

$$(s_2 - s_1t_i)xy - (s_1 - 3t_i)x - (3s_3 - s_2t_i)y + (s_2 - s_1t_i) = 0$$

or

$$(7.82) \qquad (s_2xy - s_1x - 3s_3y + s_2) - t_i(s_1xy - 3x - s_2y + s_1) = 0, \qquad i = 1, 2, 3.$$

That these circles belong to a pencil is seen by noting that the radical axis of *any two* of them is the line

$$(7.83) \qquad (s_1^2 - 3s_2)x - (s_2^2 - 3s_1s_3)y = 0,$$

and this radical axis is the Brocard line (see 7.55) through the circumcenter and the symmedian point.

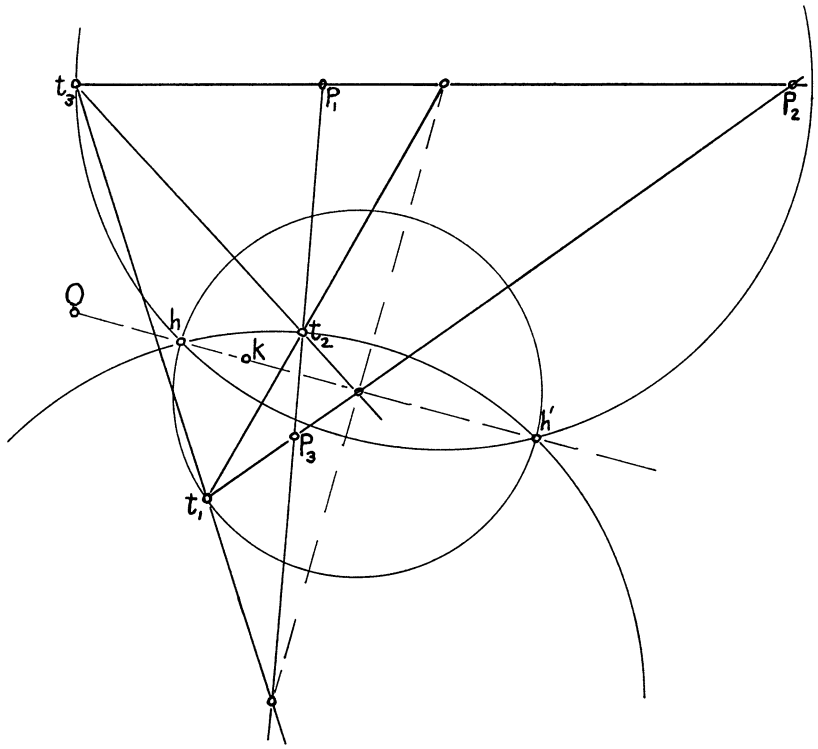


FIG. 22

That any two of the Apollonian circles must intersect in two points is intuitively obvious upon examination of the figure; and since the three circles belong to a pencil there must be two points common to all three circles and their radical axis. We shall find these two points in the next section, and show that their conjugate coordinates satisfy the equations*

$$(7.84) \qquad \begin{aligned} s_2xy - s_1x - 3s_3y + s_2 &= 0 \\ s_1xy - 3x - s_2y + s_1 &= 0, \end{aligned}$$

* These equations do not represent curves because they are not self-conjugate.

and hence the equations 7.82 of all three circles.

Since the Apollonian circles are in a pencil their centers must lie on a line. This line is called the *Lemoine line* of the triangle, and it is perpendicular to the radical axis of the pencil which is the Brocard line.

If the triangle is isosceles, say sides t_1t_2 and t_1t_3 are equal, we have $t_1^2 = t_2t_3$, or $s_2 - s_1t_1 = 0$. More generally, for any isosceles triangle we must have

$$(7.85) \qquad (s_2 - s_1t_1)(s_2 - s_1t_2)(s_2 - s_1t_3) = s_2^3 - s_1^3s_3 = 0.$$

In this case certain denominators in the preceding discussion will be zero, but the geometric facts in this special case are fairly obvious. If, for instance, sides t_1t_2 and t_1t_3 are equal, there will be no Apollonian circle through the vertex t_1; but the other two circles will exist and their radical axis, the axis of symmetry of the whole figure, will be both the Euler line and the Brocard line of the triangle. The reader may verify that when $s_2^3 - s_1^3s_3 = 0$, the equations 7.14 and 7.55 represent the same line.

7.9. In this section we shall assume that the vertices t_1, t_2, t_3, of our triangle are on the unit circle in *counter-clockwise order*, and that the triangle is not equilateral (Figure 23). Let us seek a point h whose reflections, h_1, h_2, h_3, in the sides t_2t_3, t_3t_1, t_1t_2, respectively are vertices of an equilateral triangle. Since h and h_1 are inverse points with respect to the side t_2t_3 with the equation (see 7.11)

$$\frac{x}{t_2t_3} + y = \frac{1}{t_2} + \frac{1}{t_3},$$

we must have (see 3.52)

$$\frac{h_1}{t_2t_3} + \bar{h} = \frac{1}{t_2} + \frac{1}{t_3},$$

or

$$h_1 = t_2 + t_3 - \bar{h}t_2t_3;$$

and similarly for h_2 and h_3. Then h_1, h_2, h_3 will be vertices of an equilateral triangle *in counter-clockwise* order (see 1.73) if

$$(t_2 + t_3 - \bar{h}t_2t_3) + \omega(t_3 + t_1 - \bar{h}t_3t_1) + \omega^2(t_1 + t_2 - \bar{h}t_1t_2) = 0,$$

or, using the relation $1 + \omega + \omega^2 = 0$ (see 1.51), if

$$\bar{h} = -\frac{t_1 + \omega t_2 + \omega^2 t_3}{t_2t_3 + \omega t_3t_1 + \omega^2 t_1t_2},$$

or if

$$(7.91) \qquad h = -\frac{t_2t_3 + \omega^2 t_3t_1 + \omega t_1t_2}{t_1 + \omega^2 t_2 + \omega t_3}.$$

If we set

$$p = t_1 + \omega t_2 + \omega^2 t_3, \qquad q = t_1 + \omega^2 t_2 + \omega t_3, \tag{7.92}$$

with

$$\bar{p} = \frac{t_2 t_3 + \omega^2 t_3 t_1 + \omega t_1 t_2}{s_3}, \qquad \bar{q} = \frac{t_2 t_3 + \omega t_3 t_1 + \omega^2 t_1 t_2}{s_3},$$

we can write

$$h = -\frac{\bar{p} s_3}{q}, \qquad \bar{h} = -\frac{p}{\bar{q} s_3}. \tag{7.93}$$

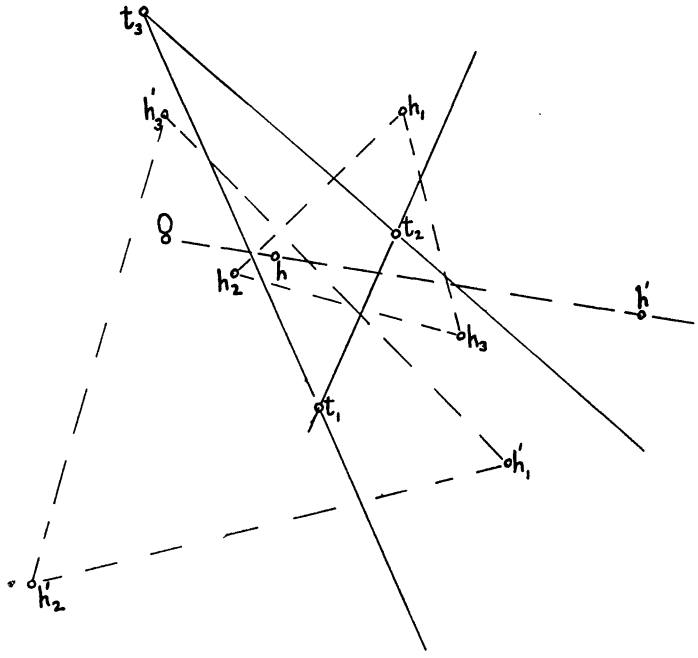


FIG. 23

Similarly, the reflections h_1', h_2', h_3' of a point h' in the sides of the triangle will be vertices of an equilateral triangle *in clockwise order* if

$$h' = -\frac{\bar{q} s_3}{p}, \qquad \bar{h}' = -\frac{q}{\bar{p} s_3}. \tag{7.94}$$

These expressions for p and q and the coordinates of the points h and h' are not symmetric functions of t_1, t_2, t_3; but certain combinations of them are. The reader should verify the following relations.

$$
\begin{aligned}
pq &= s_1^2 - 3s_2 \\
s_3^2 p\bar{p}q\bar{q} &= s_2^2 - 3s_1s_3 \\
s_3(p\bar{p} + q\bar{q}) &= 9s_3 - s_1s_2 \\
h + h' &= -\frac{9s_3 - s_1s_2}{s_1^2 - 3s_2} \\
hh' &= \frac{s_2^2 - 3s_1s_3}{s_1^2 - 3s_2}.
\end{aligned} \tag{7.95}
$$

Hence, although h and h' separately are not rational symmetric functions of t_1, t_2, t_3, they are roots of a quadratic equation,

$$(s_1^2 - 3s_2)x^2 + (9s_3 - s_1s_2)x + (s_2^2 - 3s_1s_3) = 0,$$

whose coefficients are such rational symmetric functions.

The points h and h' are called the *isodynamic points* of the triangle $t_1t_2t_3$. The reader may verify readily that they are inverse points with respect to the circumcircle $xy-1=0$. They lie on the Brocard line, and are in fact the points of intersection of the Apollonian circles (see section 7.8). These facts are established by showing that the coordinates $(h, \bar{h})$, and also $(h', \bar{h}')$, satisfy the equations (see 7.84)

$$
\begin{aligned}
s_2xy - s_1x - 3s_3y + s_2 &= 0, \\
s_1xy - 3x - s_2y + s_1 &= 0.
\end{aligned}
$$

For $(h, \bar{h})$ we have

$$
\begin{aligned}
\frac{s_2\bar{p}p}{\bar{q}q} + \frac{s_1s_3\bar{p}}{q} + \frac{3p}{\bar{q}} + s_2 &= \frac{s_2(\bar{p}p + \bar{q}q) + s_1s_3\bar{p}\bar{q} + 3pq}{q\bar{q}} \\
&= \frac{s_2(9s_3 - s_1s_2) + s_1(s_2^2 - 3s_1s_3) + 3s_3(s_1^2 - 3s_2)}{q\bar{q}s_3} = 0;
\end{aligned}
$$

and also

$$
\begin{aligned}
\frac{s_1\bar{p}p}{\bar{q}q} + \frac{3s_3\bar{p}}{q} + \frac{s_2p}{s_3\bar{q}} + s_1 &= \frac{s_1s_3(\bar{p}p + \bar{q}q) + 3s_3^2\bar{p}\bar{q} + s_2pq}{\bar{q}qs_3} \\
&= \frac{s_1(9s_3 - s_1s_2) + 3(s_2^2 - 3s_1s_3) + s_2(s_1^2 - 3s_2)}{\bar{q}qs_3} = 0.
\end{aligned}
$$

The reader may verify similarly that the coordinates $(h', \bar{h}')$ satisfy the two equations.

We have then established the following theorems about the Apollonian

circles and isodynamic points of a triangle (see Figures 22 and 23):

THEOREM 7.96. *The three Apollonian circles of a triangle cut each other in the two isodynamic points.*

THEOREM 7.97. *The centers of the Apollonian circles (the points where the three sides of the triangle cut the three corresponding sides of the tangential triangle) lie on the Lemoine line.*

THEOREM 7.98. *The isodynamic points lie on the Brocard line of the triangle, and they are inverse points with respect to the circumcircle.*

THEOREM 7.99. *The reflections of each of the isodynamic points in the sides of the triangle are vertices of an equilateral triangle.*

Chapter VIII

THE PARABOLA

8.1. Let F be the focus of a parabola and H the reflection of the focus in the directrix. From any point P of the parabola drop the perpendicular PZ to the directrix. From the definition of a parabola $PZ=PF$, one sees that FPZ and

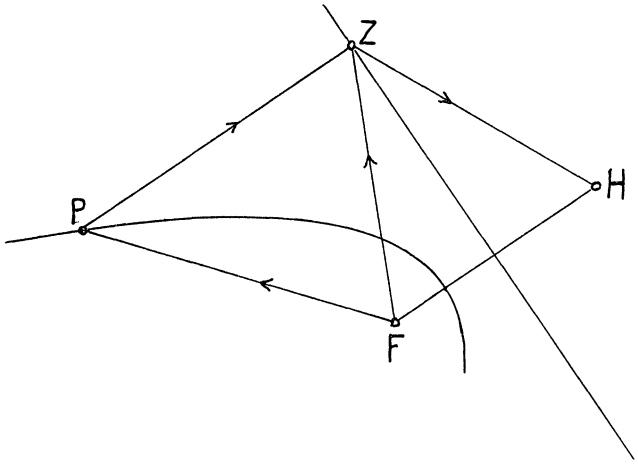


FIG. 24

FZH are similar isosceles triangles. If now in a system of conjugate coordinates, F and H correspond to the complex numbers f and h, and P and Z to the complex variables x and z, we see that vector $z-x$ is equal to the vector $x-f$ multiplied by a certain turn t, and vector $h-z$ is equal to vector $z-f$ multiplied by *the same turn t.* Hence we have

$$z - x = t(x - f)$$
$$h - z = t(z - f),$$

and eliminating z between these two equations we obtain

$$x = f + \frac{h - f}{(1 + t)^2} \tag{8.11}$$

as a general map equation of a parabola. For any parabola it is always possible to set up the coordinate system so that f and h are the points 0 and 1 respectively, and we then have the simplified map equation for the parabola

$$x = \frac{1}{(1 + t)^2} \cdot \tag{8.12}$$

With this map equation the focus of the parabola is at the origin and the directrix is the line with the self-conjugate equation

$$x + y = 1. \tag{8.13}$$

The reader may verify that the self-conjugate equation of this parabola is

$$(x - y)^2 - 2(x + y) - 1 = 0.$$

From 8.12, writing $R(t)$ for $1/(1+t)^2$, we have

$$D_t x = R'(t) = \frac{-2}{(1 + t)^3},$$

$$\overline{D_t x} = \frac{-2t^3}{(1 + t)^3},$$

and hence the clinant of the tangent to the parabola at the point $R(t_1)$ is (see 5.42) t_1. This gives a geometrical significance to the parameter t in the map equation 8.12.

The clinant of the focal radius to the point $R(t_1)$ of the parabola is (see 3.42)

$$\tau = -\frac{\overline{R(t_1)}}{R(t_1)} = -\frac{\dfrac{t_1^2}{(1 + t_1)^2}}{\dfrac{1}{(1 + t_1)^2}} = -t_1^2,$$

and the clinant of the line through the point $R(t_1)$ parallel to the axis of reals is -1; therefore the clinants of the bisectors of the angles between these lines (see 3.32) are

$$\pm \sqrt{(-t_1^2)(-1)} = \pm t_1.$$

We have then the well known result.

THEOREM 8.14. *The tangent and normal at any point P of a parabola bisect the angles made by the focal radius FP and the line through P parallel to the axis of the parabola.*

The proofs of the two following theorems may be left as exercises for the reader.

THEOREM 8.15. *The points of tangency of two perpendicular tangents to a parabola are collinear with the focus.*

THEOREM 8.16. *A circle having a focal chord of a parabola as a diameter is tangent to the directrix.*

8.2. Consider now the map equation

$$x = \frac{1}{(1+t_1)(1+t)}, \tag{8.21}$$

in which t_1 is any *fixed* turn other than -1. This is a map equation of a straight line (see 3.11), and the reader can readily verify that the clinant of the line is t_1. But it is obvious that this line goes through the point $R(t_1)$ of the parabola when $t=t_1$; and since its clinant is the same as that of the tangent at $R(t_1)$, it follows that 8.21 is a map equation of the tangent to the parabola 8.12 at the point $R(t_1)$.

From the form of the equation 8.21 it is evident that the point

$$\frac{1}{(1+t_1)(1+t_2)} \tag{8.22}$$

is the point of intersection of the two tangents to the parabola at the points $R(t_1)$ and $R(t_2)$. If the two tangents are perpendicular to each other, $t_2=-t_1$, and the point of intersection is

$$x = \frac{1}{1-t_1^2}.$$

But when $x=1/(1-t_1^2)$, $y=\bar{x}=-t_1^2/(1-t_1^2)$, and $x+y=1$, which is the equation of the directrix (see 8.13). Hence

THEOREM 8.23. *The locus of intersections of pairs of perpendicular tangents to a parabola is the directrix.*

8.3. Consider the triangle formed by three tangents, l_i, with clinants t_i and map equations

$$x = \frac{1}{(1+t_i)(1+t)}, \qquad i = 1, 2, 3.$$

The vertex where l_2 and l_3 intersect is (see 8.22)

$$v_1 = \frac{1}{(1+t_2)(1+t_3)} = \frac{1+t_1}{(1+t_1)(1+t_2)(1+t_3)} = \frac{1+t_1}{1+s_1+s_2+s_3},$$

where s_1, s_2, s_3 are the symmetric functions of the three clinants. We shall write s for $1+s_1+s_2+s_3$, and note that since $t_i \neq -1$, $s \neq 0$; and also

$$\bar{s} = 1 + \frac{s_2}{s_3} + \frac{s_1}{s_3} + \frac{1}{s_3} = \frac{s}{s_3}. \tag{8.31}$$

Then the three vertices of the triangle are

$$v_i = \frac{1 + t_i}{s}, \qquad i = 1, 2, 3; \tag{8.32}$$

and these vertices evidently lie on the circle

$$x = \frac{1}{s} + \frac{1}{s}t, \tag{8.33}$$

which is the circumcircle of the triangle, with center at $1/s$ and radius $|1/s|$, and it passes through the origin when $t = -1$. Hence

THEOREM 8.34. *The circumcircle of a triangle formed by any three tangents to a parabola passes through the focus of the parabola.*

Now consider any four tangents to the parabola, l_i, with clinants t_i, $i = 1, 2, 3, 4$. Let $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ be the symmetric functions of the four clinants, and s_1, s_2, s_3 the symmetric functions of three of them when any one, say t_k, is omitted. We have

$$s = 1 + s_1 + s_2 + s_3 = \frac{(1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)}{1 + t_k}$$
$$= \frac{1 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4}{1 + t_k} = \frac{\Sigma}{1 + t_k},$$

where Σ is written for $1+\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4$. Then the circumcenter of the triangle of three tangents, omitting the tangent l_k, is

$$\frac{1}{s} = \frac{1 + t_k}{\Sigma}.$$

The four such circumcenters lie on the circle

$$x = \frac{1}{\Sigma} + \frac{1}{\Sigma}t, \tag{8.35}$$

and this circle also passes through the origin. We have then

THEOREM 8.36. *The circumcenters of the four triangles formed by any four tangents to a parabola lie on a circle which passes through the focus of the parabola.*

We now find the orthocenter of a triangle of three tangents. The vertex v_1 is (see 8.32) $(1+t_1)/s$, and the clinant of the altitude through this vertex is $-t_1$. Hence the equation of this altitude is

$$-t_1 x + y = \frac{-t_1(1 + t_1)}{s} + \frac{(1 + t_1)s_3}{t_1 s} = \frac{-t_1^3 + s_3 - t_1^2 + s_3 t_1}{t_1 s}.$$

Using the identity $t_1^3 - s_1 t_1^2 + s_2 t_1 - s_3 = 0$ (see 6.15) this becomes

$$-t_1 x + y = \frac{-s_1 t_1 + s_2 - t_1 + s_3}{s},$$

or

$$t_1(-sx + s_1 + 1) + (sy - s_2 - s_3) = 0.$$

Then the equations of the three altitudes are

$$t_i(-sx + s_1 + 1) + (sy - s_2 - s_3) = 0, \qquad i = 1, 2, 3,$$

and all three equations are obviously satisfied by the pair of *conjugate* values

$$x = \frac{1 + s_1}{s}, \qquad y = \frac{s_2 + s_3}{s}.$$

Hence the point

$$x = \frac{1 + s_1}{s} \tag{8.37}$$

is the orthocenter of the triangle. Adding the coordinates of this point we have

$$x + y = \frac{1 + s_1 + s_2 + s_3}{s} = 1,$$

and since $x+y=1$ is the equation of the directrix of the parabola (see 8.13) we have

THEOREM 8.38. *The orthocenter of any triangle formed by three tangents to a parabola lies on the directrix.*

8.4. By a *proper n-line* we shall mean a set of n lines no two of which are parallel and no three concurrent. Thus a proper 3-line is a triangle. It is well-known, from the point of view of projective geometry, that a proper 4-line determines a unique parabola to which each of the four lines is tangent. Hence for any proper 4-line the coordinate system may be set up in such a way that the four lines will be tangents to the parabola with the map equation 8.12, and therefore the preceding theorems about tangents to a parabola may be stated as theorems about any proper 4-line. Thus we have

THEOREM 8.41. *For any proper 4-line the circumcircles of the four triangles (omitting one line at a time) pass through a point called the node of the 4-line, this node being the focus of the parabola tangent to the four lines; and the four circumcenters of the triangles lie on a circle, the centric-circle of the 4-line, and this centric-circle also passes through the node.*

THEOREM 8.42. *For any proper 4-line the orthocenters of the four triangles (omitting one line at a time) lie on a line which is the directrix of the parabola tangent to the four lines.*

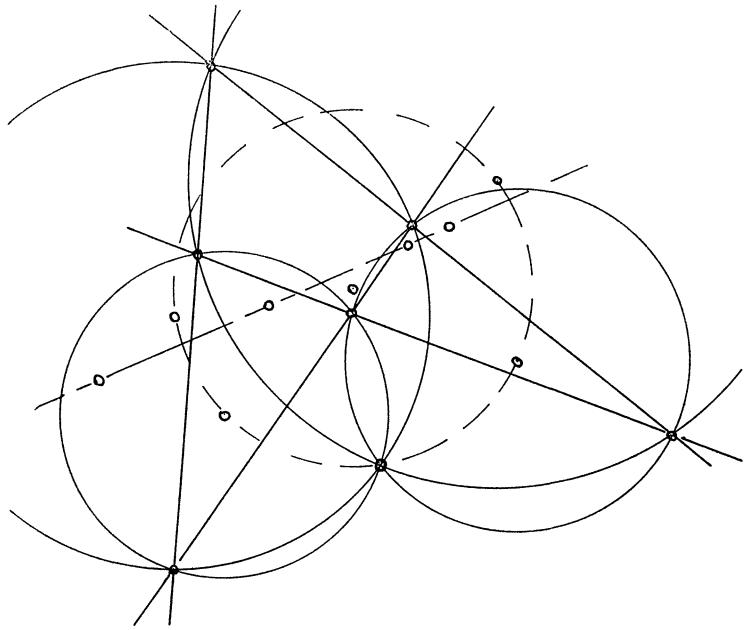


FIG. 25

We prove also the following

THEOREM 8.43. *If one line of a proper 4-line is parallel to the Euler line of the triangle formed by the other three lines, the same is true for every line of the 4-line.**

We regard the four lines as tangents to the parabola 8.12, map equations of the tangents being

$$x = \frac{1}{(1 + t_i)(1 + t)} \qquad i = 1, 2, 3, 4.$$

We have seen (see 8.33 and 8.37) that the circumcenter and orthocenter of a triangle of three tangents are respectively

$$\frac{1}{s} \quad \text{and} \quad \frac{1 + s_1}{s}.$$

* This problem, proposed by Howard Eves, appeared as problem 617 in the May, 1945, issue of the National Mathematics Magazine, and again as problem 54 in the January–February, 1950, issue of the Mathematics Magazine. A synthetic proof is given in a paper, The complete quadrilateral, by H. E. Fettis in the September–October, 1948, issue of the same magazine.

Then the clinant of the Euler line through these two points (see 3.42) is

$$-\frac{\frac{\bar{s}_1}{\bar{s}}}{\frac{s_1}{s}} = -\frac{\frac{s_2}{s}}{\frac{s_1}{s}} = -\frac{s_2}{s_1}.$$

Hence this Euler line is parallel to the remaining line if

$$-\frac{s_2}{s_1} = t_k,$$

or if

$$s_2 + s_1 t_k = 0.$$

But (see 6.21) $s_2 + s_1 t_k = \Sigma_2$, and the condition for parallelism becomes

$$\Sigma_2 = 0.$$

The symmetry of this result for the four lines proves the theorem.

Chapter IX

HYPOCYCLOIDS

9.1. If a circle rolls around a fixed base circle with no slipping at the point of contact, a point P on the circumference of the rolling circle describes a cycloidal curve, a *hypocycloid* if the rolling circle is inside the base circle and an *epicycloid* if the rolling circle is outside the base circle.

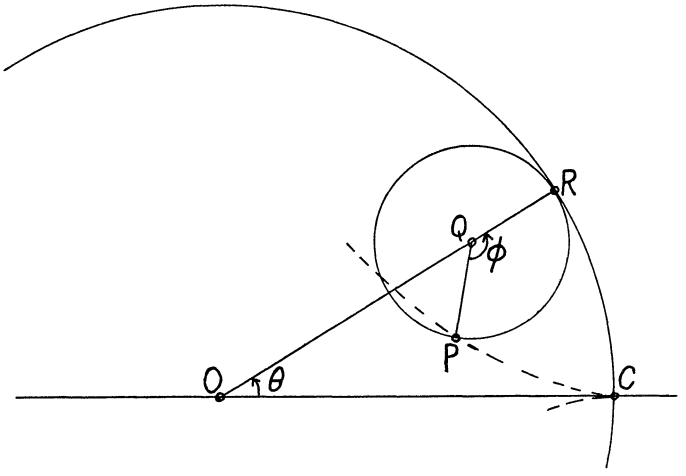


FIG. 26

Consider first the simple hypocycloid for which the radius of the base circle is n times that of the rolling circle, n an integer greater than 1. It is rather obvious that the point P describing the curve will come into contact with the base circle at n equally spaced points, resulting in a closed curve of n cusps.* We may think of the rolling motion as starting with the point P in contact with the base circle at a cusp C. We let r and r_1 be the radii of the base and rolling circles respectively, $r = nr_1$, and we set up the coordinate system with the origin O at the center of the base circle and the cusp C on the axis of reals at the point r. Let Q and R be respectively the moving center of the rolling circle and the moving point of contact of the two circles; and let θ be the variable angle from the vector OC to the vector OR, and ϕ the variable angle from the vector QP to the vector QR. Since the arc CR of the base circle must be of the same length as the arc PR of the rolling circle, we have $r\theta = r\phi/n$, or $\phi = n\theta$. Then

$$\text{vector } OP = \text{vector } OQ + \text{vector } QP,$$

* The reader is probably familiar with the hypocycloid of four cusps whose equation in Cartesian coordinates is

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

$$\text{vector } OQ = \left(r - \frac{r}{n}\right)e^{i\theta} = \frac{r}{n}(n-1)e^{i\theta},$$

$$\text{vector } QP = \frac{r}{n}e^{-i(\phi-\theta)} = \frac{r}{n}e^{-i\theta(n-1)}.$$

If now we put $e^{i\theta}=t$, and let x be the complex number corresponding to the point P (and therefore also to the vector OP), we have

$$x = \frac{r}{n}\left\{(n-1)t + \frac{1}{t^{n-1}}\right\}, \qquad n > 1, \tag{9.11}$$

as a map equation of the hypocycloid of n cusps. Letting $\epsilon = e^{2\pi i/n}$, so that ϵ^s, $s=1, 2, \cdots, n$, are the nth roots of unity (see section 1.5), it is geometrically obvious that the cusps occur at the points $r\epsilon^s$, $s=1, 2, \cdots, n$. These are the points where the derivative $D_t x$ vanishes; for we have from 9.11,

$$D_t x = \frac{r(n-1)(t^n-1)}{nt^n},$$

and this derivative vanishes when $t=\epsilon^s$. The corresponding points of the curve are

$$x = \frac{r}{n}\left\{(n-1)\epsilon^s + \frac{1}{\epsilon^{s(n-1)}}\right\} = \frac{r}{n}\left\{(n-1)\epsilon^s + \left(\frac{1}{\epsilon^{n-1}}\right)^s\right\}$$

$$= \frac{r}{n}\{(n-1)\epsilon^s + \epsilon^s\} = r\epsilon^s,$$

and hence the cusps occur at the points where $D_t x = 0$ (see section 5.4).

We now consider some special cases of interest.

9.2. The map equation of the hypocycloid of two cusps is

$$x = \frac{r}{2}\left(t + \frac{1}{t}\right). \tag{9.21}$$

Since t and $1/t$ are conjugates, their sum is real, and the point x moves back and forth on the axis of reals from $x=r$ when $t=1$ to $x=-r$ when $t=-1$. Thus the hypocycloid of two cusps is a line segment of length $2r$, the "cusps" in this case being the ends of the segment. The derivative $D_t x$ vanishes when $t=\pm 1$, the values of t that give the ends of the segment.

If t is eliminated from the equation 9.21 and the conjugate equation

$$y = \frac{r}{2}\left(\frac{1}{t} + t\right),$$

we get the self-conjugate equation $x-y=0$ of the axis of reals. It should be noted that this self-conjugate equation is satisfied not only by all the points of the

segment given by the map equation 9.21 but also by points of the axis of reals outside this segment (see 5.13).

The more general map equation (see 5.21)

$$x = a + b\left(t + \frac{1}{t}\right), \tag{9.22}$$

where $b = \rho e^{i\theta} \neq 0$, represents a line segment of length 4ρ, making an angle θ with the axis of reals, and having its center at the point a.

9.3. The hypocycloid of three cusps is known as the *deltoid* and its map equation is

$$x = \frac{r}{3}\left(2t + \frac{1}{t^2}\right).$$

In order to study the properties of any one such deltoid, we may choose our unit of length in such a way that $r = 3$, and we then have the simpler map equation

$$x = 2t + \frac{1}{t^2}, \tag{9.31}$$

and in the study of the deltoid we shall write $R(t)$ for $2t+1/t^2$. For this curve we have

$$D_t x = 2 - \frac{2}{t^3},$$

$$\overline{D_t x} = 2 - 2t^3,$$

and hence the clinant of the tangent to the curve at the point $R(t_1)$ (see 5.42) is

$$\frac{2 - 2t_1^3}{t_1^2\left(2 - \frac{2}{t_1^3}\right)} = -t_1.$$

It follows that no two tangents to the deltoid can be parallel.

The cusp radius (radius of the base circle) is 3, and the cusps are at the points 3, 3ω, and $3\omega^2$, given by the values 1, ω, and ω^2 for t. The reader may verify the fact (which is rather obvious mechanically) that the deltoid is tangent to the unit circle at the points -1, $-\omega$, and $-\omega^2$.

Consider now the map equation

$$x = t_1 + t + \frac{1}{t_1 t}, \tag{9.32}$$

in which t_1 is a fixed turn. It is obvious that when t takes on the value t_1 the deltoid 9.31 and the curve 9.32 have the point $R(t_1)$ in common. Moreover, at this common point the tangents to the two curves have the same direction; because for the curve 9.32,

$$D_t x = 1 - \frac{1}{t_1 t^2},$$

$$\overline{D_t x} = 1 - t_1 t^2,$$

and the clinant of the tangent at the point where $t = t_1$ is

$$\frac{1 - t_1^3}{t_1^2\left(1 - \frac{1}{t_1^3}\right)} = -t_1,$$

the same as the clinant of the tangent to the deltoid at the common point. Hence for any fixed value of the turn t_1, the map equation 9.32 represents a curve tangent to the deltoid at the point $R(t_1)$. For different values of t_1 we have a family of curves one of which is tangent to the deltoid at each point of the deltoid. We call this family of curves, given by the map equation 9.32, *penosculants** of the deltoid. The similar penosculants of the parabola (see section 8.2) were the straight lines tangent to the parabola.

Letting T_1 represent either one of the square roots of t_1, so that $T_1^2 = t_1$, the equation 9.32 may be written

$$x = t_1 + \frac{1}{T_1}\left(T_1 t + \frac{1}{T_1 t}\right).$$

Comparing this with 9.22, and noting that $T_1 t$ is a turn (see section 5.2), we see that the penosculant is a hypocycloid of two cusps, a line segment, 4 units in length and with its center at the point t_1 on the unit circle. If $T_1 = e^{i\phi}$ (and hence $t_1 = e^{2i\phi}$), the segment makes an angle $-\phi$ with the axis of reals. The ends of the segment are the points $x = t_1 \pm 2/T_1 = T_1^2 \pm 2/T_1$, but these are the points $R(\pm 1/T_1)$ *on the deltoid.* We have then proved

THEOREM 9.33. *If at any point of a deltoid a tangent line segment is drawn with its ends on the deltoid, the mid-point of the segment will lie on the inscribed circle of the deltoid and its length will be four times the radius of this circle.*

The results of this paragraph may be stated in another way. If the center of a line segment 4 units long moves around a circle of unit radius in the counter-clockwise direction while the segment itself rotates about its center half as fast in the clockwise direction, the segment will describe a deltoid *in three distinct*

* The name penosculant was introduced by Morley because of the analogy with Study's theory of osculants; *cf.* Morley, Metric geometry of the plane n-line, Trans. Amer. Math. Soc., vol. 1, 1900, p. 97.

ways—each end of the segment will describe the deltoid and the segment will envelop the deltoid.

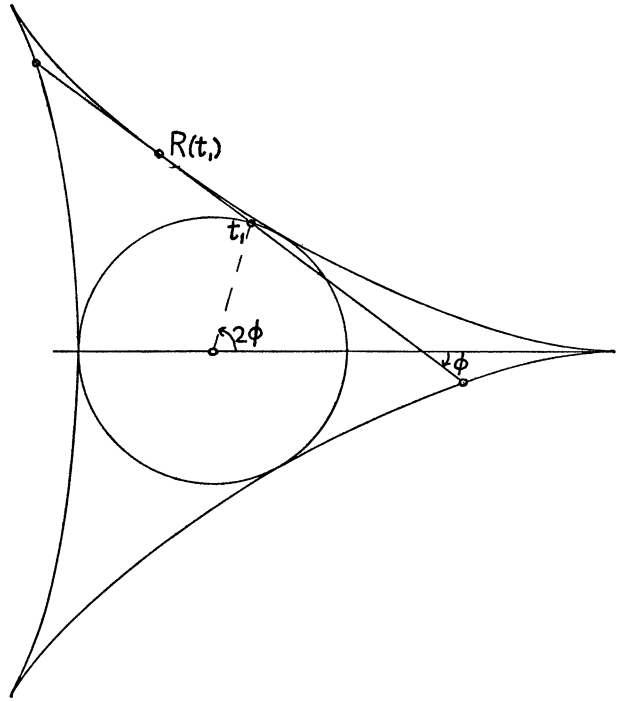


FIG. 27

9.4. The method of the preceding section may be extended to prove a remarkable theorem* about a simple hypocycloid of any number of cusps.

The map equation of a hypocycloid of n cusps, H_n, is

$$x = (n-1)t + \frac{1}{t^{n-1}} = R(t),$$

and it has a family of penosculant curves with map equations

$$x = t_1 + (n-2)t + \frac{1}{t_1 t^{n-2}}. \tag{9.41}$$

For any fixed t_1 the penosculant curve is tangent to the hypocycloid H_n at the point $R(t_1)$. If now T_1 is any one of the $(n-1)$th roots of t_1, so that $T_1^{n-1} = t_1$, the map equation of the penosculant curve may be written

* This theorem was given in a paper, On adjustable cycloidal and trochoidal curves, by Morley, Amer. J. of Math., vol. 16, 1894, p. 188.

$$x = t_1 + \frac{1}{T_1}\left\{(n-2)T_1 t + \frac{1}{(T_1 t)^{n-2}}\right\}. \tag{9.42}$$

Since $T_1 t$ is a turn, we see that this is the map equation (see 5.21) of a hypocycloid of $n-1$ cusps, H_{n-1}, with its center at the point t_1 of the unit circle. Its $n-1$ cusps occur for those values of t for which $D_t x = 0$. From equation 9.41,

$$D_t x = n - 2 - \frac{n-2}{t_1 t^{n-1}} = (n-2)\left(1 - \frac{1}{t_1 t^{n-1}}\right),$$

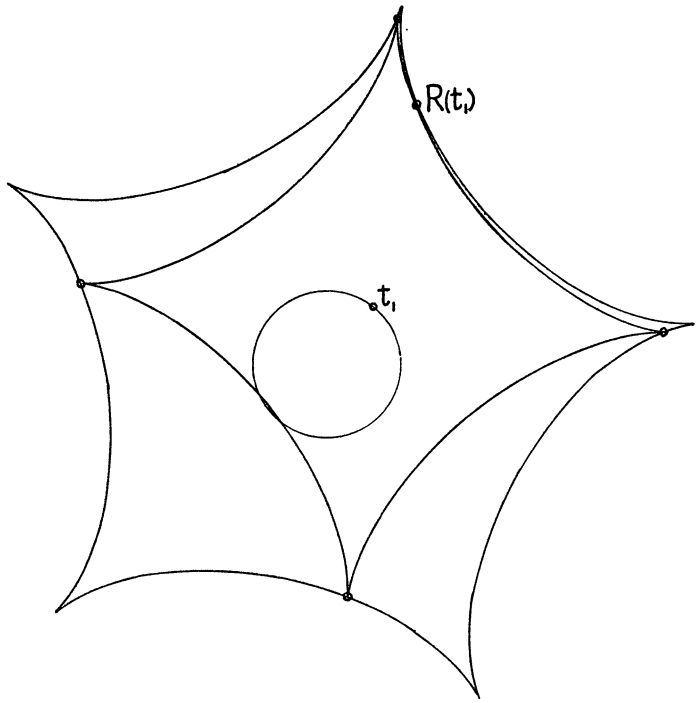


FIG. 28

and $D_t x$ vanishes when t is any one of the $(n-1)$th roots of $1/t_1$, that is, when $t = 1/T_1$. Putting this value of t in 9.41, we see that there is a cusp of H_{n-1} at the point

$$x = t_1 + (n-2)\frac{1}{T_1} + \frac{T_1^{n-2}}{t_1} = T_1^{n-1} + (n-2)\frac{1}{T_1} + \frac{1}{T_1}$$

$$= (n-1)\frac{1}{T_1} + \frac{1}{\left(\frac{1}{T_1}\right)^{n-1}}.$$

But this is the point $R(1/T_1)$ on the hypocycloid H_n; and hence each cusp of the penosculant hypocycloid H_{n-1} lies on the hypocycloid H_n. As the turn t_1 varies, the penosculant hypocycloid rotates with its $n-1$ cusps always on the hypocycloid H_n, its center on the unit circle, and always tangent to the hypocycloid H_n at the point $R(t_1)$. Figure 28 shows the case $n=5$. Theorem 9.33 is merely the special case for $n=3$ of this more general theorem.

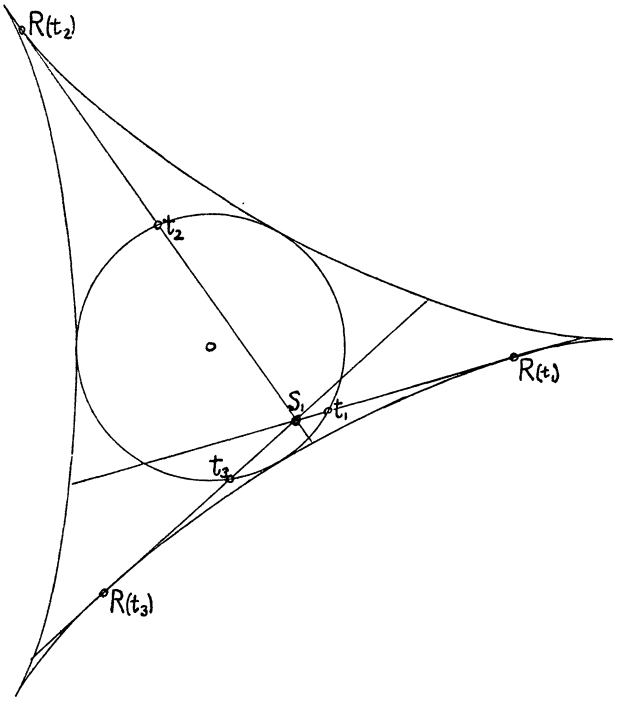


FIG. 29

9.5. Returning to a consideration of the deltoid, it is obvious that the two tangent line segments

$$x = t_1 + t + \frac{1}{t_1 t} \quad \text{and} \quad x = t_2 + t + \frac{1}{t_2 t}$$

intersect at the point

$$x = t_1 + t_2 + \frac{1}{t_1 t_2}.$$

The clinants of these two tangents (see section 9.3) are $-t_1$ and $-t_2$, and hence they are perpendicular if $t_2 = -t_1$. In this case their point of intersection is $x = -1/t_1^2$, a point on the unit circle. Thus the locus of intersections of pairs of

perpendicular tangents to the deltoid is the unit circle, the inscribed circle of the deltoid.

Three tangents which do not meet in a point form a triangle with vertices

$$x = s_1 - t_i + \frac{t_i}{s_3}, \qquad i = 1, 2, 3.$$

Then

$$x = s_1 + \left(\frac{1}{s_3} - 1\right)t \tag{9.51}$$

is the map equation of the circumcircle of such a triangle of tangents. The circumcenter of the triangle is the point s_1 which is the orthocenter (see 7.13) of the triangle $t_1t_2t_3$. If $s_3 = 1$, the radius of this circle becomes zero; and hence under the condition $s_3 = 1$, the three tangents to the deltoid at the points $R(t_i)$, $i = 1, 2, 3$, meet at the point s_1, the orthocenter of the triangle $t_1t_2t_3$ (Figure 29).

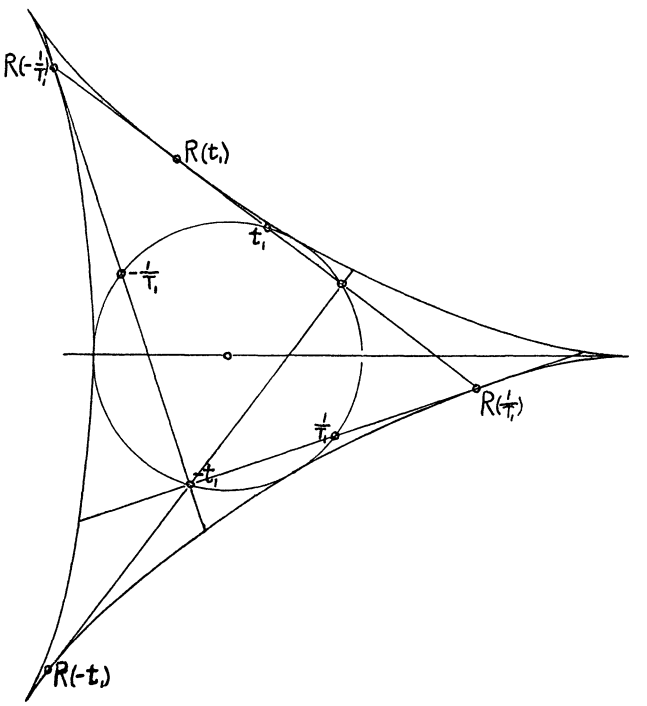


FIG. 30

The reader may show that the self-conjugate equation of the tangent to the deltoid $x = 2t + 1/t^2$ at the point where $t = t_1$ is

$$t_1^3 - t_1^2 x + t_1 y - 1 = 0,$$

and if three tangents meet at the point (x_1, y_1), the clinants of the tangents will be the three roots of the equation

$$t^3 + x_1t^2 + y_1t + 1 = 0.$$

We have seen that the ends of the line segment

$$x = t_1 + t + \frac{1}{t_1t}$$

tangent to the deltoid at the point $R(t_1)$, are on the deltoid at the points $R(\pm 1/T_1)$, where $T_1^2 = t_1$. The clinants of the tangents at these two ends are $\mp 1/T_1$, and hence the two tangents are perpendicular to each other. Moreover, these tangents intersect at the point $-T_1^2 = -t_1$ on the unit circle, and the third tangent through this point is the tangent at $R(-t_1)$, perpendicular to the tangent at (Rt_1) and having $-t_1$ as the mid-point of the segment (Figure 30).

9.6. It has been pointed out that there is a unique parabola tangent to the four lines of any proper 4-line (no two of the lines parallel, no three concurrent). Similarly there is a unique deltoid tangent to the four lines of any proper 4-line, and hence any theorem about a proper 4-line may be proved by assuming that the four lines are all tangent to the deltoid 9.31. Thus the theorems proved in section 8.4 by regarding the four lines as tangents to a parabola may also be proved by regarding the lines as tangents to a deltoid. We prove Theorem 8.41 and leave the others as exercises for the reader.

The map equation 9.51 gives the circumcircle of the triangle formed by three tangents at the points $R(t_i)$. If then we have four tangents, and write Σ_1, Σ_2, Σ_3, Σ_4 for the symmetric functions of the four turns t_i, the circumcircles of the four triangles are given by the map equations

$$x = \Sigma_1 - t_i + \left(\frac{t_i}{\Sigma_4} - 1\right)t, \qquad i = 1, 2, 3, 4.$$

For $t = \Sigma_4$ each of these circles passes through the point $\Sigma_1 - \Sigma_4$, the node of the 4-line. Also the four circumcenters, $\Sigma_1 - t_i$, lie on the circle $x = \Sigma_1 - t$, the centric-circle of the 4-line, and this centric-circle passes through the node when $t = \Sigma_4$.

Note that the radius of the centric-circle is 1, which is the radius of the inscribed circle of the deltoid tangent to the four lines.

Given a triangle of three tangents to the deltoid, the clinant of the side tangent at the point $R(t_1)$ is $-t_1$ and the opposite vertex is the point $t_2 + t_3 + 1/(t_2t_3)$, and therefore the equation of the corresponding altitude is

$$t_1x + y = t_1t_2 + t_1t_3 + \frac{t_1}{t_2t_3} + \frac{1}{t_2} + \frac{1}{t_3} + t_2t_3$$

or

$$t_1x + y = s_2 + \frac{s_1t_1}{s_3}.$$

The three such altitudes have the equations

$$t_ix + y = s_2 + \frac{s_1t_i}{s_3}, \qquad i = 1, 2, 3,$$

and they obviously meet at the point $(s_1/s_3, s_2)$; and hence the orthocenter of a triangle of three tangents is the point s_1/s_3. We have seen that the circumcenter of this triangle is the point s_1. Since s_3 is a turn, $|s_3| = 1$ and $|s_1/s_3| = |s_1|$, and it follows that the circumcenter and orthocenter of any triangle of three tangents to a deltoid are equidistant from the center of the deltoid. This gives a method of locating the center of the deltoid tangent to the lines of a given proper 4-line. For each of the four triangles we draw the perpendicular bisector of the segment from the circumcenter to the orthocenter, and these four lines must meet at the center of the tangent deltoid. The size of the deltoid is fixed by the fact that the centric-circle of the 4-line is the same size as the inscribed circle of the deltoid, and the base circle of the deltoid is three times as large. Thus, given a proper 4-line, the inscribed circle and the base circle of the deltoid touching the four lines can be readily constructed. But there remains the rather difficult problem of locating a cusp of the deltoid on the base circle. For the reader who may wish to work at this problem it should be pointed out that the locating of a cusp involves the trisection of a general angle. The trisection of an angle presents no *practical* difficulty in making a drawing, but it cannot be accomplished by a strict compass and straight-edge construction.

9.7. We have considered only those simple hypocycloids for which the ratio r/r_1 is an integer. The reader may be interested in proving that when $r/r_1 = n/(n-1)$ we get the same *simple* hypocycloid as when $r/r_1 = n$. Thus the same deltoid will be described by a point on a rolling circle whose radius is either one-third or two-thirds that of the base circle. However, if the ratio r/r_1 is a rational fraction, $r/r_1 = p/q$, p, q integers, $q \neq p-1$, a nodal (or star) hypocycloid is obtained. Replacing n in equation 9.11 by p/q will give a map equation for such a curve. The star hypocycloid of 5 cusps, with $r/r_1 = 5/2$, is an interesting curve to study and to draw.

Chapter X

EPICYCLOIDS AND TROCHOIDS

10.1. As noted in section 9.1, if a circle rolls around a fixed base circle, the rolling circle *externally* tangent to the base circle, a point P on the circumference of the rolling circle describes a curve called an *epicycloid.* For the *simple* epicycloid the radius of the base circle is n times that of the rolling circle, n any positive integer. As in the case of the hypocycloid, it is obvious that the point P comes in contact with the base circle at n equally spaced points, these points in this case being cusps pointing in toward the center of the base circle.

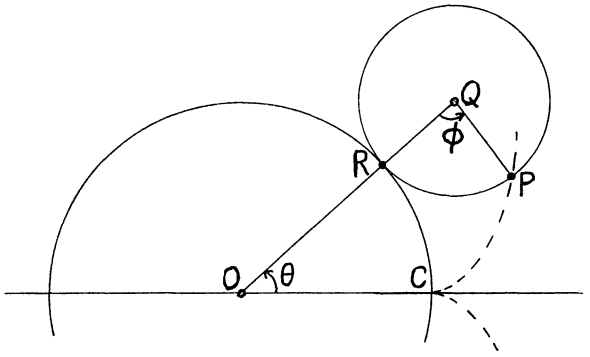


FIG. 31

Again we let r and r_1 be the radii of the base and rolling circles respectively, $r = nr_1$, and we set up the coordinate system with the origin O at the center of the base circle and a cusp C on the axis of reals at the point r. With Q as the center of the rolling circle, R the point of contact with the base circle, θ the angle from the vector OC to the vector OR, and ϕ the angle from the vector QR to the vector QP, we have

$$n\theta = \phi,$$

$$\begin{aligned}\text{vector } OP &= \text{vector } OQ + \text{vector } QP \\ &= \frac{r}{n}(n+1)e^{i\theta} + \frac{r}{n}e^{i(\theta+\phi-\pi)} \\ &= \frac{r}{n}(n+1)e^{i\theta} - \frac{r}{n}e^{i\theta(n+1)}.\end{aligned}$$

Then writing r for $e^{i\theta}$ we have the map equation of the curve

$$x = \frac{r}{n}\{(n+1)t - t^{n+1}\}. \tag{10.11}$$

It may be noted that this is the map equation we would get if we would replace n in equation 9.11 by $-n$. Thus if we allow n to take on negative as well as positive integral values, either equation 9.11 or equation 10.11 will include all simple hypocycloids and epicycloids of $|n|$ cusps.

10.2. We could not have a hypocycloid of one cusp, but the epicycloid of one cusp is the familiar *cardioid* whose equation in polar coordinates is $\rho = a(1-\cos\theta)$. Putting $n=1$ in 10.11 we have

$$x = r(2t - t^2);$$

or, if we take the radius of the base circle as the unit of length, we have for the cardioid the simple map equation

$$x = 2t - t^2. \tag{10.21}$$

In the study of the properties of this curve in this section we may write $R(t)$ for $2t-t^2$.

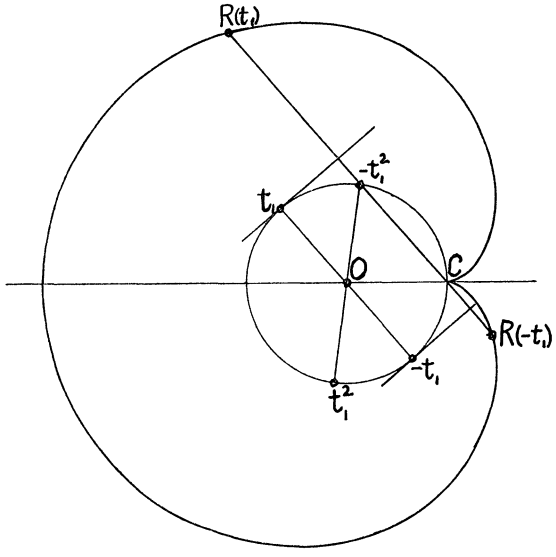


FIG. 32

Consider the reflections of the point 1 in the tangents to the unit circle. The equation of the tangent to the unit circle at the point t_1 (see 7.41) is

$$\frac{1}{t_1^2}x + y - \frac{2}{t_1} = 0$$

or

$$x + t_1^2 y - 2t_1 = 0.$$

Then the reflection of the point 1 in this tangent (see 3.54) is

$$x_2 = 2t_1 - t_1^2,$$

and this is the point $R(t_1)$ on the cardioid $x = R(t)$. Hence we have

THEOREM 10.22. *If C is a fixed point on a circle, the locus of the reflections of C in the tangents to the circle is a cardioid with its cusp at C and the given circle as its base circle.*

We see also that the perpendicular bisector of the line segment from the cusp to the point $R(t_1)$ of the cardioid $x = R(t)$ is tangent to the base circle at the point t_1, and this gives a simple graphical method of finding the turn t_1 which corresponds to any point $R(t_1)$ of the cardioid. Moreover the vector from the cusp to the point $R(t_1)$, $t_1 \neq 1$, is $2t_1 - t_1^2 - 1 = t_1(2 - t_1 - 1/t_1)$. But $t_1 + 1/t_1$ is real and less than 2, and hence $2 - t_1 - 1/t_1$ is positive. Hence the vector from the cusp to the point $R(t_1)$ has the same direction (arrowhead direction) as the vector t_1. Also the vector from the cusp to the point $R(-t_1)$ has the opposite direction, and thus the two points $R(t_1)$ and $R(-t_1)$ are collinear with the cusp, the cusp lying between the two points. The mid-point of the segment from $R(t_1)$ to $R(-t_1)$ is

$$\frac{(2t_1 - t_1^2) + (-2t_1 - t_1^2)}{2} = -t_1^2,$$

a point on the base circle. The vector from $R(-t_1)$ to $R(t_1)$ is $(2t_1 - t_1^2) - (-2t_1 - t_1^2) = 4t_1$, and the length of this vector is 4. We have then

THEOREM 10.23. *Any chord of a cardioid through the cusp is of length four times the radius of the base circle, and the mid-point of such a chord lies on the base circle; and, conversely, if a line segment passes through two points C and P of a circle, the segment having P as its mid-point and its length four times the radius of the circle, then as P moves around the circle and C remains fixed the ends of the segment describe a cardioid with C as its cusp and the circle as its base circle.*

From the map equation 10.21 we have

$$D_t x = 2(1 - t),$$

and hence the clinant of the tangent to the cardioid at the point $R(t_1)$ (see 5.42) is

$$\tau = \frac{2\left(1 - \frac{1}{t_1}\right)}{2t_1^2(1 - t_1)} = -\frac{1}{t_1^3}.$$

The equation of this tangent is then

$$-\frac{1}{t_1^3}x + y = -\frac{2t_1 - t_1^2}{t_1^3} + \frac{2t_1 - 1}{t_1^2}$$

or

$$t_1^3 y - 3t_1^2 + 3t_1 - x = 0. \tag{10.24}$$

We call t_1 the *parameter* of this tangent at the point $R(t_1)$. Because of the relation 10.24 it follows that if any tangent of the cardioid passes through a given point (x_1, y_1) of the plane, the parameter of this tangent must satisfy this equation in t:

$$t^3 - \frac{3}{y_1}t^2 + \frac{3}{y_1}t - \frac{x_1}{y_1} = 0.$$

If this equation is satisfied by three turns, t_1, t_2, t_3, we must have

$$s_1 = \frac{3}{y_1}, \qquad s_2 = \frac{3}{y_1}, \qquad s_3 = \frac{x_1}{y_1},$$

from which we must have

$$s_1 = s_2 \neq 0, \qquad x_1 = \frac{3s_3}{s_1}, \qquad y_1 = \frac{3}{s_2} = \frac{3}{s_1}, \qquad y_1 = \bar{x}_1.$$

Suppose now that we start with any three turns which satisfy the relation $s_1 = s_2 \neq 0$, and let $x_1 = 3s_3/s_1$, $y_1 = 3/s_1 = \bar{x}_1$. Then the tangents to the cardioid $x = R(t)$ at the three points $R(t_1)$, $R(t_2)$, $R(t_3)$ will be concurrent at the point $(3s_3/s_1, 3/s_1)$. For the tangent at $R(t_i)$, $i = 1, 2, 3$, has the equation

$$t_i^3 y - 3t_i^2 + 3t_i - x = 0$$

and the point $(3s_3/s_1, 3/s_1)$ lies on this tangent since

$$\frac{3t_i^3}{s_1} - 3t_i^2 + 3t_i - \frac{3s_3}{s_1} = \frac{3}{s_1}(t_i^3 - s_1 t_i^2 + s_2 t_i - s_3) = 0.$$

Thus we have the following theorems:

THEOREM 10.25. *If t_1, t_2, t_3 are three turns such that $s_1 = s_2 \neq 0$, then the tangents to the cardioid $x = 2t - t^2$ at the points $R(t_1)$, $R(t_2)$, $R(t_3)$ are concurrent at the point $(3s_3/s_1, 3/s_1)$.*

THEOREM 10.26. *If three tangents to the cardioid $x = 2t - t^2$ are concurrent at the point (x_1, y_1), the parameters of the tangents are the three roots of the equation*

$$y_1 t^3 - 3t^2 + 3t - x_1 = 0.$$

The reader may prove also that through a point (not the cusp) of the cardioid $x=2t-t^2$ two tangents will pass, the tangent at the point and a second tangent whose parameter is $(2t_1-t_1^2)/(2t_1-1)$.

If we have three turns of the form t_1, ωt_1, $\omega^2 t_1$, we have at once $s_1=s_2=0$, the case excluded in theorem 10.25. Since the clinant of the tangent at the point $R(t_1)$ is $-1/t_1^3$, it follows that the tangents at the three points $R(t_1)$, $R(\omega t_1)$ and $R(\omega^2 t_1)$ have the same clinant. Also the vectors t_1, ωt_1, and $\omega^2 t_1$ make angles of 120° with each other, and the same must be true of the vectors from the cusp to the three points of contact. Hence we have

THEOREM 10.27. *For each direction in the plane there are three parallel tangents to the cardioid, and the vectors from the cusp to the three points of contact make angles of* 120° *with each other.*

As special cases of this theorem, the three tangents parallel to the axis of reals have the points of contact $R(1)=1$, $R(\omega)=2\omega-\omega^2=1+3\omega$, and $R(\omega^2)=2\omega^2-\omega^4=1+3\omega^2$; while the three tangents perpendicular to the axis of reals have the points of contact $R(-1)=-3$, $R(-\omega)=-2\omega-\omega^2=1-\omega$, and $R(-\omega^2)=-2\omega^2-\omega^4=1-\omega^2$. In the latter case the tangents at $1-\omega$ and $1-\omega^2$ coincide as the double tangent of the cardioid with the equation $x+y=3$.

Associated with the cardioid $x=2t-t^2$ is a family of penosculant curves with the map equation

$$x = t_1 + t - t_1 t = t_1 + (1 - t_1)t. \tag{10.28}$$

As in preceding cases (see sections 8.2, 9.3, 9.4) it is easily verified that for any fixed t_1 the penosculant curve is tangent to the cardioid at the point $R(t_1)$. Also the penosculant curve is obviously a circle with its center at the point t_1 and passing through the cusp (the point 1) when $t=1$. Thus we have

THEOREM 10.29. *A circle passing through the cusp of the cardioid $x=2t-t^2$ and having its center at the point t_1 on the base circle will be tangent to the cardioid at the point $R(t_1)$; and, conversely, if a family of circles have their centers on a fixed base circle and all pass through a fixed point C of this base circle, they will envelop a cardioid with its cusp at C.*

10.3. Two penosculant circles, C_1 and C_2, of the cardioid $x=2t-t^2$ with their centers at t_1 and t_2 evidently intersect at the cusp and at the point $t_1+t_2-t_1t_2$. For three circles with centers at t_1, t_2, t_3 the three intersections (other than the cusp) will be

$$x_i = s_1 - t_i - \frac{s_3}{t_i}, \qquad i = 1, 2, 3,$$

and these three intersections lie on the curve with the map equation

$$x = s_1 - t - \frac{s_3}{t}.$$

If T is one of the square roots of s_3, $T^2 = s_3$, this equation may be written

$$x = s_1 - T\left(\frac{t}{T} + \frac{T}{t}\right). \tag{10.31}$$

Comparing this with 9.22 and noting that t/T is a turn, one sees that we have the map equation of a line segment 4 units in length and center at the point s_1. If now we consider four penosculant circles, we will have four such line segments l_i, the line l_1 passing through the intersections of the circles C_2, C_3, C_4, *etc.* The intersection of circles C_3 and C_4 is the intersection of lines l_1 and l_2, and C_1 is the circumcircle of the triangle formed by lines l_2, l_3, and l_4. We have then again the figure of the proper 4-line (see sections 8.4 and figure, and 9.6), with circles C_i as the circumcircles of the four triangles, the cusp C as the node, and the base circle of the cardioid as the centric-circle of the 4-line. Conversely, if we start with any given proper 4-line, the associated cardioid is uniquely determined since its cusp and base circle are the node and centric-circle of the 4-line. In the cases of the parabola and deltoid associated with a 4-line, the four lines were tangent to the curves, but in the case of the cardioid it is the four circumcircles which are tangent to the curve.

The reader may be interested in drawing an accurate figure of a proper 4-line, its four circumcircles, its node and centric-circle, and the associated parabola, deltoid, and cardioid. As it is rather difficult to locate the deltoid when the 4-line is given, it is best to start with a deltoid, using four of its tangents for the 4-line. Then the parabola is fixed by its focus and directrix and the cardioid by its base circle and cusp.

10.4. If a circle rolls around inside or outside a fixed base circle (Figure 33), a point P attached to the rolling circle, but not necessarily on its circumference, describes a curve called a *trochoid.* Hypocycloids and epicycloids are included in the more general class of trochoids. Simple trochoids are those for which the radius r of the base circle is $|n|$ times the radius r_1 of the rolling circle, n an integer positive or negative but not zero.

Let the point P be attached to the rolling circle at a distance kr_1 from its center, $0 < k \leqq 1$*. Then by the same procedure that was used to obtain map equations 9.11 and 10.11 we find for the general map equation for all simple trochoids

$$x = \frac{r}{n}\left[(n-1)t + \frac{k}{t^{n-1}}\right]. \tag{10.41}$$

This equation gives hypo-trochoids for n positive and epi-trochoids for n negative (see section 10.1), and it includes the simple hypocycloids and epicycloids

* Any curve that can be constructed with $k > 1$ can also be constructed with a value of $k < 1$ and a different r and r_1.

when $k=1$. The *simple* trochoids ($r/r_1=n$ or $n/(n-1)$) may be convex curves for values of k near 0, but for values of k near 1 they have rounded points which are nearly cusps.

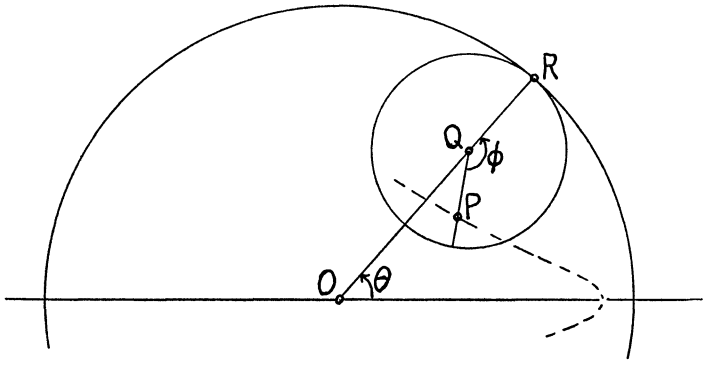


FIG. 33

10.5. We examine the rather special hypo-trochoid for $n=2$, $0<k<1$, with the map equation

$$x = \frac{r}{2}\left[t + \frac{k}{t}\right] = R(t). \tag{10.51}$$

We shall show that this curve is an ellipse with foci at the points $\pm r\sqrt{k}$ on the axis of reals. The vector from the point $r\sqrt{k}$ to the point $R(t_1)$ of the curve is

$$\frac{r}{2}\left[t_1 + \frac{k}{t_1}\right] - r\sqrt{k}$$

or

$$\frac{r}{2t_1}(t_1 - \sqrt{k})^2.$$

The length of this vector is

$$\sqrt{\frac{r}{2t_1}(t_1 - \sqrt{k})^2 \cdot \frac{rt_1}{2}\left(\frac{1}{t_1} - \sqrt{k}\right)^2} = \frac{r}{2t_1}(t_1 - \sqrt{k})(1 - t_1\sqrt{k}).$$

Similarly the distance from the point $-r\sqrt{k}$ to the point $R(t_1)$ is

$$\frac{r}{2t_1}(t_1 + \sqrt{k})(1 + t_1\sqrt{k}),$$

and the sum of these distances is

$$\frac{r}{2t_1}(2t_1 + 2kt_1) = r(1 + k).$$

Since this sum is a constant for all t_1, the curve must be an ellipse with $\pm r\sqrt{k}$ as its foci. Putting $t = \pm 1$ in the map equation 10.51 we see that the ellipse cuts the axis of reals at the points $\pm\frac{1}{2}r(1+k)$, and for $t = \pm i$ it cuts the axis of pure imaginaries at the points $\pm\frac{1}{2}r(1-k)i$, so that the semi-major and minor axes are

$$a = \frac{r}{2}(1 + k), \qquad b = \frac{r}{2}(1 - k).$$

From these relations we have

$$r = a + b, \qquad k = \frac{a - b}{a + b},$$

giving the values of r and k which correspond to an ellipse of any specified dimensions.

Writing the map equation of the ellipse in the form

$$x = \frac{t^2 + k}{2t},$$

we have a family of penosculant curves

$$x = \frac{t_1 t + k}{t_1 + t}.$$

The reader may verify that these penosculants are straight lines (see section 5.3) tangent to the ellipse.

A line segment through the center of an ellipse with its ends on the curve is called a *diameter* of the ellipse. From the form of the map equation 10.51 it is obvious that the points $R(\pm t_1)$ are the ends of a diameter.

From 10.51 we have

$$D_t x = \frac{r}{2}\left(1 - \frac{k}{t^2}\right),$$

and hence the clinant of the tangent to the ellipse at each of the points $R(\pm t_1)$ is

$$\tau = \frac{\frac{r}{2}(1 - kt_1^2)}{\frac{r}{2}\left(1 - \frac{k}{t_1^2}\right)t_1^2} = \frac{kt_1^2 - 1}{k - t_1^2}. \tag{10.52}$$

The clinant of the focal radius from the focus $r\sqrt{k}$ to the point $R(t_1)$ of the ellipse is

$$-\frac{\frac{r}{2}\left(\frac{1}{t_1}+kt_1\right)-r\sqrt{k}}{\frac{r}{2}\left(t_1+\frac{k}{t_1}\right)-r\sqrt{k}}=-\frac{(1-t_1\sqrt{k})^2}{(t_1-\sqrt{k})^2}.$$

Similarly the clinant of the focal radius from the focus $-r\sqrt{k}$ to the same point is

$$-\frac{(1+t_1\sqrt{k})^2}{(t_1+\sqrt{k})^2}.$$

The clinants of the bisectors of the angles between these focal radii (see 3.32) are

$$\pm\sqrt{\frac{(1-t_1\sqrt{k})^2(1+t_1\sqrt{k})^2}{(t_1-\sqrt{k})^2(t_1+\sqrt{k})^2}}=\pm\frac{1-t_1^2k}{t_1^2-k},$$

and comparing this with 10.52 we see that these are the clinants of the tangent and normal at the point. This proves the well known

THEOREM 10.53. *The tangent and normal at any point of an ellipse bisect the angles made by the focal radii through the point.*

Chapter XI

THE RECTANGULAR HYPERBOLA

11.1. It is easily verified (see section 5.3) that the map equations

$$x = \frac{i+t}{1+t} \quad \text{and} \quad x = \frac{1+t}{i+t}$$

represent two perpendicular lines, l_1 and l_2, through the origin, with self-conjugate equations $ix+y=0$ and $-ix+y=0$ respectively. Consider now the map equation

$$x = \frac{i+t}{1+t} + \frac{1+t}{i+t}. \tag{11.11}$$

We shall show that this equation represents a rectangular hyperbola with the lines l_1 and l_2 as asymptotes. If we write z for $(i+t)/(1+t)$, we see that z is a variable point on the line l_1, or the vector from the origin to this point; and similarly $1/z$ is the vector from the origin to the point $1/z$ on the line l_2; and the sum of these two vectors is the vector x from the origin to the point x of the curve. But the product of the lengths of the vectors z and $1/z$ is

$$\sqrt{z\bar{z}} \cdot \sqrt{\frac{1}{z} \cdot \frac{1}{\bar{z}}} = 1,$$

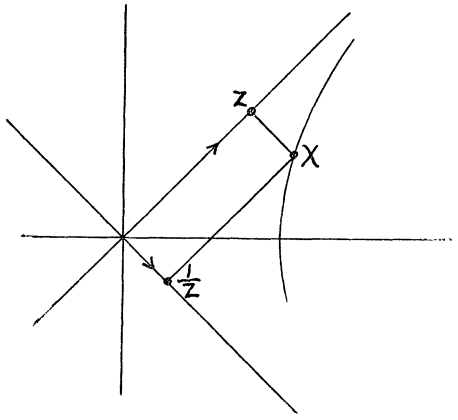


FIG. 34

and since this product is a constant the curve must be a rectangular hyperbola with the lines l_1 and l_2 as asymptotes. The reader may verify that the vertices of the hyperbola are the points $\pm\sqrt{2}$ on the axis of reals, and the foci are the points ± 2.

Since

$$\bar{z} = \frac{\frac{1}{i} + \frac{1}{t}}{1 + \frac{1}{t}} = \frac{i + t}{i(1 + t)} = \frac{z}{i},$$

we may write equation 11.11 and its conjugate as

$$x = z + \frac{1}{z} \quad \text{and} \quad y = \frac{z}{i} + \frac{i}{z},$$

and eliminating z we have

(11.12) $$x^2 + y^2 = 4 \quad \text{or} \quad (ix + y)(-ix + y) = 4$$

as the self-conjugate equation of this particular rectangular hyperbola. For a general map-equation of *any* rectangular hyperbola (see 5.21) we have

$$x = a + c\left(z + \frac{1}{z}\right), \qquad a, c \text{ arbitrary complex constants,}$$

and from this we deduce the general self-conjugate equation

$$\left(\frac{x - a}{c}\right)^2 + \left(\frac{y - \bar{a}}{\bar{c}}\right)^2 = 4,$$

or

(11.13) $$\bar{c}^2(x - a)^2 + c^2(y - \bar{a})^2 = 4c^2\bar{c}^2,$$

or

(11.14) $$\{\bar{c}(x - a) + ic(y - \bar{a})\}\{\bar{c}(x - a) - ic(y - \bar{a})\} = 4c^2\bar{c}^2.$$

This hyperbola has the point a as its center, and the lines

$$\bar{c}(x - a) \pm ic(y - \bar{a}) = 0$$

as its asymptotes.

In this chapter we are primarily interested in the *rectangular* hyperbola; but the general hyperbola may be treated in a similar manner, the general map equation being

$$x = a + c\left(\frac{t_0 + t}{1 + t} + \frac{1 + t}{t_0 + t}\right).$$

11.2. Four points, of which no three are collinear, ordinarily determine a unique rectangular hyperbola which passes through them, but there are two exceptional cases. If the set of points is an orthocentric set (see section 7.2)

there are infinitely many rectangular hyperbolas through them, a case which will be treated in the next section. If the set of points is not orthocentric, but if the line through one pair of them is perpendicular to the line through the other pair, there is, strictly speaking, no rectangular hyperbola through the points; but the pair of perpendicular lines through them may be regarded as a degenerate case of a rectangular hyperbola.

We find now the equation of the rectangular hyperbola through four points of the unit circle. Equation 11.13 may be written in the form

$$\bar{c}^2x^2 - 2a\bar{c}^2x + (a^2\bar{c}^2 + \bar{a}^2c^2 - 4c^2\bar{c}^2) - 2\bar{a}c^2y + c^2y^2 = 0, \tag{11.21}$$

and if we solve this simultaneously with the equation of the unit circle $xy=1$, eliminating y, we have

$$\bar{c}^2x^4 - 2a\bar{c}^2x^3 + (a^2\bar{c}^2 + \bar{a}^2c^2 - 4c^2\bar{c}^2)x^2 - 2\bar{a}c^2x + c^2 = 0.$$

If the hyperbola passes through four points t_i, $i=1, 2, 3, 4$, of the unit circle, these four turns t_i are the roots of the above quartic in x; and if we write Σ_1, Σ_2, Σ_3, Σ_4 for the elementary symmetric functions of the t_i, we must have

$$\begin{cases} \Sigma_1 = 2a, & \Sigma_3 = \dfrac{2\bar{a}c^2}{\bar{c}^2}, \\ \Sigma_2 = \dfrac{a^2\bar{c}^2 + \bar{a}^2c^2 - 4c^2\bar{c}^2}{\bar{c}^2}, & \Sigma_4 = \dfrac{c^2}{\bar{c}^2}. \end{cases} \tag{11.22}$$

Putting these results in equation 11.21 we see that the equation of the rectangular hyperbola through four points t_i of the unit circle is

$$x^2 - \Sigma_1x + \Sigma_2 - \Sigma_3y + \Sigma_4y^2 = 0. \tag{11.23}$$

Also from relations 11.22 we have $a=\Sigma_1/2$, $c/\bar{c} = \pm\sqrt{\Sigma_4}$, and hence it follows that the center of the rectangular hyperbola 11.23 is $\Sigma_1/2$, and its asymptotes are

$$\left(x - \frac{\Sigma_1}{2}\right) \pm i\sqrt{\Sigma_4}\left(y - \frac{\Sigma_3}{2\Sigma_4}\right) = 0. \tag{11.24}$$

Four points of a circle can obviously not be an orthocentric set; but it may happen that the line joining two of them is perpendicular to the line joining the other two. The line through t_1 and t_2 has the clinant $1/t_1t_2$ (see section 7.1), and hence this special case will occur when

$$\left(\frac{1}{t_1t_2} + \frac{1}{t_3t_4}\right)\left(\frac{1}{t_1t_3} + \frac{1}{t_2t_4}\right)\left(\frac{1}{t_1t_4} + \frac{1}{t_2t_3}\right) = 0,$$

or when

$$(t_3t_4 + t_1t_2)(t_2t_4 + t_1t_3)(t_2t_3 + t_1t_4) \equiv \Sigma_1^2\Sigma_4 + \Sigma_3^2 - 4\Sigma_2\Sigma_4 = 0.$$

But equation 11.23 may be written

$$\left\{\left(x-\frac{\Sigma_1}{2}\right)+i\sqrt{\Sigma_4}\left(y-\frac{\Sigma_3}{2\Sigma_4}\right)\right\}\left\{\left(x-\frac{\Sigma_1}{2}\right)-i\sqrt{\Sigma_4}\left(y-\frac{\Sigma_3}{2\Sigma_4}\right)\right\}$$

$$=\frac{\Sigma_1^2\Sigma_4+\Sigma_3^2-4\Sigma_2\Sigma_4}{4\Sigma_4},$$

and hence in this exceptional case equation 11.23 represents a pair of perpendicular lines, the lines given by equations 11.24.

11.3. We may now study the family of rectangular hyperbolas passing through the vertices of any triangle. We take the circumcircle of the triangle as the unit circle, the vertices corresponding to three turns, t_1, t_2, t_3. A hyperbola cutting a circle in three points must cut it also in a fourth point, though this fourth point may coincide with one of the three points. Therefore we get *all* rectangular hyperbolas through the vertices of the triangle if we consider the rectangular hyperbolas determined by the vertices and a variable fourth point on the unit circle. For convenience in what follows we take the fourth point as t^2 where t is a variable turn. The hyperbolas will then be given by the equation 11.23 if the Σ's are symmetric functions of t_1, t_2, t_3, t^2. If we write s_1, s_2, s_3 for the symmetric functions of t_1, t_2, t_3 and use the relations 6.21, equation 11.23 becomes

$$x^2-(s_1+t^2)x+(s_2+s_1t^2)-(s_3+s_2t^2)y+s_3t^2y^2=0, \tag{11.31}$$

and this is then the equation of the family of all rectangular hyperbolas through the vertices of the triangle.

The coordinates of the orthocenter of the triangle (see 7.13) are $(s_1, s_2/s_3)$, and substituting these coordinates for x and y in equation 11.31 we find that the equation is satisfied identically. Thus all the rectangular hyperbolas through the vertices of a triangle pass also through the orthocenter; or, to state it more symmetrically, all rectangular hyperbolas through three points of an orthocentric set pass also through the fourth point. This is the exceptional case, mentioned in the preceding section, of a set of four points which do not determine a *unique* rectangular hyperbola.

For any fixed value of t the center of the hyperbola is

$$a=\frac{s_1+t^2}{2};$$

and hence the locus of the centers of all the hyperbolas is the circle whose map equation is

$$x=\frac{s_1}{2}+\frac{t^2}{2},$$

and this is the nine-point circle (see 7.21) of the triangle.

Referring to 11.24 we see that the asymptotes of the hyperbola 11.31 for a fixed t are

$$\left(x - \frac{s_1 + t^2}{2}\right) \pm i\sqrt{s_3 t^2}\left(y - \frac{s_3 + s_2 t^2}{2s_3 t^2}\right) = 0.$$

If we let T be any one of the sixth roots of s_3, so that

$$T^6 = s_3, \qquad \pm\sqrt{s_3 t^2} = \pm T^3 t,$$

the equation of one asymptote* may be written

$$\left(x - \frac{s_1 + t^2}{2}\right) + iT^3 t\left(y - \frac{s_3 + s_2 t^2}{2s_3 t^2}\right) = 0$$

or

$$\frac{x}{iT^3 t} + y - \frac{s_1}{2iT^3 t} - \frac{t}{2iT^3} - \frac{1}{2t^2} - \frac{s_2}{2s_3} = 0. \tag{11.32}$$

We wish to obtain the equation of the envelope of these asymptotes as t varies, and the usual procedure would be to differentiate 11.32 partially with respect to the parameter t and then eliminate t between the equation thus obtained and equation 11.32. But since the partial differentiation with respect to t gives us an equation independent of y, we may solve at once for x in terms of t to obtain a map equation of the envelope. Thus, differentiating 11.32 partially with respect to t, we have

$$-\frac{x}{iT^3 t^2} + \frac{s_1}{2iT^3 t^2} - \frac{1}{2iT^3} + \frac{1}{t^3} = 0,$$

from which we obtain

$$x = \frac{s_1}{2} + \frac{iT^3}{t} - \frac{t^2}{2}$$

or

$$x = \frac{s_1}{2} + \frac{T^2}{2}\left\{2\left(\frac{iT}{t}\right) + \left(\frac{t}{iT}\right)^2\right\}. \tag{11.33}$$

If we compare this result with 9.31 and note that iT/t runs through all turn values as t runs through all turn values, we see that this envelope of the asymptotes is a deltoid with center at $s_1/2$, the nine-point center of the triangle, and having the nine-point circle as its inscribed circle.

* For any value of t, say $t=t_0$, equation 11.31 represents a certain hyperbola and equation 11.32 one of its asymptotes; and when $t=-t_0$, 11.31 gives the *same* hyperbola and 11.32 its *other* asymptote. As t runs through all turn values we get each hyperbola twice but each asymptote only once.

The line given by equation 11.32 for any particular value of t, say $t=t_0$ touches the deltoid at the point

$$x = \frac{s_1}{2} + \frac{iT^3}{t_0} - \frac{t_0^2}{2}.$$

For three special values of t^2 equation 11.31 represents two perpendicular lines rather than a proper hyperbola, namely, when $t^2 = -s_3/t_j^2$, $j=1, 2, 3$. Each of these pairs of lines consists of a side of the triangle and the corresponding altitude. These sides and altitudes are included in the family of lines given by equation 11.32, the sides when $t=-iT^3/t_j$ and the altitudes when $t=iT^3/t_j$. Thus the sides of the triangle are tangent to the deltoid at the points

$$\frac{s_1}{2} - t_j + \frac{s_3}{2t_j^2},$$

and the altitudes at the points

$$\frac{s_1}{2} + t_j + \frac{s_3}{2t_j^2}.$$

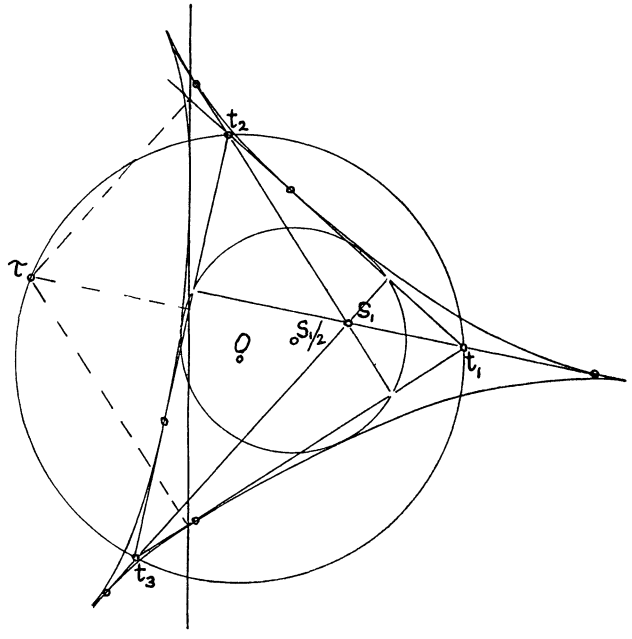


FIG. 35

Since four tangents determine a deltoid uniquely, this is the only deltoid tangent to the sides and altitudes of the triangle. Thus we have proved

THEOREM 11.34. *All the rectangular hyperbolas through the vertices of a triangle pass also through the orthocenter; the locus of their centers is the nine-point circle of the triangle, and the envelope of their asymptotes is the deltoid which is tangent to the sides and altitudes of the triangle and has the nine-point circle as its inscribed circle.*

11.4. The lines of theorem 11.34 which are the asymptotes of the rectangular hyperbolas and the tangents to the deltoid are also related to the triangle in another way. If from any point τ of the circumcircle of the triangle $t_1t_2t_3$ perpendiculars are dropped to the three sides, the feet of these perpendiculars will lie on a line. To see this we note that the equation of the side t_2t_3 of the triangle (see 7.11) is

$$\frac{x}{t_2t_3} + y = \frac{1}{t_2} + \frac{1}{t_3}$$

and the line perpendicular to this side from the point τ is

$$-\frac{x}{t_2t_3} + y = \frac{-\tau}{t_2t_3} + \frac{1}{\tau}.$$

Solving these equations for the point of intersection we have

$$x_1 = \frac{1}{2}\left(t_2 + t_3 + \tau - \frac{t_2t_3}{\tau}\right)$$

or

$$x_1 = \frac{1}{2}\left(s_1 + t_1 + \tau - \frac{s_3}{t_1\tau}\right).$$

Then the three feet of the perpendiculars from the point τ to the three sides will lie on the curve whose map equation is

$$x = \frac{1}{2}\left(s_1 + \tau - t - \frac{s_3}{\tau t}\right). \tag{11.41}$$

Following the method that was used in the case of equation 10.31, one could show that this is the map equation of a line segment (hypocycloid of two cusps). But it will suit our present purpose better to obtain the self-conjugate equation of the line by eliminating the parameter t between equation 11.41 and the conjugate equation

$$y = \frac{1}{2}\left(\frac{s_2}{s_3} + \frac{1}{\tau} - \frac{1}{t} - \frac{\tau t}{s_3}\right).$$

The resulting linear equation is

$$-\frac{\tau x}{s_3}+y+\frac{s_1\tau}{2s_3}+\frac{\tau^2}{2s_3}-\frac{s_2}{2s_3}-\frac{1}{2\tau}=0. \tag{11.42}$$

For any point τ on the circumcircle of a triangle there is such a line, called the Simson* line for the point τ. As τ varies we have a whole family of such Simson lines given by equation 11.42, and we shall now show that these Simson lines are exactly the asymptotes of the hyperbolas of theorem 11.34. If we simply substitute in equation 11.32 giving the asymptotes and in equation 11.42 giving the Simson lines, remembering that $T^6=s_3$, we see that the asymptote for $t=iT^3/\tau$ is the Simson line for τ, and the Simson line for $\tau=iT^3/t$ is the asymptote for t. It follows that the Simson lines for any triangle are the tangents to the deltoid of theorem 11.34; and for this reason the deltoid is sometimes called the *Simson deltoid* (or Simson quartic) of the triangle.

* There is considerable doubt as to whether Simson was the discoverer of these lines. See the historical note in Johnson's Modern Geometry, p. 137, and the paper by Mackay to which Johnson refers.

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Brown, Patricia Ann, **122.**
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Browne, D. H., *254, 665, 666, 725.*
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DeVol, David, *665*.
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Dixon, L. J., *726*.
Dodes, I. A., *43*, *44*, *44*, *121*, *187*, *344*, *346*, *581*.
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Duarte, F. J., *255*, *665*.
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Duncan, D. C., *121*.
Eby, E. S., *344*.
Elkin, J. M., *343*, *344*, *423*, *493*, *725*.
Ellis, David, *432*, *581*, *582*, *588*.
Epstein, A. L., *189*.
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Feldman, H. M., *43*, *44*, *254*, *255*, *256*.
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Findley, G. B., *48*, *128*, *260*, *261*, *432*.
Fine, N. J., **43**, *43*, *44*, **51**, **128**, **194**, *260*, *500*, *500*, *583*, **665**, *666*.
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Fleisher, Edward, *254*, **255**, *423*.
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Fletcher, H. J., *48*.
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Frey, A. H., Jr., *500*.
Friedlen, D. M., *44*.
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Gold, B. K., *122*.
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Goldberg, Michael, *44*, **45**, 47, *48*, 186, *187*, *188*, *188*, **188**, *190*, *254*, *255*, *256*, *343*, *344*, *346*, *493*, *494*, *495*, *496*, *665*, *666*, *668*, *672*, *725*, *728*, **728**.
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Golomb, S. W., *260*, *263*, *348*, *350*, 420, *431*, 584.
Goode, James, *344*.
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Gordon, R. D., 186, *588*, *591*, *666*.
Gordon, W. O., *260*.
Gorman, C. D., *122*, *124*, *262*, *350*, *432*, *588*, *672*.
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Gould, Peter, *494*.
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Grace, A. G., Jr., *255*, *256*.
Grainger, G. R., *187*.
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Grimm, C. A., *254*, *254*, *255*.
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Grossman, E. H., *665*, *666*.
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Rao, K. Subba, *254*, *255*.
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Raynor, G. E., *48*.
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ON PICTURE-WRITING*

G. PÓLYA, Stanford University

To write "sun", "moon" and "tree" in picture-writing, one draws simply a circle, a crescent and some simplified, conventionalized picture of a tree, respectively. Picture-writing was used by some tribes of red Indians and it may well be that more advanced systems of writing evolved everywhere from this primitive system. And so picture-writing may be the ultimate source of the Greek, Latin and Gothic alphabets, the letters of which we currently use as mathematical symbols. I wish to observe that also the primitive picture-writing may be of some use in mathematics. In what follows, I wish to show how the method of generating functions, important in Combinatory Analysis, can be quite intuitively evolved from "figurate series" the terms of which are pictures (or, more precisely, variables represented by pictures).

Picture-writing is easy to use on paper or blackboard, but it is clumsy and expensive to print. Although I have presented several times the contents of the following pages orally, I hesitated to print it.† I am indebted to the editor of the MONTHLY who encouraged me to publish this article.

I shall try to explain the general idea by discussing three particular examples the first of which, although the easiest, will be very broadly treated.

1.1. In how many ways can you change one dollar? Let us generalize the proposed question. Let P_n denote the number of ways of paying the amount of n cents with five kinds of coins: cents, nickels, dimes, quarters and half-dollars. The "way of paying" is determined if, and only if, it is known how many coins of each kind are used. Thus, $P_4=1$, $P_5=2$, $P_{10}=4$. It is appropriate to set $P_0=1$. The problem stated at the outset requires us to compute P_{100}. More generally, we wish to understand the nature of P_n and eventually devise a procedure for computing P_n.

It may help to visualize the various possibilities. We may use no cent, or just 1 cent, or 2 cents, or 3 cents, or $\cdots$. These alternatives are schematically pictured in the first line of Figure 1;** "no cent" is represented by a square which may remind us of an empty desk. The second line pictures the alternatives: using no nickel, 1 nickel, 2 nickels, $\cdots$. The following three lines represent in the same way the possibilities regarding dimes, quarters and half-dollars. We have to choose one picture from the first line, then one picture from the second line, and so on, choosing just one picture from each line; combining (juxtaposing) the five pictures so selected, we obtain a manner of paying. Thus, Figure 1 exhibits directly the alternatives regarding each kind of coin and, indirectly, all manners of paying we are concerned with.

* Address presented at the meeting of the Association in Athens, Ga., March 16, 1956.

† I used it, however, in research. See **2**, especially p. 156, where the "figurate series" are introduced in a closely related, but somewhat different, form. (Numerals in boldface indicate the references at the end of the paper.)

** A photo of actual coins would be more effective here but too clumsy in the following figures.

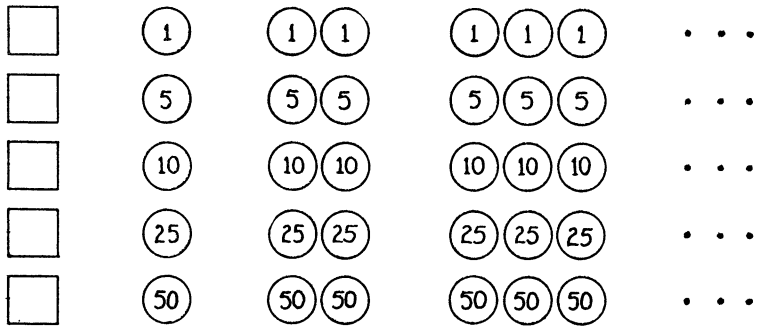


FIG. 1. A complete survey of alternatives.

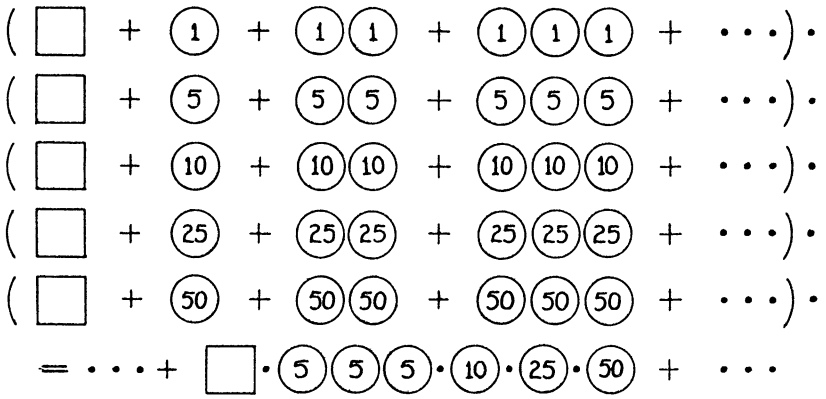


FIG. 2. Genesis of the figurate series.

The main discovery consists in observing that, in fact, we combine the pictures in Figure 1 according to certain rules of algebra: if we conceive each line of Figure 1 as the *sum* of the pictures contained in it and we consider the *product* of these five (infinite) sums, in short, if we pass from Figure 1 to Figure 2, and we develop the product, the terms of this development will represent the various manners of paying we are concerned with. The one term of the product exhibited in the last line of Figure 2 as an example represents one manner of paying one dollar (putting down no cents, three nickels, one dime, one quarter and one half-dollar). The sum of all such terms is an infinite series of pictures; each picture exhibits one manner of paying, different terms represent different manners of paying, and the whole series of pictures, appropriately called the *figurate series*, displays all manners of paying that we have to consider when we wish to compute the numbers P_n.

1.2. Yet this way of conceiving Figure 2 raises various difficulties. First, there is a theoretical difficulty: in which sense can we add and multiply pictures? Then, there is a practical difficulty: how can we pick out conveniently from the whole figurate series the terms counted by P_n, that is, those cases in which the

sum paid amounts to just n cents?

We avoid the theoretical difficulty if we employ the pictures, these symbols of a primitive writing, as we are used to employing the letters of more civilized alphabets: we regard each picture as the symbol for a variable or *indeterminate.*†

To master the other difficulty, we need one more essential idea: we substitute for each "pictorial" variable (that is, variable represented by a picture) a power of a new variable x, the *exponent* of which is the *joint value of the coins* represented by the picture, as it is shown in detail by Figure 3. The third line of Figure 3 shows a lucky coincidence: we have conceived the three juxtaposed nickels as *one picture*, as the symbol of one variable (corresponding to the use of precisely three nickels). For this variable we have to substitute x^{15} according to our general rule; yet even if we substitute for each of the juxtaposed coins the correct power of x and consider the product of these juxtaposed powers, we arrive at the same final result x^{15}.

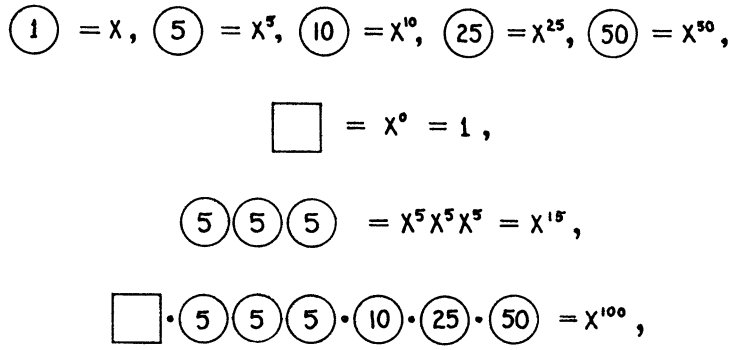


FIG. 3. Powers of one variable substituted for variables represented by pictures.

The last line of Figure 3 is very important. It shows by an example (see the last line of Fig. 2) how the described substitution affects the general term of the figurate series. Such a term is the product of 5 pictures (pictorial variables). For each factor a power of x is substituted whose exponent is the value in cents of that factor; the exponent of the product, obtained as a sum of 5 exponents, will be the joint value of the factors. And so the substitution indicated by Figure 3 changes each term of the figurate series into a power x^n. As the figurate series represents each manner of paying just once, the exponent n arises precisely P_n times so that (after suitable rearrangement of the terms) the whole figurate series goes over into

† In a formal presentation it may be advisable to restrict the term "picture" to denote a (visible, written or printed) symbol that stands for an indeterminate; in the present introductory, rather informal, address the word is now and then more loosely used.

Let us pass over two somewhat touchy points: the infinity of variables and the convergence of the series in which they arise. Both are considered in certain advanced theories and both are momentary. They will be eliminated by the next step.

$$P_0 + P_1x + P_2x^2 + \cdots + P_nx^n + \cdots. \tag{1}$$

In this series the coefficient of x^n enumerates the different manners of paying the amount of n cents, and so (1) is suitably called the *enumerating series.*

The substitution indicated by Figure 3 changes the first line of Figure 2 into a geometric series:

$$1 + x + x^2 + x^3 + \cdots = (1 - x)^{-1}. \tag{2}$$

In fact, this substitution changes each of the first five lines of Figure 2 into some geometric series and the equation indicated by Figure 2 goes over into

$$(1 - x)^{-1}(1 - x^5)^{-1}(1 - x^{10})^{-1}(1 - x^{25})^{-1}(1 - x^{50})^{-1} = P_0 + P_1x + P_2x^2 + \cdots + P_nx^n + \cdots. \tag{3}$$

We have succeeded in expressing the sum of the enumerating series. This sum is usually termed the *generating function*; in fact, this function, expanded in powers of x, generates the numbers $P_0, P_1, \cdots, P_n, \cdots$, the combinatorial meaning of which was our starting point.

1.3. We have reduced a combinatorial problem to a problem of a different kind: expanding a given function of x in powers of x. In particular, we have reduced our initial problem about changing a dollar to the problem of computing the coefficient of x^{100} in the expansion of the left hand side of (3). Our main goal was to show how picture-writing can be used for this reduction. Yet let us add a brief indication about the numerical computation.

The left hand side of (3) is a product of five factors. The well known expansion of the first factor is shown by (2). We proceed by adjoining successive factors, one at a time. Assume, for example, that we have already obtained the expansion of the product of the first two factors:

$$(1 - x)^{-1}(1 - x^5)^{-1} = a_0 + a_1x + a_2x^2 + \cdots,$$

and we wish to go on hence to three factors:

$$(1 - x)^{-1}(1 - x^5)^{-1}(1 - x^{10})^{-1} = b_0 + b_1x + b_2x^2 + \cdots.$$

It follows that

$$(b_0 + b_1x + b_2x^2 + \cdots)(1 - x^{10}) = a_0 + a_1x + a_2x^2 + \cdots.$$

Comparing the coefficient of x^n on both sides, we find that

$$b_n = b_{n-10} + a_n \tag{4}$$

(set $b_m = 0$ if $m < 0$). By (4), we can conveniently compute the coefficients b_n by recursion if the a_n are already known, and the series (3) can be obtained from (2) in four successive steps each of which is similar to the one we have just discussed.

We add a table that shows the computation of P_{50}. This table exhibits the coefficient of x^n for some values of n in five different expansions. The head of

each column shows the value of n, the beginning of each row the last factor taken into account; the bottom row would show P_n for $n=0, 5, 10, \cdots, 50$ *if* we had computed it. Yet the table registers only the steps needed for computing the answer to our initial question and yields $P_{50}=50$; that is, one can pay 50 cents in exactly 50 different ways. We leave it to the reader to continue the computation and verify that $P_{100}=292$; he can also try to justify the procedure of computation directly without resorting to the enumerating series.*

Table to compute P_{50}

$n=0$	5	10	15	20	25	30	35	40	45	50
$(1-x)^{-1}$	1	1	1	1	1	1	1	1	1	1	1
$(1-x^5)^{-1}$	1	2	3	4	5	6	7	8	9	10	11
$(1-x^{10})^{-1}$	1	2	4	6	9	12	16		25		36
$(1-x^{25})^{-1}$	1					13					49
$(1-x^{50})^{-1}$	1										50

2.1. Dissect a convex polygon with n sides into $n-2$ triangles by $n-3$ diagonals and compute D_n, the number of different dissections of this kind. Examining first the simplest particular cases helps to understand the problem. We easily see that $D_4=2$, $D_5=5$; of course $D_3=1$.

The solution is indicated by the parts (I), (II), and (III) of Figure 4. After the broad discussion of the foregoing solution it should not be difficult to understand the indications of Figure 4.

Part (I) of Figure 4 hints the key idea: we build up the dissections of any polygon that is not a triangle from the dissections of other polygons which have fewer sides. For this purpose, we emphasize one of the sides of the polygon, place it horizontally at the bottom and call it the *base*. One of the triangles into which the polygon is dissected has the base as side; we call this triangle Δ. In the given polygon there are two smaller polygons, one to the left, the other to the right, of Δ. For example, the top line of Figure 4 (I) shows an octagon in which there is a quadrilateral to the left, and a pentagon to the right, of Δ, both suitably dissected. As the figure suggests, we can generate this dissection of the octagon by starting from Δ and placing on it, from both sides, the two other appropriately pre-dissected polygons. We may hope that building up the dissections in this manner will be useful.

In exploring the prospects of this idea, we may run into an objection: there are cases, such as the one displayed in the second line of Figure 4 (I), in which the partial polygon on a certain side of Δ does not exist. Yet we can parry this objection: yes, the partial polygon on that side of Δ (the left side in the case of the figure) *does* exist, but it is degenerate; it is reduced to a mere *segment*.

* For the usual method of deriving the generating function, *cf.* 1, Vol. 1, p. 1, Problem 1.

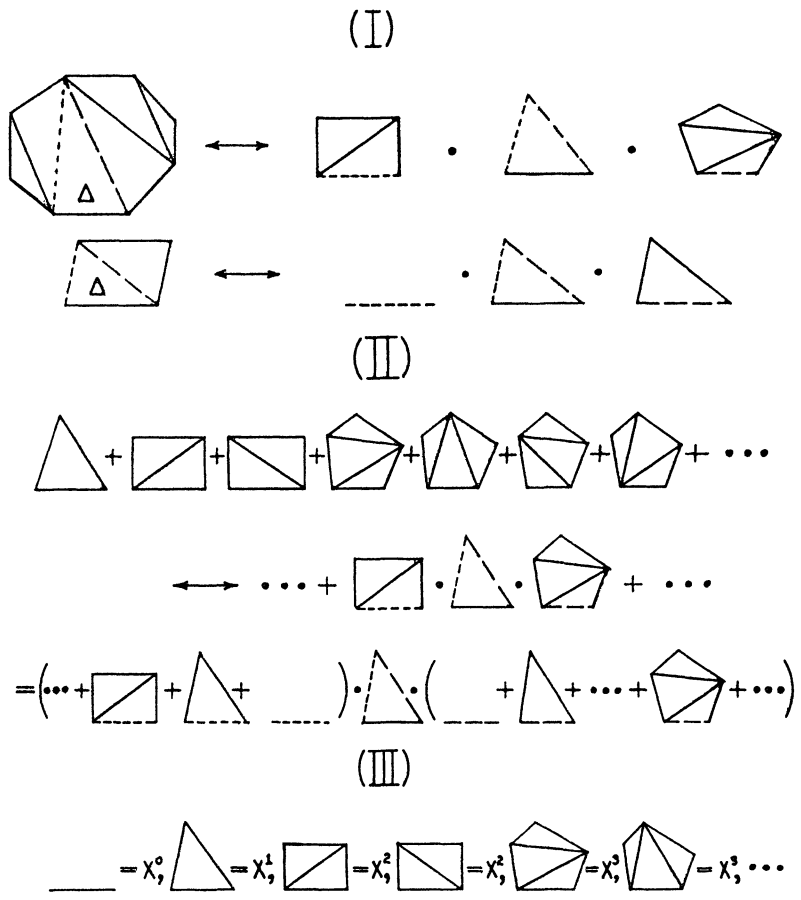


FIG. 4. Key idea, figurate series, transition.

Part (II) of Figure 4 shows the genesis of the figurate series. This series, which occupies the first line, is the sum of all possible dissections of polygons with 3, 4, 5, · · · sides. According to Part (I) (as the next line reminds us) each term of the figurate series can be generated by placing two pre-dissected polygons on a triangle Δ, one from the left and one from the right (one or the other of which, or possibly both, may be degenerate). Therefore, as the next line (the last of Figure 4 (II)) indicates, the terms of the figurate series are in one-one correspondence with the expansion of a product of three factors: the middle factor is just a triangle, the other two factors are equal to the figurate series augmented by the segment.

2.2. Part (III) of Figure 4 hints the transition from the figurate series to the enumerating series. Following the pattern set by Figure 3 and Section 1.2, we substitute for each dissection (more precisely, for the variable represented by that dissection) a power of x the exponent of which is the number of triangles

in that dissection. This substitution, indicated by Figure 4 (III), changes the figurate series into

$$D_3x + D_4x^2 + D_5x^3 + \cdots + D_nx^{n-2} + \cdots = E(x), \tag{5}$$

where $E(x)$ stands for enumerating series. The relation displayed by Figure 4 (II) goes over into

$$E(x) = x[1 + E(x)]^2. \tag{6}$$

This is a quadratic equation for $E(x)$ the solution of which is

$$\begin{aligned} E(x) &= D_3x + D_4x^2 + D_5x^3 + \cdots + D_nx^{n-2} + \cdots \\ &= \frac{1 - 2x - [1 - 4x]^{1/2}}{2x} \\ &= x + 2x^2 + \cdots . \end{aligned} \tag{7}$$

In fact, to arrive at (7), we have to discard the other solution of the quadratic equation (6) which becomes ∞ for $x=0$.

2.3. We have reduced our original problem which was to compute D_n to a problem of a different kind: to find the coefficient of x^{n-2} in the expansion of the function (7) in powers of x.* This latter is a routine problem which we need not discuss broadly. We obtain from (7), using the binomial formula and straightforward transformations, that for $n \geqq 3$

$$D_n = -\frac{1}{2}\binom{1/2}{n-1}(-4)^{n-1} = \frac{2}{2}\,\frac{6}{3}\,\frac{10}{4}\cdots\frac{4n-10}{n-1}.$$

3.1. A (topological) *tree* is a connected system of two kinds of objects, *lines* and *points*, that contains no closed path. A certain point of the tree in which just one line ends is called the *root* of the tree, the line starting from the root the *trunk*, any point different from the root a *knot*. In Figure 5 the root is indicated by an arrow, and each knot by a small circle. Our problem is: *compute* T_n, *the number of different trees with n knots.*

It makes no difference whether the lines are long or short, straight or curved, drawn on the paper to the left or to the right: only the difference in (topological) connection is relevant. Examining the simplest cases may help the reader to understand the intended meaning of the problem; it is easily seen that $T_1=1$, $T_2=1$, $T_3=2$, $T_4=4$, $T_5=9$.†

* For a more usual method *cf.* **3**, Vol. 1, p. 102, Problems 7, 8, and 9.

† The trees here considered should be called more specifically *root-trees;* see **4**, Vol. 11, p. 365. Their definition which is merely hinted here is elaborated in **2**, pp. 181–191; see also the passages there quoted of **5**. It may be, however, sufficient and in some respects even advantageous if, at a first reading, the reader takes the definition "intuitively" and supplements it by examples. Observe that in Cayley's first paper on the subject, **4**, Vol. 3, pp. 242–246, the definition of a tree is not even attempted. Chemistry is one of the sources of the notion "tree": if the points stand for atoms and the connecting lines for valencies, the tree represents a chemical compound.

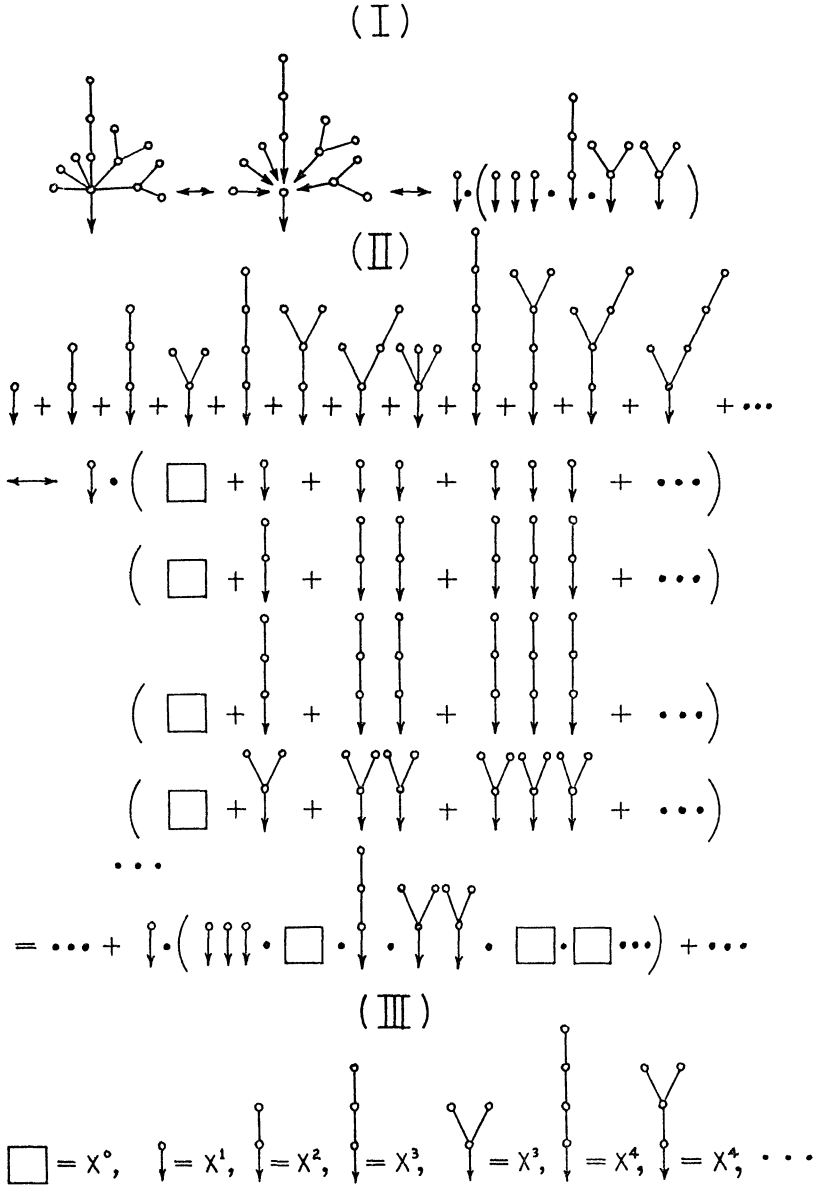


FIG. 5. Key idea, figurate series, transition.

The solution is indicated by the three parts of Figure 5 the general arrangement of which is closely similar to that of Figure 4. The reader should try to understand the solution by merely looking at Figure 5 and observing relevant analogies with all the foregoing figures. He may, however, fall back upon the following brief comments.

The simplest tree consists of root, trunk and just one knot. The key idea is to build up any tree different from the simplest tree from other trees which have fewer knots. For this purpose we conceive, as Figure 5 (I) shows, the "main branches" of any tree as trees (with fewer knots) inserted into the upper endpoint (the only knot) of the trunk. Therefore, as Figure 5 (I) further shows, we can conceive of any tree as the juxtaposition of the simplest tree and of several pictures, each of which consists of one, or two, or more *identical* trees; observe the analogy with the last line of Figure 2.

Part (II) of Figure 5 displays the figurate series: the infinite sum of all different trees. Its genesis is similar to, but more complex than, that of the figurate series of Figure 2. In Figure 2 we see a product of five "virtually geometric" series; in Figure 5 we see a product of an infinity of "virtually geometric" series, multiplied by an initial one term factor (the simplest tree, the common trunk of all trees).

3.2. Part (III) of Figure 5 displays the substitution that changes the figurate series into the enumerating series. By this substitution, each "virtually geometric" series arising in Figure 5 (II) goes over into a proper geometric series the sum of which is known, and the whole relation displayed by Figure 5 (II) goes over into the remarkable relation due to Cayley*

$$T_1x + T_2x^2 + T_3x^3 + \cdots + T_nx^n + \cdots = x(1-x)^{-T_1}(1-x^2)^{-T_2}(1-x^3)^{-T_3}\cdots(1-x^n)^{-T_n}\cdots. \tag{8}$$

3.3. By expanding the right hand side of Equation (8) in powers of x and comparing the coefficient of x^n on both sides, we obtain a recursion formula, that is, an expression for T_n in terms of $T_1, T_2, \cdots, T_{n-1}$ for $n \geqq 2$. The reader should work out the first cases and verify by analytical computation the values T_n for $n \leqq 5$ which he found before by geometrical experimentation.

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* This form is slightly different from that given in **4**, Vol. 3, pp. 242–246. For other forms see **2**, p. 149.

IMPROVING THE CONVERGENCE OF SERIES: APPLICATION TO SOME ELLIPTIC INTEGRALS

LEO F. EPSTEIN AND NANCY E. FRENCH, Knolls Atomic Power Laboratory†

1. Introduction. In a recent study of a physical problem [6] it became necessary to determine methods for evaluating a large number of incomplete elliptic integrals of the first and second kinds,

$$F(k, \phi) \equiv \int_0^{\infty} (1 - k^2 \sin^2 x)^{-1/2} dx, \tag{1.1a}$$

$$E(k, \phi) \equiv \int_0^{\infty} (1 - k^2 \sin^2 x)^{+1/2} dx, \tag{1.1b}$$

and some complete integrals $K(k) \equiv F(k, \pi/2)$, and $\mathbf{E}(k) \equiv E(k, \pi/2)$. The results obtained do not seem to have been reported previously, and these, along with the technique used, may be of some general interest. The method employed is to improve the rate of convergence of an absolutely convergent power series by writing

$$f(x) = \sum_0^{\infty} \alpha_n x^n = \sum_0^{\infty} \alpha_n^* x^n + \sum_0^{\infty} (\alpha_n - \alpha_n^*) x^n \tag{1.2}$$

where α_n^* is some conveniently chosen function of n, asymptotic with α_n (*i.e.* $\lim_{n\to\infty} \alpha_n^*/\alpha_n = 1$) whose sum is known, and where the second sum on the right in (1.2) converges much more rapidly than that on the left. It will often be possible to write the modified solution in the form

$$f_N(x) = \sum_0^N \alpha_n' x^n; \qquad f_\infty(x) = f(x), \tag{1.3}$$

where

$$\alpha_0' = \alpha_0 - \alpha_0^* + \sum_0^{\infty} \alpha_n^* x^n \quad \text{and} \quad \alpha_n' = \alpha_n - \alpha_n^* \text{ for } n \neq 0.$$

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2. Complete elliptic integrals of the first and second kinds. The values of $K(k)$ and $\mathbf{E}(k)$ are given by the series [1]

$$K(k) = \sum_0^\infty a_n k^{2n}; \tag{2.1}$$

$$\mathbf{E}(k) = \sum_0^\infty a_n k^{2n}/(1-2n); \tag{2.2}$$

where $a_n = (\pi/2)(1/4)^{2n}\binom{2n}{n}^2$. Applying Stirling's formula to the binomial coefficients, for large values of n,

$$a_n = +1/2n - 1/2^3n^2 + 1/2^6n^3 + \cdots. \tag{2.3}$$

To compute $K(k)$, a useful approximation for $n \geqq 1$ is

$$a_n^* = +1/2(n)_1 - 1/2^3(n)_2 - 7/2^6(n)_3 \tag{2.4}$$

in terms of the symbol [8]

$$(n)_m \equiv (n+m-1)!/(n-1)! = n(n+1)(n+2)\cdots(n+m-1). \tag{2.5}$$

To compute $\mathbf{E}(k)$, on the other hand, the approximation

$$a_n^*/(1-2n) = -1/2^2(n)_2 - 5/2^4(n)_3 - 93/2^7(n)_4 \tag{2.6}$$

is more useful. The sum in α_0' is then found in closed form by repeated integration of $\ln(1-t)$. Carrying this through, the complete elliptic integrals can be represented by

$$K_N(k) = \sum_0^N b_n k^{2n}, \tag{2.7a}$$

$$\mathbf{E}_N(k) = \sum_0^N c_n k^{2n}, \tag{2.7b}$$

with the values for the first few coefficients given in the table below. The higher coefficients are obtained by the method of the preceding section. The equations

n	b_n	c_n
0	$\pi/2-(1/256k^4)[(53k^2-14)k^2$ $+2(41k^4+30k^2-7)\ln(1-k^2)]$	$\pi/2-(1/1536k^6)[(1085k^4-705k^2+186)k^2$ $+6(135k^4-102k^2+31)(1-k^2)\ln(1-k^2)]$
1	$(48\pi-161)/384$	$-(384\pi-637)/3072$
2	$(36\pi-115)/512$	$-(120\pi-311)/5120$

(2.7) give astonishingly high accuracy even for $N=2$ as shown in Table I. An approximation $\overline{\mathbf{E}}(k)$ better than $\mathbf{E}_3(k)$ is obtained by replacing $c_3 = -(150\pi-431)/15360$ by the coefficient $-(900\pi-2817)/2560$, which makes the formula fit exactly at $k=1$.

TABLE I

ACCURACY OF APPROXIMATION FORMULAS FOR $K(k)$ AND $\mathbf{E}(k)$

k^2	$K_2(k)$	$K(k)$	$\mathbf{E}_2(k)$	$\bar{\mathbf{E}}(k)$	$\mathbf{E}(k)$
0	1.570 796	1.570 796	1.570 796	1.570 796	1.570 796
0.05	1.591 004	1.591 003	1.550 974	1.550 973	1.550 973
.10	1.612 442	1.612 441	1.530 760	1.530 756	1.530 758
.15	1.635 260	1.635 257	1.510 131	1.510 117	1.510 122
.20	1.659 633	1.659 624	1.489 057	1.489 025	1.489 035
.25	1.685 768	1.685 750	1.467 507	1.467 443	1.467 462
.30	1.713 921	1.713 889	1.445 441	1.445 331	1.445 363
.35	1.744 402	1.744 351	1.422 818	1.422 643	1.422 691
.40	1.777 598	1.777 519	1.399 584	1.399 324	1.399 392
.45	1.813 998	1.813 884	1.375 681	1.375 310	1.375 402
.50	1.854 236	1.854 075	1.351 035	1.350 525	1.350 644
.55	1.899 145	1.898 925	1.325 556	1.324 878	1.325 024
.60	1.949 862	1.949 568	1.299 133	1.298 253	1.298 428
.65	2.007 984	2.007 598	1.271 625	1.270 505	1.270 707
.70	2.075 860	2.075 363	1.242 844	1.241 446	1.241 671
.75	2.157 148	2.156 516	1.212 535	1.210 816	1.211 056
.80	2.258 002	2.257 205	1.180 333	1.178 246	1.178 490
.85	2.390 010	2.389 016	1.145 668	1.143 166	1.143 396
.90	2.579 323	2.578 092	1.107 554	1.104 583	1.104 775
.95	2.909 855	2.908 337	1.063 848	1.060 354	1.060 474
1.00	$+\infty$	$+\infty$	1.004 076	1.000 000	1.000 000

The columns headed $K(k)$ and $\mathbf{E}(k)$ were obtained from the Appendix to [2].

3. Computation of $F(k, \phi)$ and $E(k, \phi)$ for small k and ϕ. For these cases it is convenient to start with the series [2, 17]

$$F(k, \phi) = \sum_0^\infty \binom{-1/2}{n} (-k^2)^n A_n(\phi), \tag{3.1a}$$

$$E(k, \phi) = \sum_0^\infty \binom{+1/2}{n} (-k^2)^n A_n(\phi), \tag{3.1b}$$

with

$$A_n(\phi) = \int_0^\phi \sin^{2n} x dx = (1/2n)[(2n-1)A_{n-1}(\phi) - \sin^{2n-1}\phi \cos\phi] \tag{3.2}$$

and $A_0(\phi) = \phi$. By the method of steepest descents [3, 4 I p. 54, 5, 9, 14, 16] taking $y = \phi - x$ and expanding it is found that

$$A_n(\phi) = (\sin^{2n}\phi) \int_0^\phi [1 - (n \csc^2\phi)y^2 + \cdots] \exp(-2ny \cot\phi) dy. \tag{3.3}$$

Changing the upper limit to ∞, integrating and replacing n^m in the result by $(n)_m$, the approximation function

$$A_n^*(\phi) = (\tan\phi \sin^{2n}\phi)[1/2(n)_1 - (\sec^2\phi)/4(n)_2] \tag{3.4}$$

is obtained. With this, a closed analytic expression for the terms entering into $\sum_1^\infty \alpha_n^* x^n$ is found by integrating $\zeta^{-1}[(1-\zeta)^{\pm 1/2}-1]$ repeatedly from zero to x. Then writing

$$F(k, \phi) = \sum_0^\infty k^{2n} B_n(\phi), \tag{3.5a}$$

$$E(k, \phi) = \sum_0^\infty k^{2n} C_n(\phi), \tag{3.5b}$$

it is found that

$$B_0(\phi) = \phi - (\sin\phi/4\cos^3\phi)[1 + 2(2\cos^2\phi - 1)\ln(1/2)(1+\Delta) - (2/x)(1-\Delta)], \tag{3.6a}$$

$$C_0(\phi) = \phi + (\sin\phi/4\cos^3\phi)[1 - 2(2\cos^2\phi - 1)\ln(1/2)(1+\Delta) - 4(1-\Delta)\cos^2\phi + (2/3x)\{1-(1+2x)\Delta\}]. \tag{3.6b}$$

In these equations, $x \equiv k^2 \sin^2\phi$ and $\Delta \equiv \sqrt{1-x}$ (*cf.* ref. [11]). For higher values of n, the coefficients are $\binom{\pm 1/2}{n}(-1)^n[A_n(\phi) - A_n^*(\phi)]$ where the lower sign corresponds to $B_n(\phi)$, the upper to $C_n(\phi)$.

While the series (3.5a) is best for small k and ϕ values, $F(k, \phi)$ will be computed using $k^2 = 0.92$ and $\phi = 30°$ since these constants have been used by Kennedy [12] in a recent study of elliptic integrals. For these parameters, 7-point Lagrangian interpolation [15] in the Legendre-Pearson tables [13] yields the value 0.54698 6214. The rapid convergence of the sum is shown in Table II. To check equation (3.5b), $E(k, \phi)$ for $k = \sin 30°$, $\phi = 45°$ was computed; the C_0 term gives this quantity accurate to 0.3%; inclusion of the C_0 and C_1 terms yields 0.003% accuracy; and the use of terms through C_2 results in a value good to 0.000 08%.

TABLE II

CALCULATION OF $F(k, \phi)$ FOR $k^2 = 0.92$, $\phi = 30°$

n	Formula (3.5a) (B_n)	Formula (4.1), modified (D_n')	Kennedy [12]	Frame [10]
0	0.54832 02081	0.54662 65378	(0.5493)	0.54674 25032
1	.54702 32447	.54698 76606	.54696	.54696 89662
2	.54698 83542	.54698 62028	.54698 67	
3	.54698 63890	.54698 62144	.54698 621	
4	.54698 62319	.54698 62143	.54698 62140	
5	.54698 62164			
6	.54698 62146			
7	.54698 62144			
8	.54698 62143			

4. Computation of $F(k, \phi)$ for small $(1-k^2)$ and ϕ. For this case, it is convenient to start with [2, 17]

$$F(k, \phi) = \sum_0^\infty \binom{-1/2}{n} (1 - k^2)^n D_n(\phi) \tag{4.1}$$

where

$$D_n(\phi) = \int_0^\phi \sec x \tan^{2n} x dx = (1/2n)[\csc \phi \tan^{2n} \phi - (2n - 1)D_{n-1}(\phi)] \tag{4.2}$$

and $D_0(\phi) = \ln (\sec \phi + \tan \phi)$. Application of the method of steepest descents as in the preceding paragraph leads to (*cf.* 3.4)

$$D_n^*(\phi) = (\sin \phi \tan^{2n} \phi)[1/2(n)_1 - (\cos^2 \phi)/4(n)_2]. \tag{4.3}$$

The $\sum \alpha_n^* z^n$ term in this case is derived by repeated integration of $\zeta^{-1}[(1+\zeta)^{-1/2} - 1]$ from zero to z. This procedure results in a summation identical in form with (4.1) except that $D_n'(\phi)$ is substituted for $D_n(\phi)$ where

$$\begin{aligned} D_0'(\phi) = D_0(\phi) &- (1/4)(\sin \phi \cos^2 \phi)[1 + (2/z)(1 - \delta) \\ &+ 2(2 \sec^2 \phi - 1) \ln (1/2)(1 + \delta)], \end{aligned} \tag{4.4}$$

with $z \equiv (1-k^2) \tan^2 \phi$ and $\delta \equiv \sqrt{1+z}$. For $n \neq 0$, $D_n'(\phi) = D_n(\phi) - D_n^*(\phi)$.

These relations are better adapted to the computation of $F(\sqrt{0.92}, 30°)$ than those considered in section 3 above, as can readily be seen from Table II. Four terms in the D_n' series are sufficient to yield 10-place accuracy. Table II also includes a comparison with the formulas of Kennedy [12] and the interesting approximations recently proposed by Frame [10]:

$$F_N(k, \phi) = \frac{(3 - a\phi^2/10) \sin \phi}{1 + \Delta + \cos \phi - a\phi^2/10} \tag{4.5}$$

where $a = 0$ for the zero-th approximation and $a = (1-k^2+k^4)/(1+k^2)$ for $N = 1$. It will be observed that in the zero order approximation, Frame's result is outstandingly the best of the three considered here; but to the first approximation, the D_n' equation above is more accurate.

5. Concluding remarks. The use of elliptic integrals to illustrate the technique of improving the convergence of series described in section 1 will not divert the alert reader from a realization that this method has quite general applicability and is a very powerful device. Its use will be shown in one more example, which will bring out some further points. In computing the third virial coefficient of a Lennard-Jones 6–9 gas [7], the value of

$$\mathbf{A}_1(s) = 2 \int_0^1 t^{-4}(t - 2)[\ln t^{-s} - (1 + t^6) \ln (1 + t^{-s})] dt \tag{5.1}$$

is required for $s = 7, 8, 9 \cdots$. This function may be written in completely closed

form in terms of $\psi(x)$, the logarithmic derivative of the gamma function [4] but for convenience in calculation by digital computing machine methods, it is preferable to use the relation, readily derived from (5.1),

$$\mathbf{A}_1(s) = +(23s/72) + \sum_1^\infty (-1)^n(4/n)\left[\frac{ns+1}{(ns-2)(ns+4)} - \frac{2ns}{(ns+3)(ns-3)}\right]. \tag{5.2}$$

Expanding the summand in powers of $1/(ns)$, it is found to be

$$-(-1)^n(4/sn^2)\sum_{i=1}^\infty \gamma_i/(ns)^{i-1} \tag{5.3}$$

with $\gamma_1=\gamma_2=1$, $\gamma_3=8$, $\gamma_4=28$, $\gamma_5=26\cdots$. Thus in the limit of large n, the summand behaves like $4/sn^2$ and to obtain 10-place accuracy with $s=7$ using this formula would require about 10^5 terms. Application of the technique of section 1 however yields

$$\mathbf{A}_1(s) = +(23s/72) + 4\sum_{i=1}^4 (\gamma_i t_{i+1})/s_i + \sum_1^\infty (-1)^n(4/n)\left[\frac{ns+1}{(ns-2)(ns+4)} - \frac{2ns}{(ns)^2-9} + \sum_{i=1}^4 \frac{\gamma_i}{(ns)^i}\right]. \tag{5.4}$$

The quantities $t_i \equiv \sum_1^\infty (-1)^{n-1}/n^i$ appearing in this equation have been extensively and accurately tabulated [4, v. II, pp. 196, 247]. The residual infinite sum in (5.4) behaves in the limit of large n like the first term omitted in (5.3), $104/s^5n^6$, and with $s=7$, twenty terms are sufficient to yield 10-place accuracy. This rather amazing reduction from about 10^5 to 20 terms is indicative of the usefulness of the method of section 1.

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ON INVOLUTORY MATRICES

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1. Let I denote an identity matrix, and O a null matrix whose order will be clear from the context. A square matrix $A=(a_{ij})$ is called involutory if $A^2=I$. Throughout this paper the elements of A will belong to the field F of real numbers.

If X and Y be two matrices of the same order and if all the elements of $X-Y$ be integral multiples of p, where p is an integer greater than 1, then we shall write $X\equiv Y \pmod p$. Following Lehmer we shall call an integer p a pseudoprime (for results on pseudoprimes see [2] and references given therein), if $2^p\equiv 2 \pmod p$ and p is not a prime.

In section 2 of this paper, two theorems on the Theory of Numbers have been proved, one of which will be used to prove a theorem of section 4. In section 3, some relationships between involutory and idempotent matrices have been established.

2. THEOREM 1. $2^{k-1}+1\not\equiv 0 \pmod k$, *for any integral value of* $k>1$. To prove this theorem, let us consider the following two cases:

Case (i). *Let k be a positive even integer.*

Suppose there exists an integer y such that

$$2^{k-1}+1=ky. \tag{1}$$

For all positive even integers k, the left-hand side of (1) is always an odd integer. But the right-hand side is an even integer. Hence our supposition is wrong and $2^{k-1}+1\not\equiv 0 \pmod k$, when k is a positive even integer.

Case (ii).* *Let k be an odd integer greater than* 1.

Suppose $k=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, where the p_r's are all distinct primes different

* Professor H. Davenport, F. R. S., was generous enough to help me in proving this part of the result.

from 2. Now $p_i-1=2^{m_i}t_i$, t_i being odd ($i=1, 2, \cdots, r$). Without loss of generality we can suppose that $m_1=\min(m_1, m_2, \cdots, m_r)$. Therefore, since $p_i\equiv 1 \pmod{2^{m_1}}$, ($i=1, 2, \cdots, r$),

$$k-1=2^{m_1}u, \text{ for some integer } u.$$

Now suppose that $2^{k-1}\equiv -1 \pmod{k}$, and therefore $2^{2^{m_1}u}\equiv -1 \pmod{p_1}$. Raising both the sides of the above congruence to the odd power t_1, we have

$$2^{2^{m_1}ut_1} \equiv -1 \pmod{p_1},$$

or

$$2^{(p_1-1)u} = -1 \pmod{p_1}.$$

But, by Fermat's Theorem $2^{p_1-1}\equiv 1 \pmod{p_1}$, which gives

$$2^{(p_1-1)u} \equiv 1 \pmod{p_1}.$$

This gives a contradiction, and the result is therefore proved in this case also.

THEOREM 2. $2^{p_1}+2^{p_2}\not\equiv 0 \pmod{p_1p_2}$, *where* p_1 *and* p_2 *are primes, except when* $p_1p_2=4$ *or* 6.

Proof. To prove the theorem consider the following cases:

Case (i). p_1 *and* p_2 *are odd.*

Assume that $2^{p_1}+2^{p_2}\equiv 0 \pmod{p_1p_2}$, so that

$$2^{p_1}+2^{p_2}\equiv 0 \pmod{p_1}. \tag{2}$$

Also, by Fermat's Theorem, we have

$$2^{p_1}-2\equiv 0 \pmod{p_1}. \tag{3}$$

(2) and (3) give

$$2^{p_2}+2\equiv 0 \pmod{p_1},$$

or

$$2^{p_2} \equiv -2 \pmod{p_1},$$

or

$$(2^{p_2})^{p_1} \equiv -2^{p_1} \pmod{p_1}, \text{ since } p_1 \text{ is odd},$$

or

$$2^{p_1p_2}+2^{p_1}\equiv 0 \pmod{p_1}.$$

The above congruence with the help of (3) gives

$$2^{p_1p_2-1}+1\equiv 0 \pmod{p_1}. \tag{4}$$

Similarly, we have

$$(5)\qquad \begin{aligned} &2^{p_1 p_2 - 1} + 1 \equiv 0 \pmod{p_2}, \text{ and so} \\ &2^{p_1 p_2 - 1} + 1 \equiv 0 \pmod{p_1 p_2}, \end{aligned}$$

contrary to Theorem 1.

Case (ii). $p_1 = 2$, $p_2 > 3$.

Assuming that $2^{p_1} + 2^{p_2} \equiv 0 \pmod{p_1 p_2}$, we have $2^{p_1} + 2^{p_2} \equiv 0 \pmod{p_2}$, so that $2^{p_2} + 4 \equiv 0 \pmod{p_2}$. But, by Fermat's Theorem $2^{p_2} - 2 \equiv 0 \pmod{p_2}$. Hence, $6 \equiv 0 \pmod{p_2}$, which contradicts $p_2 > 3$.

This completes the proof of the theorem.

3. THEOREM 3. *The involutory matrix A of order n is similar to $I_p \dotplus (-I_{n-p})$, where p depends on A and $\dotplus$ denotes the direct sum.*

Proof. Since $A^2 = I$, A satisfies $x^2 - 1 = 0$, and the minimum polynomial of A divides $x^2 - 1$. It can be either $x-1$, $x+1$ or x^2-1. But, if A is neither the identity matrix I nor is it $-I$, then it satisfies neither $x - 1 = 0$, nor $x + 1 = 0$. Therefore, assuming that $A \neq I$, $-I$, its minimal polynomial is $x^2 - 1$.

Thus the elementary divisors of A are the monic polynomials $x-1$ and $x+1$ and they are relatively prime. The companion matrix for each factor $x-1$ of the characteristic polynomial of A is the matrix (1) and for $x+1$ it is the matrix (-1). Hence, by a well known result, [3; p. 241], A is similar to $I_p \dotplus (-I_q)$, where $p + q = n$. This completes the proof.

By simple calculations it can be seen that the matrices $(I+A)/2$ and $(I-A)/2$ are idempotent. Therefore for any positive integer k, the matrices $[(I+A)/2]^k$ and $[(I-A)/2]^k$ are also idempotent. Moreover, since $[(I+A)/2]^k \cdot [(I-A)/2]^k = [(I-A)/2]^k [(I+A)/2]^k = 0$, $[(I+A)/2]^k$ and $[(I-A)/2]^k$ are orthogonal idempotent. Thus we have:

THEOREM 4. *If the matrix A is involutory, the matrices $[(I+A)/2]^k$ and $[(I-A)/2]^k$ are idempotent, k being an arbitrary positive integer.*

THEOREM 5. *The idempotent matrices $[(I+A)/2]^k$ and $[(I-A)/2]^k$ are orthogonal.*

Since the rank of the sum of two orthogonal idempotent matrices is the sum of their ranks, [1; p. 89],

$$\rho\left\{\left(\frac{I+A}{2}\right)^k\right\} + \rho\left\{\left(\frac{I-A}{2}\right)^k\right\} = \rho\left\{\left(\frac{I+A}{2}\right)^k + \left(\frac{I-A}{2}\right)^k\right\},$$

where ρ denotes the rank

$$= \rho\left[2^{-k}\left\{2I + 2\binom{k}{2}A^2 + 2\binom{k}{4}A^4 + \cdots\right\}\right]$$

$$= \rho\{I\} = n.$$

But $\rho\{(I+A)^k\} = \rho\{[(I+A)/2]^k\}$ and $\rho\{(I-A)^k\} = \rho\{[(I-A)/2]^k\}$. Hence we have:

THEOREM 6. $\rho\{(I+A)^k\}+\rho\{(I-A)^k\}=n$, *the order of* A, *for any positive integer* k.

We now establish the following:

THEOREM 7. *For an involutory matrix* A *of order* n, *such that* $\rho(I+A)=r$, *the characteristic polynomial* $|\lambda I-A|=(\lambda-1)^r(\lambda+1)^{n-r}$.

Proof. From Theorem 3, A is similar to $I_p\dotplus(-I_{n-p})$. Therefore, $|\lambda I-A|=(\lambda-1)^p(\lambda+1)^{n-p}$. Since $(I+A)/2$ is idempotent and $\rho(I+A)/2=\rho(I+A)=r$, it is similar to $I_r\dotplus O_{n-r}$, [1; p. 88]. Hence

$$\left|\lambda I-\frac{I+A}{2}\right|=(\lambda-1)^r\lambda^{n-r}.$$

But,

$$\left|\lambda I-\frac{I+A}{2}\right|=2^{-n}\,|(2\lambda-1)I-A|=2^{-n}(2\lambda-1-1)^p(2\lambda-1+1)^{n-p}$$

$$=(\lambda-1)^p\lambda^{n-p}.$$

Therefore, $p=r$, the rank of $(I+A)$.

4. THEOREM 8. *Let* A *be an involutory matrix whose elements are integers (positive, negative or zero). If* p *is a prime or a pseudoprime, then both congruences*

$$2(I\pm A)^p\mp 2A\equiv 2I \pmod{p}$$

hold, and conversely.

Proof. If p is a prime or a pseudoprime, $2^p\equiv 2 \pmod{p}$, and by hypothesis the elements of A are all integers. Hence

$$2(I\pm A)^p\mp 2A-2I$$

$$=2\left[I\pm\binom{p}{1}A+\binom{p}{2}A^2\pm\binom{p}{3}A^3+\cdots\right]\mp 2A-2I$$

$$=2\left[\left\{\binom{p}{0}+\binom{p}{2}+\binom{p}{4}+\cdots\right\}I\pm\left\{\binom{p}{1}+\binom{p}{3}+\binom{p}{5}+\cdots\right\}A\right]+2A-2I$$

$$=(2^p-2)I\pm(2^p-2)A$$

$$\equiv 0 \pmod{p}.$$

i.e., $2(I\pm A)^p\mp 2A\equiv 2I \pmod{p}$.

Conversely, if $2(I\pm A)^p\mp 2A\equiv 2I \pmod{p}$, the elements of the matrix $2(I\pm A)^p\mp 2A-2I$ or $(2^p-2)I\pm(2^p-2)A$ are divisible by p. Hence $2^p\equiv 2 \pmod{p}$, and so either p is a prime or it must be a pseudoprime.

THEOREM 9. *Let* A *be an involutory matrix whose elements are integers (posi-*

tive, negative or zero). If p is a prime or a pseudoprime, then both congruences

$$\{2(I \pm A)\}^p \mp 2A \equiv 2I \pmod{p}$$

hold, and conversely.

Proof. By hypothesis the elements of A are all integers. Moreover, if p is a prime or a pseudoprime, then $2^p \equiv 2 \pmod{p}$, and

$$\{2(I \pm A)\}^p \mp 2A - 2I$$

$$= 2^p\left[I \pm \binom{p}{1}A + \binom{p}{2}A^2 \pm \binom{p}{3}A^3 + \cdots\right] \mp 2A - 2I$$

$$= 2^p\left[\left\{\binom{p}{0} + \binom{p}{2} + \binom{p}{4} + \cdots\right\} I \pm \left\{\binom{p}{1} + \binom{p}{3} + \binom{p}{5} + \cdots\right\} A\right] \mp 2A - 2I$$

$$= (2^{2p-1} - 2)I \pm (2^{2p-1} - 2)A$$

$$= (2^p - 2)(2^{p-1} + 1)I \pm (2^p - 2)(2^{p-1} + 1)A$$

$$\equiv 0 \pmod{p}.$$

Hence $\{2(I \pm A)\}^p \mp 2A \equiv 2I \pmod{p}$.

Conversely, if $\{2(I \pm A)\}^p \mp 2A \equiv 2I \pmod{p}$, the elements of the matrix $(2^p - 2)(2^{p-1} + 1)(I \pm A)$ are all (integer) multiples of p. It has been proved in Theorem 1 that $2^{p-1} \not\equiv -1 \pmod{p}$. Therefore

$$2^p - 2 \equiv 0 \pmod{p}, \quad \text{or} \quad 2^p \equiv 2 \pmod{p}.$$

Hence, either p is a prime or it must be a pseudoprime.

Similarly, we can prove the following theorem:

THEOREM 10. *Let A be an involutory matrix whose elements are integers (positive, negative or zero). If p is a prime or a pseudoprime, then both congruences*

$$2(I \pm A)^{2p-1} \mp 2A \equiv 2I \pmod{p},$$

hold, and conversely.

5. To show that there exist matrices satisfying the hypotheses of Theorems 8, 9 and 10, we consider the following examples:

Example 1.

$$p = 7, \qquad n = 2 \quad \text{and} \quad A = \begin{pmatrix} 5 & 3 \\ -8 & -5 \end{pmatrix}.$$

Then

$$2(I + A)^7 - 2A = \begin{pmatrix} 768 & 384 \\ -1024 & -512 \end{pmatrix} - \begin{pmatrix} 10 & 6 \\ -16 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} 758 & 378 \\ -1008 & -502 \end{pmatrix} \equiv 2I \pmod{7}.$$

Also

$$2(I-A)^7+2A=\begin{pmatrix}-512 & -384\\ 1024 & 768\end{pmatrix}+\begin{pmatrix}10 & 6\\ -16 & -10\end{pmatrix}$$
$$=\begin{pmatrix}-502 & -378\\ 1008 & 758\end{pmatrix}\equiv 2I \pmod 7.$$

Example 2.

$$p=5,\qquad n=3 \text{ and } A=\begin{bmatrix}-1 & 4 & -2\\ -4 & 9 & -4\\ -8 & 16 & -7\end{bmatrix}.$$

Then

$$\{2(I+A)\}^5-2A=\begin{bmatrix}0 & 2048 & -1024\\ -2048 & 5120 & -2048\\ -4096 & 8192 & -3072\end{bmatrix}+\begin{bmatrix}-2 & 8 & -4\\ -8 & 18 & -8\\ -16 & 32 & -14\end{bmatrix}$$
$$=\begin{bmatrix}2 & 2040 & -1020\\ -2040 & 5102 & -2040\\ -4080 & 8160 & -3058\end{bmatrix}\equiv 2I \pmod 5.$$

Also

$$2(I-A)^{2p-1}+2A=\begin{bmatrix}1024 & -2048 & 1024\\ 2048 & -4096 & 2048\\ 4096 & -8192 & 4096\end{bmatrix}+\begin{bmatrix}-2 & 8 & -4\\ -8 & 18 & -8\\ -16 & 32 & -14\end{bmatrix}$$
$$=\begin{bmatrix}1022 & -2040 & 1020\\ 2040 & -4078 & 2040\\ 4080 & -8160 & 4082\end{bmatrix}\equiv 2I \pmod 5.$$

Finally, I am grateful to Professor S. M. Shah for his help and guidance. I am also thankful to the referee for suggesting shortcuts in certain proofs.

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A CHARACTERISTIC PROPERTY OF THE CIRCLE, ELLIPSE, AND HYPERBOLA

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If C is a piecewise differentiable curve and p is a fixed "pedal point," then the pencil of lines on p cuts the one-sided tangents to C orthogonally in a locus $P(C)$ which is "the pedal curve of C with respect to p." When C is an ellipse or hyperbola, and p is a focus, then $P(C)$ is a circle which has the vertices of the conic as the end points of a diameter. In either case, the conic has the same pedal curve with respect to either focus, and this is characteristic in the following sense.

THEOREM. *If a piecewise differentiable curve C has the same pedal curve $P(C)$ with respect to two points p_1 and p_2, then every arc of C belongs to a conic whose foci are p_1 and p_2, and every arc of $P(C)$ belongs to a circle whose center O is the midpoint of p_1 and p_2.*

Proof. Let $L=p_1\times p_2$ be the line of p_1 and p_2, and $d(p_1, p_2)=2\lambda$ be the distance between them. Take $R(p, \alpha)$ to denote the circle whose center is p and whose radius is α. Consider the points of $P(C)$ not on L. These cannot all belong to $R(O, \lambda)$ because C would then consist of the two isolated points p_1 and p_2. Therefore there exists a point q on $P(C)-L$, for which $d(O, q)\neq\lambda$. Let $R_q=R[O, d(O, q)]$ intersect L at v_1 and v_2. There is then an "associated" conic H_q, with vertices v_1 and v_2 and foci p_1 and p_2, whose pedal curve is R_q. Assume that v_i is the vertex nearest to p_i, $i=1, 2$. Now take the line T_1 perpendicular to $p_2\times q$ at q and let q_1 be the foot of p_1 on T_1. Because T_1 is a one-sided tangent to C, the point q_1 is on $P(C)$. Repeating, let T_2 be normal to $p_2\times q_1$ at q_1 and take q_2 to be the foot of p_1 on T_2. Continuing in this way, a sequence $\{q_i\}$ is obtained on $P(C)\cap R_q$ which progresses monotonically in the direction from v_2 to v_1 on the semi-circle, and so converges. Hence as $i\to\infty$, $d(q_i, q_{i+1})\to 0$. But $d(q_i, q_{i+1})=2\lambda\cos\theta_i$, where θ_i is the angle between L and $q_i\times q_{i+1}$. Therefore $\cos\theta_i\to 0$, which implies that $\theta_i\to 90°$ and that $\{q_i\}$ converges to v_1. Now let s be any point of $P(C)$ which is not on R_q. Then there cannot be an arc of $P(C)$ joining q and s. For if there were, there would exist a subarc, from q to some point x, which did not intersect $R(O, \lambda)$. For each point y of this subarc, the associated conic H_y would have a vertex v_y, nearest to p_1, belonging to C, hence the interval from v_1 to v_x would belong to C. Since at each such point the tangent to C is perpendicular to L, the curve C would intersect itself at right angles at every point of an interval. But this is impossible since C is a curve. By the same reasoning, it is clear that $P(C)$ cannot intersect $R(O, \lambda)$. The only remaining possibility is

for $P(C)$ to consist of circular arcs which have center O and which do not intersect $R(O, \lambda)$. Thus C consists of elliptic and hyperbolic pieces.

It is clear that one can further restrict C in a variety of ways (*e.g.*, differentiability) to obtain an explicit characterization of the ellipse alone or the hyperbola alone.

One can define a modified pedal curve $P'(C)$ as the feet of the pedal point on the lines of support to C (instead of the tangents). This introduces the case where C consists of p_1, p_2 and part of the interval between them. Then $P'(C) = R(O, \lambda)$. No other new case arises.

Another generalization would be the use of "local lines of support" (*i.e.*, those lines through a point r of C which are lines of support at r to the part of C within some neighborhood of r). The only new cases here arise when C is the closed interval from p_1 to p_2, or else is the line L minus the open interval from p_1 to p_2. In both cases the new type of pedal curve is $R(O, \lambda)$.

ANOTHER PROOF OF THE INFINITUDE OF PRIMES

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It is believed that the proof of the infinitude of primes presented here has some feature of novelty.

Let A_0, A_1, and A_2 be positive integers which are relatively prime in pairs, and for $n \geqq 3$ set $A_n \equiv A_0A_1 \cdot \cdots \cdot A_{n-3}A_{n-1} + A_{n-2}$. By induction we can show that $A_0, A_1, \cdots, A_n$ are relatively prime in pairs for all values of $n \geqq 3$ and hence relatively prime in pairs for all values of $n \geqq 1$. Since $A_n > 1$ at least for $n \geqq 3$, it follows that $A_0, A_1, \cdots$ contains an infinite set of positive integers, larger than unity, no two with a common divisor $d > 1$. Therefore the number of primes is infinite.

The recursion formula for A_n is that for the numerators of the approximants of the regular infinite continued fraction

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots}}, \qquad b_n = A_0A_1 \cdot \cdots \cdot A_{n-3}, \qquad n \geqq 3.$$

If b_0, b_1, and b_2 are positive integers determined so that $A_0 (= b_0)$, $A_1 (= b_0b_1 + 1)$, and $A_2 (= b_0b_1b_2 + b_0 + b_2)$ are relatively prime in pairs, then the numerators of the approximants form a set as described above. In particular, if $b_0 = b_1 = b_2 = 1$, the first few partial denominators and approximants are:

n	0	1	2	3	4	5	6	7	8
b_n	1	1	1	1	2	6	30	390	32,370
A_n	1	2	3	5	13	83	2503	976,253	31,601,312,113
B_n	1	1	2	3	8	51	1538	599,871	19,417,825,808

Since 31,601,312,113 is divisible by 7 (but is not a power of 7), we have proved that the ninth prime is less than 31,601,312,113.

A MAPPING OF COPLANAR LINE VECTORS ON 3-SPACE

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A line vector $\mathbf{u}$ acting in the line whose vector equation is $\mathbf{r}\times\mathbf{u}=\mathbf{u}_0$, is fully specified by the Plücker coordinates of the line $(\mathbf{u}, \mathbf{u}_0)$. Here $\mathbf{u}_0$ is the moment of $\mathbf{u}$ about the origin; and $\mathbf{u}\cdot\mathbf{u}_0=0$. The symbol $(\mathbf{u}, \mathbf{u}_0)$ thus denotes a

Proper line vector when $\mathbf{u} \neq \mathbf{0}$;

Couple of moment $\mathbf{u}_0$ when $\mathbf{u} = \mathbf{0}$, $\mathbf{u}_0 \neq \mathbf{0}$;

Null vector when $\mathbf{u} = \mathbf{0}$, $\mathbf{u}_0 = \mathbf{0}$.

The line vectors and couples of the xy-plane, when written

$$(\mathbf{u}, \mathbf{u}_0) = (u_1\mathbf{i} + u_2\mathbf{j}, u_3\mathbf{k}),$$

suggest the one-to-one correspondence

$$(\mathbf{u}, \mathbf{u}_0) \leftrightarrow [u_1, u_2, u_3] = \mathbf{U}, \tag{1}$$

with the space vectors $\mathbf{U}$ drawn from the origin:

Line vectors $(\mathbf{u} \neq \mathbf{0}, \mathbf{u}_0) \leftrightarrow [u_1, u_2, u_3]$,

Couples $(\mathbf{0}, \mathbf{u}_0 \neq \mathbf{0}) \leftrightarrow [0, 0, u_3]$,

Null vector $(\mathbf{0}, \mathbf{0}) \leftrightarrow [0, 0, 0]$.

Of the space vectors localized at the origin, those lying on the z-axis correspond to couples, all others to line vectors. Addition of coplanar line vectors now corresponds to the addition of space vectors:

$$(\mathbf{u}, \mathbf{u}_0) + (\mathbf{v}, \mathbf{v}_0) \leftrightarrow [u_1 + v_1, u_2 + v_2, u_3 + v_3] = \mathbf{U} + \mathbf{V}. \tag{2}$$

All line vectors parallel to $(\mathbf{u}, \mathbf{u}_0)$ are given by $(\lambda\mathbf{u}, \lambda'\mathbf{u}_0)$; hence the corresponding space vectors lie in a plane parallel to $\mathbf{u}$ through the z-axis.

If the line vector $(\mathbf{u}, \mathbf{u}_0)$ passes through the point (x, y), the corresponding

$$\mathbf{U} = [u_1, u_2, xu_2 - yu_1]$$

is a space vector perpendicular to $[y, -x, 1]$; hence all line vectors through the point (x, y) correspond to space vectors $\mathbf{U}$ lying in the plane perpendicular to $[y, -x, 1]$.

We can now readily solve the following important problem which in statics is known as the *Problem of Three Forces*:

Express a given line vector as the sum of three coplanar line vectors acting along specified lines, not concurrent and not all parallel.

Solution. Write the given line vector $(\mathbf{u}, \mathbf{u}_0)$ in the form (1) and let the three lines given by their Plücker coordinates

$$(\mathbf{a}, \mathbf{a}_0) \leftrightarrow \mathbf{A}, \qquad (\mathbf{b}, \mathbf{b}_0) \leftrightarrow \mathbf{B}, \qquad (\mathbf{c}, \mathbf{c}_0) \leftrightarrow \mathbf{C}.$$

Since the lines are not concurrent and not all parallel, the corresponding space vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are not coplanar and admit a reciprocal set $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$. Now

$$\mathbf{U} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C},$$

where $\alpha = \mathbf{U}\cdot\mathbf{A}'$, $\beta = \mathbf{U}\cdot\mathbf{B}'$, $\gamma = \mathbf{U}\cdot\mathbf{C}'$. This equality corresponds to

$$(\mathbf{u}, \mathbf{u}_0) = \alpha(\mathbf{a}, \mathbf{a}_0) + \beta(\mathbf{b}, \mathbf{b}_0) + \gamma(\mathbf{c}, \mathbf{c}_0)$$

and the problem is solved. The solution moreover is unique.

This gives a swift analytic solution of a problem of three forces. For example, let us resolve the force $\mathbf{f} = [1, 2]$ acting through the point $(-3, 0)$ into forces along the lines $y = 0$, $x = 0$, $x + y = 2$.

Since $\mathbf{f}_0 = -6\mathbf{k}$, $\mathbf{F} = [1, 2, -6]$. The three lines correspond to

$$\mathbf{A} = [1, 0, 0], \qquad \mathbf{B} = [0, 1, 0], \qquad \mathbf{C} = [-1, 1, 2],$$

and since the box product $[\mathbf{ABC}] = 2$, the reciprocal set is

$$\mathbf{A}' = \tfrac{1}{2}[2, 0, 1], \qquad \mathbf{B}' = \tfrac{1}{2}[0, 2, -1], \qquad \mathbf{C}' = \tfrac{1}{2}[0, 0, 1];$$

hence

$$\mathbf{F} = \mathbf{F}\cdot\mathbf{A}'\ \mathbf{A} + \mathbf{F}\cdot\mathbf{B}'\ \mathbf{B} + \mathbf{F}\cdot\mathbf{C}'\ \mathbf{C} = -2\mathbf{A} + 5\mathbf{B} - 3\mathbf{C}.$$

Thus the given force $(\mathbf{f}, \mathbf{f}_0)$ is equivalent to a force $-2\mathbf{i}$ along the x-axis, $5\mathbf{j}$ along the y-axis, $3\mathbf{i} - 3\mathbf{j}$ along the line $x + y = 2$.

Since any three non-coplanar vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ form a basis in space, the corresponding $(\mathbf{a}, \mathbf{a}_0)$, $(\mathbf{b}, \mathbf{b}_0)$, $(\mathbf{c}, \mathbf{c}_0)$ form a basis in the xy-plane. Thus the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in space corresponds to two line vectors $(\mathbf{i}, \mathbf{0})$, $(\mathbf{j}, \mathbf{0})$ and the couple $(\mathbf{0}, \mathbf{k})$. Any three line vectors form a basis provided they are not concurrent in a finite or infinite point.

In space we have the Euclidean metric

$$|\mathbf{U}|^2 = u_1^2 + u_2^2 + u_3^2;$$

the corresponding metric in the plane is given by

$$|\mathbf{u}, \mathbf{u}_0|^2 = |\mathbf{u}|^2 + |\mathbf{u}_0|^2.$$

Parallel space vectors $\mathbf{U}$, $\lambda\mathbf{U}$ correspond to line vectors

$$(\mathbf{u}, \mathbf{u}_0), \qquad (\lambda\mathbf{u}, \lambda\mathbf{u}_0)$$

which have the same axis. If such line vectors are called "parallel," the geometry in the xy-plane is elliptic non-euclidean; for *no* parallel can be drawn to a line from a point without.

Corresponding to the inner product

$$\mathbf{U}\cdot\mathbf{V} = u_1v_1 + u_2v_2 + u_3v_3$$

in space we have

$$(\mathbf{u}, \mathbf{u}_0) \cdot (\mathbf{v}, \mathbf{v}_0) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u}_0 \cdot \mathbf{v}_0$$

in the plane. Perpendicular space vectors $\mathbf{U}$, $\mathbf{V}$ for which $\mathbf{U} \cdot \mathbf{V} = 0$ thus correspond to "orthogonal" line vectors

$$\mathbf{u} \cdot \mathbf{v} + \mathbf{u}_0 \cdot \mathbf{v}_0 = 0. \tag{3}$$

Such vectors have the interesting property:

If two line vectors $(\mathbf{u}, \mathbf{u}_0)$ *and* $(\mathbf{v}, \mathbf{v}_0)$ *are "orthogonal," the line* $(\mathbf{v}, \mathbf{v}_0)$ *passes through the antipole of* $(\mathbf{u}, \mathbf{u}_0)$ *with respect to the unit circle* $x^2+y^2=1$; *and vice versa.*

Proof. The perpendicular $\mathbf{p} = \overrightarrow{OP}$ on the line $\mathbf{r} \times \mathbf{u} = \mathbf{u}_0$ satisfies

$$\mathbf{p} \times \mathbf{u} = \mathbf{u}_0, \qquad \mathbf{p} \cdot \mathbf{u} = 0;$$

hence

$$\mathbf{p} = \frac{\mathbf{u} \times \mathbf{u}_0}{\mathbf{u} \cdot \mathbf{u}} \quad \text{and} \quad \overrightarrow{OQ} = \mathbf{q} = \frac{\mathbf{u} \times \mathbf{u}_0}{\mathbf{u}_0 \cdot \mathbf{u}_0}$$

locates the pole Q of $(\mathbf{u}, \mathbf{u}_0)$ with respect to the unit circle; for $\mathbf{p}$ and $\mathbf{q}$ have the same direction and $\mathbf{p} \cdot \mathbf{q} = 1$. The antipole Q' of $(\mathbf{u}, \mathbf{u}_0)$, whose position vector is $-\mathbf{q}$, now lies on the line $\mathbf{r} \times \mathbf{v} = \mathbf{v}_0$; for since $\mathbf{v}_0 = \lambda \mathbf{u}_0$,

$$-\mathbf{q} \times \mathbf{v} = \frac{(\mathbf{u}_0 \times \mathbf{u}) \times \mathbf{v}}{\mathbf{u}_0 \cdot \mathbf{v}_0} = \frac{-\mathbf{u} \cdot \mathbf{v}}{\mathbf{u}_0 \cdot \mathbf{u}_0} \mathbf{u}_0 = \frac{\mathbf{u}_0 \cdot \mathbf{v}_0}{\mathbf{u}_0 \cdot \mathbf{u}_0} \mathbf{u}_0 = \mathbf{v}_0.$$

The reciprocity of poles and polars corresponds to the commutativity of the scalar product.

SOME MISSING THEOREMS ON THE ANTICOMPLEMENTARY TETRAHEDRON

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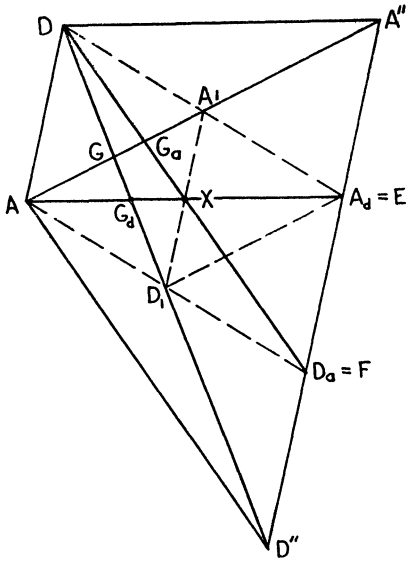
1. Let $(T'') = A''B''C''D''$ be the anticomplementary, or, briefly, the an-ry tetrahedron of a given tetrahedron $(T) = ABCD$. The two tetrahedrons are homothetic, the center of the homothecy being their common centroid G, and the ratio equal to $(-1:3)$ [1; p. 53].

Let G_d, G_a be the centroids of the triangles ABC, DBC. The lines DA'', AG_d are the traces of the plane DAG on the two parallel planes $A''B''C''$, ABC, hence the two lines are parallel. On the other hand, the two lines DA, $D''A''$ are parallel due to the homothecy, hence the trace E of the line $A''D''$ in the plane ABC, or more precisely, on the line AG_d is the fourth vertex of the parallelogram $A''DAE$. Thus $A''E = DA$.

A similar argument may show that if F denotes the trace of the line $D''A''$ on the median DG_a, we have $D''F = DA$. Moreover, since $D''A'' = 3DA$, due to

the homothecy, we also have $EF=DA$.

Likewise for the other edges of (T''). Hence: *An edge of the an-ry tetrahedron of a given tetrahedron (T) is trisected by the two faces of (T) which that edge meets.*



2. Let $A_dB_dC_d$, $B_aC_aD_a$, $C_bD_bA_b$, $D_cA_cB_c$ be the an-ry triangles of the triangles ABC, BCD, CDA, DAB, the vertex A_d corresponding to the vertex A of ABC, $\cdots$.

The quadrilateral $ADEF$ is a parallelogram, for its two opposite sides AD, EF are equal and parallel (art. 1). Hence the points E, F are the symmetrics of the vertices A, D with respect to the point of intersection X of the diagonals AG_dE, DG_aF. Now the lines AG_d, DG_a are the medians of the two triangles ABC, DBC relative to the common base BC of these triangles, hence the point X is the midpoint of the edge BC of (T). Consequently the points E, F coincide with the vertices A_d, D_a of the an-ry triangles of the triangles ABC, DBC, respectively, and correspond to the vertices A, D.

Similarly for the other edges of (T''). Thus: *The twelve vertices of the four an-ry triangles of the four face triangles of a tetrahedron (T) lie in pairs on the six edges of the an-ry tetrahedron (T'')* of (T).

3. The three points D'', $A_d=E$, A'', are collinear, and we have (arts. 1 and 2): $D''A_d:D''A''=2:3$.

Considering the edges $D''B''$, $D''C''$ of (T) we may show in a like manner that

$$D''B_d:D''B'' = D''C_d:D''C'' = 2:3.$$

Similarly for the other faces of (T). Thus: *The an-ry triangle of the face of a tetra-*

hedron (T) *is homothetic to the corresponding face of the an-ry tetrahedron* (T'') *of* (T), *the center of the homothecy being the fourth vertex of* (T''), *and the ratio* (2:3).

4. Let A_1, D_1 be the points of intersection of the pairs of diagonals DA_d, AGA''; AD_a, DGD'' of the two parallelograms $A''DAA_d$, $D''ADD_a$.

The points A_1, D_1 are thus the midpoints of two opposite sides of the parallelogram DAD_aA_d, hence DAD_1A_1 is a parallelogram; its diagonals AA_1, DD_1 meet in the centroid G of (T). Thus the points A_1, D_1 are the symmetrics of the vertices A, D of (T) with respect to the centroid G of (T), that is to say, A_1, D_1 are two vertices of the twin tetrahedron $(T_1)=A_1B_1C_1D_1$ of (T) [1; p. 58, art. 191].

The three points D, A_1, A_d, are collinear. We may show in a like manner that the triads of points D, B_1, B_d; D, C_1, C_d, are collinear.

Now the lines DA_1, DB_1, DC_1 are the edges, passing through D, of the Monge circumscribed parallelepiped of (T) [1; p. 58, art. 189], hence: *The three edges of the Monge parallelepiped of a tetrahedron* (T) *which pass through a given vertex of* (T) *meet the face of* (T) *opposite that vertex, in the vertices of the an-ry triangle of that face of* (T).

It is readily seen that the triangles $A_1B_1C_1$, $A_dB_dC_d$ are homothetic, the center of the homothecy being the vertex D of (T) and the ratio (1:2).

5. a. The common vertex D of the three triangles DBC, DCA, DAB has for its corresponding points in the an-ry triangles of those triangles, the points D_a, D_b, D_c. *The triangle* $D_aD_bD_c$ *is symmetric to the triangle* ABC; the center of symmetry is the vertex D_1, of the tetrahedron (T_1), corresponding to the vertex D of (T). Similarly for the other vertices of (T).

b. Let U, V, W be the midpoints of the edges DA, DB, DC, of (T). i. *The three lines* UA_d, VB_d, WC_d, *are concurrent*; ii. *They pass, respectively, through the centroids of the faces* DBC, DCA, DAB, *of* (T).

The proofs are left to the interested reader. He may also formulate more propositions connected with this figure.

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AN ALGORITHM FOR DISTINCT REPRESENTATIVES*

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1. Introduction. A theorem of Philip Hall's [3] states that sets $S_1, \cdots, S_n$ which are finite subsets of a set W will have distinct representatives if and only if the following condition holds. Condition C: *For each* $k=1, 2, \cdots, n$ *any* k *distinct sets contain between them at least* k *distinct elements.* This theorem has been generalized in various ways and has been treated by linear programming [1, 2, 4, 5]. In general it may be difficult to verify whether or not Condition C

* This research was supported by the National Science Foundation.

holds, since a direct verification would involve examining 2^n-1 combinations of subsets. In this note an algorithm is given which requires no preliminary verifications and terminates either with the desired set of distinct representatives or exhibits k sets which do not contain as many as k distinct elements between them.

2. The algorithm. Choose an arbitrary element a_1 from S_1, and as long as we can, choose an arbitrary a_i from S_i different from the previously chosen a's. This will yield distinct representatives if this can be continued to S_n. This process will fail if we reach a set S_r all of whose elements $b_1, \cdots, b_t$ have been used as representatives. Now we construct some auxiliary sets. T_1 shall consist of $b_1, b_2, \cdots, b_t$ which we regard as listed in order. T_2 consists of those elements in the set whose representative is b_1, excluding b_1 itself and any other elements of T_1. T_2 may of course be vacuous, but if it contains any elements, these are simply ordered following the elements of T_1. Continuing, if the ith element of the sequence b_i is the representative of some set S_j, we form a set T_{i+1} consisting of those elements of S_j not in the earlier T's. If b_i is not a representative of a set, then T_{i+1} is vacuous. The elements of T_{i+1}, if any, are then simply ordered following those of the earlier T's. In this way we construct a sequence $b_1, \cdots, b_s$ in which each b_v, $v>t$, is in a set S_j whose representative is a b_u with $u<v$. If some b_v is not a representative, then there is an S_j with representative b_u, $u<v$. Take b_v as a new representative for S_j freeing b_u. We may now use b_u as a representative for the set S_m from which it arose, freeing an earlier b_w. This process must continue until we have freed one of the first t elements, which we may now assign as a representative for S_r. In this process, $S_1, \cdots, S_{r-1}$ still have distinct representatives and we may go ahead in the same way to find a representative for S_{r+1}. This method will fail if in the entire sequence constructed, $b_1, \cdots, b_s$, every element is a representative of a set. By the construction every element of the s sets whose representatives these are will be in the sequence. But then these sets and S_r will form $s+1$ sets which between them contain only s elements. In this case we have exhibited sets which violate Condition C.

This algorithm is an easy and practical one to apply and could be programmed on a digital computer.

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CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

COMBINATORIAL PROOF OF FERMAT'S "LITTLE" THEOREM

S. W. GOLOMB, University of Oslo

It is possible to give an interesting, purely combinatorial proof of Fermat's theorem that n^p-n is divisible by p, for any positive integer n, and any prime number p.

Suppose we have beads in n different colors, and we wish to make necklaces using exactly p beads. First we put p beads on a string. Since each of the beads can be chosen in n ways, there are n^p possible strings. For each of the n colors, there is one string entirely of that color. We throw these away, leaving n^p-n strings. We will join the two ends of each of these strings to form necklaces. But we observe that if two strings differ only by a cyclic permutation of the beads, the resulting necklaces will be indistinguishable. Since there are p cyclic permutations of the p beads on a string, the number of *distinguishable* necklaces is $(n^p-n)/p$, which must therefore be an integer.

If it is permitted to flip the necklaces over, there are only $(n^p-n)/2p$ distinguishable cases, so that this too must be an integer, unless $p=2$.

If this proof is used in the classroom, it is of pedagogic value to ask the class:

1) Where is the hypothesis that p is prime used in the proof?

and

2) In view of the fact that there are $n!$ ways to permute the n colors, is it further true that $n!$ divides $(n^p-n)/p$?

A somewhat analogous combinatorial proof of Wilson's Theorem is given by R. D. Carmichael (*The Theory of Numbers*, p. 50), who shows that $((p-1)!-(p-1))/2p$ is the precise number of distinct irregular stellated p-gons.

CONGRUENCES AND CARD SHUFFLING

PAUL B. JOHNSON, Occidental College

Congruences are some of the most powerful tools in the theory of numbers, yet they are among the more difficult to motivate. To the beginning student the usual illustrations appear either absurdly simple, or merely tricky manipulations set up to illustrate this particular theory.

The majority of people in the United States will immediately recognize the following illustration as a serious and significant problem in its own right, and yet it is one easily solved using congruences but is very difficult without them. The illustration is the simple question "How thoroughly does the ordinary riffle shuffle mix up the cards in a bridge deck?" The average person thinks a pure riffle shuffle thoroughly mixes the cards, implying that all of the 52! or 8×10^{67}

possible orders will result on repeated shuffling. It is quite a stimulating shock to find how wrong this can be.

In a pure riffle shuffle the deck is cut into two equal 26-card packs. The new deck is obtained by alternately snapping one card from each pack. Under this shuffle, card 1 goes into position 2, card 2 goes into position 4, $\cdots$, card 26 goes into position 52, card 27 goes into position 1, $\cdots$, card 52 into position 51. In general card x goes into position $p(x) \equiv 2x \pmod{53}$.

Usually the shuffle is repeated several times. Clearly, after n shuffles, card x goes into position $p_n(x) \equiv 2^n x \pmod{53}$. More precisely, $p_n(x)$ is the least positive number satisfying this congruence.

It follows that for n such that $2^n \equiv 1 \pmod{53}$, n shuffles will return the deck to its original order. The smallest such n is the number of different orders into which the riffle shuffle will mix the cards. Since there are only 52 non-zero residues mod 53, the smallest n can not be greater than 52. From Fermat's theorem $2^{52} \equiv 1 \pmod{53}$. 52 is a small enough number that some students will solve the congruence by trial, obtaining smallest $n = 52$. 52 is also large enough for students to see the value of more powerful theorems. This result is so surprising that some students will start testing possible loopholes in the above argument.

It is superfluous to admit that hardly any actual riffle is a pure riffle. Almost always more than one card is snapped from one pack at some point in the riffle. Hence the above analysis is not perfectly applicable to the real-life situation. This calls for the reply of the confederate veteran when challenged by a comrade about his stories of sweeping victories over the yankees: "Another good story spoiled by a danged eyewitness."

ON DEFINING THE CONIC SECTIONS

MICHAEL PASCUAL, Siena College

A merging of the analytic and the geometric definitions of the conic sections could be done neatly when three dimensional analysis is being covered. By obtaining the equation of a right circular cone and considering the equation of the curve of intersection by a cutting plane, we arrive at the equations of the various conic sections. Now to get the different sections we may either vary the cutting plane or the vertex angle of the cone. We use the latter method, since it lends itself more readily to analysis. For convenience, we place the vertex of the cone at (0, 1, 1) and let the axis be the line $z = y, x = 0$. Letting α designate the angle formed by the axis and any ruling, we find the equation of the cone to be

$$2x^2 + (y - z)^2 = (y + z - 2)^2 \tan^2 \alpha.$$

By setting $z = 0$, we obtain the equation of the curve of intersection of the cone with the xy plane:

$$2x^2 + (1 - \tan^2 \alpha)y^2 + 4y \tan^2 \alpha - 4 \tan^2 \alpha = 0.$$

1. For $0 < \alpha < \pi/4$, an ellipse is the intersection; its equation is

$$\frac{x^2}{a^2}+\frac{(y+a^2)^2}{b^2}=1 \text{ where } a=\frac{\sqrt{2}\tan\alpha}{\sqrt{1-\tan^2\alpha}} \text{ and } b=\frac{2\tan\alpha}{1-\tan^2\alpha}.$$

2. For $\alpha=\pi/4$, a parabola is the intersection; it's equation is

$$x^2=-2(y-1).$$

3. For $\pi/4<\alpha<\pi/2$, a hyperbola is the intersection; it's equation is

$$\frac{x^2}{a^2}-\frac{(y+a^2)^2}{b^2}=1.$$

If we wish to continue further and obtain a focal point and directrix, we need only get the equation of a sphere tangent to the cone and the xy plane, and the equation of the plane determined by the circle of tangency. By showing that the line of intersection of this plane and the xy plane is such that the ratio of the distance from any point on the section to the point of tangency of the sphere and the xy plane to the distance from that same point to the line of intersection is a constant, we fulfill the requirements of the focal point and directrix.

A SIMPLE PROOF OF THE RATIO THEOREM

FRANZ HOHN, University of Illinois

In proving theorems about matrices, students often resort to scalar-type proofs when proofs employing directly the properties of matrix algebra would be more simple. In fact, the student's mastery of matrix algebra may be measured by the extent to which he avoids the scalar methods of proof in favor of those employing matrix processes. We illustrate our point by a simple demonstration of the "ratio theorem", the scalar proof of which is tedious. (Cf. [1], pp. 74–77.) The theorem itself is both striking and useful and deserves to be better known.

It is helpful to begin with two definitions. The major determinants, or simply the *majors* of a matrix are the determinants of its square submatrices of maximum order. In particular, "the" determinant of a square matrix is its only major. A major of one matrix is said to *correspond* to a major of another matrix of like (or transposed) order if the row and column indices of the first major are respectively the row and column (or column and row) indices of the second. Then the *ratio theorem* may be stated as follows:

THEOREM 1. *In an m by n matrix A whose rank r is less than m, the majors of any set of r rows are proportional to the corresponding majors of any other set of r rows. (A similar result holds for columns.)*

Let a set of r linearly independent columns of A be denoted by $L_1, L_2, \cdots, L_r$. Then every column of A is a linear combination of these r columns and we can write A as a matrix of columns, thus:

THE DETERMINANT OF A SUM OF MATRICES

L. M. WEINER, DePaul University

If $A=(a_{ij})$ and $B=(b_{ij})$ are n by n matrices such that $a_{ij}=b_{ij}$ for $i\neq s$ and $C=(c_{ij})$ is an n by n matrix such that $c_{ij}=a_{ij}$ for $i\neq s$ and $c_{sj}=a_{sj}+b_{sj}$ for $j=1, 2, \cdots, n$, then $|C|=|A|+|B|$. This property of determinants leads some students to the incorrect generalization that $|A+B|=|A|+|B|$ for any matrices, A and B.

The formula for $|A+B|$ is actually a bit more involved; since, going back to the definition of determinants, we find that $|A+B|=\sum(a_{1i_1}+b_{1i_1})\cdot(a_{2i_2}+b_{2i_2})\cdots(a_{ni_n}+b_{ni_n})$ where $i_1, i_2, \cdots, i_n$ range over all permutations of $1, 2, \cdots, n$. If, in evaluating this sum of products, we choose first all the a_{ij} and sum over the permutations of $1, 2, \cdots, n$, we obtain the value $|A|$. If we choose the b_{ij}, we obtain $|B|$. If we make some other definite choice of a_{ij} or b_{ij} from each parenthesis, and sum over the permutations of $1, 2, \cdots, n$, we obtain the determinant of a matrix whose kth row is the kth row of A or B depending upon whether we chose a_{ij} or b_{ij} from the kth parenthesis.

There are 2^n possible ways of making the above choices; therefore, we see that we may evaluate $|A+B|$ by constructing the 2^n matrices each of which is obtained by choosing for its kth row the kth row of A or B and then taking the sum of the determinants of these matrices.

This result may be extended to $|A_1+A_2+\cdots+A_t|$. We find in a similar manner that $|A_1+A_2+\cdots+A_t|$ is the sum of the determinants of t^n matrices each of which is obtained by choosing for its kth row the kth row of one of the A_i.

The student should note that if $a_1, a_2, \cdots, a_t$ are elements of some non-commutative ring, there is a strong analogy between the expressions for $|A_1+A_2+\cdots+A_t|$ and $(a_1+a_2+\cdots+a_t)^n$ in that $(a_1+a_2+\cdots+a_t)^n$ is the sum of t^n products each of which is obtained by choosing for its kth factor one of the a_i.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1241. *Proposed by C. W. Trigg, Los Angeles City College*

The holiday greeting, *MERRY XMAS TO ALL*, is a cryptarithm in which each of the letters is the unique representation of a digit, and each word is a

square integer. Find all solutions.

E 1242. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the circle orthogonal to the circles inscribed in the squares of centers A', B', C' constructed exteriorly (or interiorly) on the sides of a triangle ABC is concentric with the nine point circle of triangle $A'B'C'$.

E 1243. *Proposed by M. A. Rashid and M. A. Uppal, Lahore, West Pakistan*

Prove that the product of a number consisting of two digits and its reverse is never a square except when the two integers are equal.

E 1244. *Proposed by Leo Moser, University of Alberta*

Show that

$$\sum_{n=0}^{\infty} [5/(7n+1)(7n+6) + 3/(7n+2)(7n+5) - 1/(7n+3)(7n+4)] = \pi/\sqrt{7}.$$

E 1245. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If

$$b_{n+1} = \int_0^1 \min(x, a_n)dx, \qquad a_{n+1} = \int_0^1 \max(x, b_n)dx,$$

prove that the sequences $\{a_n\}$ and $\{b_n\}$ both converge and find their limits.

SOLUTIONS

Digits and Zero Digits

E 1211 [1956, 253]. *Proposed by Alan Wayne, Cooper Union School of Engineering*

Prove the following theorem, discovered by Cipriano Ferraris of the *New York Riddlers Club*: "For any positive integer k, the total number of digits in the sequence $1, 2, \cdots, 10^k$ equals the number of zero digits in the sequence $1, 2, \cdots, 10^{k+1}$."

I. *Solution by J. B. Johnston, Cornell University.* Let $1 \leqq N = a_t \cdots a_0 \leqq b^k$, N being an integer written in base b, $b \geqq 2$. For each m such that $0 \leqq m \leqq t$, let N_m be the integer whose base b representation is that of N except for the insertion of a zero immediately to the right of a_m. The association of a_m with this inserted zero forms a 1-1 mapping of the digits in the sequence $1, 2, \cdots, b^k$ onto the zero digits of the sequence $1, 2, \cdots, b^{k+1}$.

II. *Solution by J. A. Schumaker, New Jersey State Teachers College.* Let D be the total number of digits in the sequence $1, 2, \cdots, 10^k$ and Z the number of zero digits in the sequence $1, 2, \cdots, 10^{k+1}$. Obviously, each of the $9 \cdot 10^{n-1}$ numbers N, which satisfy $10^{n-1} \leqq N < 10^n$ for $n > 0$, has n digits. Hence

$$D = k + 1 + \sum_{n=1}^{k} 9n10^{n-1}.$$

It is also obvious that each of the $n-1$ places on the right is zero in exactly one-tenth of the $9 \cdot 10^{n-1}$ numbers N for $n>1$; hence there are exactly $9(n-1)10^{n-2}$ zeros in those numbers. Therefore

$$Z = k + 1 + \sum_{n=2}^{k+1} 9(n-1)10^{n-2} = k + 1 + \sum_{n=1}^{k} 9n10^{n-1} = D.$$

Also solved by D. H. Browne, J. E. Darraugh, Monte Dernham, J. M. Elkin, L. D. Goldberg, Michael Goldberg, A. J. Goldman, A. R. Hyde, Wallace Manheimer, D. C. B. Marsh, J. B. Muskat, M. A. Rashid and M. A. Uppal (jointly), Azriel Rosenfeld, J. W. Ross, David Rothman, H. D. Ruderman, Harry Schor, R. E. Shafer, A. V. Sylwester, J. A. Tierney, and the proposer.

Editorial Note. Solution II is easily generalized to hold for arbitrary base b. The problem may be solved also by mathematical induction on k.

A Trigonometric Inequality

E 1212 [1956, 253]. *Proposed by H. A. Osborn, University of California, Berkeley*

Show that $t>0$ implies $(2+\cos t)t>3 \sin t$.

I. *Solution by J. R. Hatcher, Fisk University.* Set

$$f(t) = t - 3(\sin t)/(2 + \cos t).$$

Then $f(0)=0$ and

$$f'(t) = [(1 - \cos t)/(2 + \cos t)]^2 \geqq 0.$$

It follows that $t>0$ implies $f(t)>0$.

II. *Solution by E. W. Marchand, Eastman Kodak Company.* For $t \geqq \pi$ the result is trivial. Let

$$f(t) = (2 + \cos t)t - 3 \sin t.$$

Then

$$f'(t) = 2 (\sin t)(\tan t/2 - t/2),$$

giving $f(0)=0$ and $f'(t)>0$ for $0<t<\pi$. Hence $f(t)>0$ in this interval, and the desired result follows.

III. *Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J.* Let $f(t)=(2+\cos t)t-3 \sin t$. Since $f(0)=f'(0)=f''(0)=0$ and $f'''(t)=t \sin t$, Maclaurin's formula gives $f(t)=(t^3\xi \sin \xi)/3!$, $0<\xi<t$, which is clearly positive for $0<t<\pi$. The inequality is trivial for $t \geqq \pi$.

IV. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* This problem and a generalization both appear in Durell and Robson, *Advanced Trigonometry*, p. 83 and p. 100. We will prove the generalization: If $a \geqq 2b>0$ and

$\pi \geqq t > 0$, then

$$(a + b \cos t)t > (a + b) \sin t.$$

If the inequality is true for the range $\pi \geqq t > 0$, it follows immediately that it will be true for all $t > 0$. Since the two sides of the inequality are equal for $t = 0$, the inequality will follow if we can show that the derivative of the left hand side is greater than the derivative of the right hand side, that is, if

$$a + b \cos t - bt \sin t > (a + b) \cos t,$$

or

$$\tan t/2 > bt/a,$$

which is true if $a \geqq 2b$.

Setting $a = 2$, $b = 1$, we get $(2 + \cos t)t > 3 \sin t$.

The similar inequality, $(2 + \cosh t)t > 3 \sinh t$, $t > 0$, is given on p. 115.

Also solved by H. W. Bailey, H. F. Bennett, A. P. Boblétt, R. F. Church, A. E. Danese, J. E. Darraugh, Hüseyin Demir, L. J. Dixon, Hazel Evans, L. D. Goldberg, A. J. Goldman, Oliver Gross, L. L. Hildilnand, P. G. Hodge, Jr., Vern Hoggatt, J. R. Holdsworth, F. A. Homann, A. R. Hyde, Martin Isaacs, Jerome Kaplan, Lee Lorch, P. J. McCarthy, Neill McShane, D. C. B. Marsh, Leo Moser, J. B. Muskat, C. S. Ogilvy, Dan Passman, J. P. Phillips, C. A. Pursel, Roger Purves, Nathaniel Queen, M. A. Rashid and M. A. Uppal (jointly), L. A. Ringenberg, Azriel Rosenfeld, David Rothman, Mohammad Salhab, E. C. Schlesinger, Harry Schor, R. E. Shafer, P. C. Shields, J. A. Tierney, Alan Wayne, David Zeitlin, and the proposer.

A Wronskian

E 1213 [1956, 253]. *Proposed by F. D. Parker, Clarkson College of Technology*

Find the Wronskian, $W(n, k)$, of the set of functions x^n, $x^n \ln x$, $x^n \ln^2 x$, $\cdots$, $x^n \ln^k x$.

Solution by J. H. Hodges and F. R. Olson, University of Buffalo. Using Leibniz's formula for the rth derivative of a product, we find

$$D^r x^n \ln^s x = \sum_{j=0}^{k} \binom{r}{r-j} D^{r-j} x^n D^j \ln^s x,$$

where

$$\binom{r}{r-j} = 0 \text{ if } j > r.$$

It now follows, from the familiar procedure for multiplying two determinants, that

$$W(n, k) = |a_{rj}| \, |b_{js}|,$$

where

$$|a_{rj}| = \left|\binom{r}{r-j} D^{r-j}x^n\right|, \qquad |b_{js}| = |D^j \ln^s x|.$$

Since $a_{rj}=0$ for $j>r$ and $a_{rr}=x^n$ for $r=0, 1, \cdots, k$, we then have

$$W(n, k) = x^{n(k+1)}|D^j \ln^s x| = x^{n(k+1)}W(0, k). \tag{1}$$

Examining $W(0, k)$ it is readily found that

$$W(0, k) = k!W(-1, k-1). \tag{2}$$

But, from (1), $W(-1, k-1)=x^{-k}W(0, k-1)$, so that (2) implies

$$W(0, k) = k!x^{-k}W(0, k-1). \tag{3}$$

Since $W(0, 0)=1$, repeated use of the recurrence formula (3), together with (1), gives

$$W(n, k) = x^{(k+1)(2n-k)/2}\prod_{t=1}^{k} t!.$$

Also solved by H. W. Bailey, A. E. Danese, J. E. Darraugh, A. J. Goldman, A. R. Hyde, M. S. Klamkin, Azriel Rosenfeld, David Zeitlin, and the proposer.

Zeitlin showed more generally that if u and v are differentiable functions of x, then the Wronskian of the set of functions $u, uv, uv^2, \cdots, uv^k$ is

$$u^{k+1}(Dv)^{k(k+1)/2}\prod_{t=1}^{k} t!.$$

Setting $u=x^n$ and $v=\ln x$ we obtain the above value for $W(n, k)$.

Envelope of a Family of Ellipses

E 1214 [1956, 253]. *Proposed by Paul Payette, École Polytechnique de Montreal*

Find the envelope of the family of ellipses of constant major axis having one focus at a given point and the other focus on a given straight line.

I. *Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J.* Let m, $(a, 0)$, and the y-axis be the constant major axis, the fixed focus, and the given straight line, respectively. An ellipse whose other focus is at $(0, k)$ has the equation

$$[x^2 + (y-k)^2]^{1/2} + [(x-a)^2 + y^2]^{1/2} = m.$$

Differentiating with respect to k according to the usual procedure gives $y=k$ and thus

$$|x| + [(x-a)^2 + y^2]^{1/2} = m$$

is the envelope. This is the loop formed by the two parabolas

$$y^2 = (m \mp a)(m \pm a \mp 2x),$$

where the upper and lower signs correspond to $x \geqq 0$ and $x \leqq 0$, respectively.

II. *Solution by Robin Robinson, Dartmouth College.* Consider two ellipses C_1 and C_2 of the family, with common focus O and equal major axes m. Let Q_1 and Q_2 be their other foci.

An intersection R of C_1 and C_2 must lie on the perpendicular bisector of Q_1Q_2, since $OR+RQ_1=OR+RQ_2=m$ and hence $RQ_1=RQ_2$. There are just two such intersections R and S, for this line may cut C_1 only twice.

As $Q_2 \to Q_1$ along line Q_1Q_2, the limiting position of RS will be the perpendicular to line Q_1Q_2 at Q_1, which meets C_1 at $\overline{R}$ and $\overline{S}$. These points are the points of contact of C_1 with the envelope of the family of ellipses.

But $O\overline{R}+\overline{R}Q_1=m$, or $O\overline{R}=m-\overline{R}Q_1$, which is the perpendicular distance from $\overline{R}$ to the line parallel to Q_1Q_2 and m units away from it on the other side of $\overline{R}$ from Q_1. So $\overline{R}$ lies on a parabola with focus at O and this line as directrix. Similarly, $\overline{S}$ lies on another such parabola, which opens, however, in the opposite direction.

The portions of these two parabolas bounded by their common chord Q_1Q_2 constitute the desired envelope.

Also solved by H. W. Bailey, Joseph Basile, Hüseyin Demir, Hazel Evans, J. Gallego-Diaz, Michael Goldberg, A. J. Goldman, Cornelius Groenewoud, A. R. Hyde, E. W. Marchand, Beckham Martin, C. S. Ogilvy, O. J. Ramler, M. A. Rashid and M. A. Uppal (jointly), Azriel Rosenfeld, E. M. Scheuer, Chih-yi Wang and David Zeitlin (jointly), and the proposer.

Some of these solutions were not complete.

Criterion for an Incircle

E 1215 [1956, 253]. *Proposed by J. P. Ballantine, University of Washington*

State a necessary and sufficient condition for an ordered set of n line segments to be the consecutive sides of an n-gon possessing an inscribed circle.

Solution by Michael Goldberg, Washington, D. C. Fold the given lengths in accordion fashion. Then, for odd n, a cut midway between the ends will cross the sides at the points of tangency with the circle since the segments adjacent to a fold are equal and the end segments are equal. For the segments so obtained, there is only one radius that will satisfy. Therefore, the necessary and sufficient condition is that all the sides be crossed by the cut.

For even n, the ends must coincide. Any cut which crosses all the sides will give a solution. Hence, the necessary and sufficient condition is that the ends must coincide (that is, the sum of the lengths of one set of alternate sides must equal the sum of the lengths of the other sides) and the cut must cross all the sides. There may be no solution, one solution, or a continuous infinity of solutions.

Also solved by D. C. B. Marsh, M. A. Rashid and M. A. Uppal (jointly), and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4713. *Proposed by Oystein Ore, Yale University*

In the number system with the base $n=12$, every square number ends in one of the digits 0, 1, 4, 9, that is, in a square digit. Find all n for which this is the case.

4714. *Proposed by Chandler Davis, Bronxville, N. Y.*

For a real polynomial $P_n(x)$ of degree n, denote the zeros of $P_n'(x)$, multiplicity counted, by ξ_i, $i=1, 2, \cdots, n-1$. Assume all ξ_i real, and let $\xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_{n-1}$. Now, to what extent are the numbers $P_n(\xi_i)$ arbitrary? More precisely, give necessary and sufficient conditions on an $(n-1)$-tuple of real numbers $\eta_1, \cdots, \eta_{n-1}$ in order that there exist P_n such that $P_n(\xi_i)=\eta_i$, $i=1, 2, \cdots, n-1$.

4715. *Proposed by R. S. Underwood, Texas Technological College*

Find real solutions of the equations

$$x^2+y^2+z^2+u+v=D, \qquad x^2+\frac{y^2}{2}+\frac{z^2}{3}+\frac{u^2}{4}+\frac{v^2}{5}=1,$$

in case (a) $D=-3$, and (b) $D=15/4$. What can be said about the solutions for other values of D?

4716. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Determine the equation of motion if $\overline{V}_s=\lambda\overline{V}_t$, where $\overline{V}_s$ and $\overline{V}_t$ are the averages of velocity with respect to distance and time, respectively, in any time interval starting at $t=0$. What is the minimum eigenvalue λ?

4717. *Proposed by Patrick Gallagher, Harvard University*

Let $f(z)$ and $g(z)$ be entire functions, each without fixed points, and not both linear. Show that $f(g(z))$ has infinitely many fixed points.

SOLUTIONS

Uniform and Absolute Convergence

4653 [1955, 585]. *Proposed by Albert Wilansky, Lehigh University*

Consider the three statements:

(a) $\sum U_n(x)$ converges uniformly, (b) $\sum |U_n(x)|$ converges,
(c) $\sum |U_n(x)|$ converges uniformly.

Consideration of $U_n(x) = (-x)^n/n$ shows that if we require (a) and (b) to hold on an open interval (in this case $0 < x < 1$) we cannot deduce (c).

Does (c) follow if (a) and (b) hold on a closed interval?

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minn. The following example shows that (c) does not necessarily follow. Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^2}{(1+x^2)^n}$$

on the interval $0 \leqq x \leqq 1$. Here

$$|R_n(x)| = \left| \sum_{r=n}^{\infty} \frac{(-1)^n x^2}{(1+x^2)^r} \right| = \frac{x^2}{(1+x^2)^{n-1}(2+x^2)}, \qquad x \neq 0,$$

and $R_n(0) = 0$. It is easily seen that $|R_n(x)| < \epsilon$ holds for any arbitrary positive number $\epsilon < 1$ and for all values of x if we choose, say, $n > 1/\epsilon \log (1+\epsilon^2)$. Thus (a) is established.

It is readily seen that the series $\sum x^2/(1+x^2)^n$ converges for all values of x, and (b) is satisfied.

However the series (b) does not converge uniformly. In fact, $R_n(x) = (1+x^2)^{1-n}$ and it is obvious that no value of n can be found sufficiently large to ensure $|R_n(x)| < \epsilon$ everywhere on $0 \leqq x \leqq 1$.

Also solved by Joshua Barlaz, P. T. Bateman, R. O. Davies, R. J. Dickson, A. J. Goldman, Fritz Herzog, G. B. Lang, D. R. Lewis, Norman Miller, L. A. Ringenberg, and Harold Shulman.

Editorial Note. If continuity of $U_n(x)$ is not desired, as several solvers point out, the Proposer's example suffices upon defining $U_n(1) = 0$. The problem has had considerable discussion in the literature. Herzog supplies the following references: Graves, *The Theory of Functions of Real Variables*, pp. 111–112; Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier Series*, vol. II, pp. 116, 119; Pierpont, *The Theory of Functions of Real Variables*, vol. II, p. 165; Shulman, On uniform and absolute convergence, this MONTHLY, 1956, p. 35.

Subsets with Odd Number of Elements

4654 [1955, 586]. *Proposed by Harley Flanders, University of California at Berkeley*

Let $X = \{1, 2, 3, \cdots\}$ be the set of whole numbers, S the class of all subsets of X. Prove that there is a subclass T of S such that (a) if $Y_1, \cdots, Y_n$ are arbitrarily chosen distinct members of T, then there is at least one integer x which belongs to exactly an odd number of the Y_j, and (b) if Y is a non-empty member of S, then there are distinct sets $Y_1, \cdots, Y_n$ in T such that Y is the

set of all integers x which belong to exactly an odd number of the Y_j.

Solution by R. O. Davies, University College, Leicester, England. Let X be any non-empty set with members x. For non-empty subsets $Y_1, \cdots, Y_n$ of S, let $O(Y_1, \cdots, Y_n)$ denote the set of all x which belong to exactly an odd number of the Y_j. Assuming the axiom of choice, define by transfinite induction as large as possible a well-ordered series of non-empty subsets of S

$$Z_1, Z_2, \cdots, Z_\alpha, \cdots, \tag{1}$$

satisfying the condition that Z_α is not equal to $O(Z_{\beta(1)}, \cdots, Z_{\beta(n)})$ for any finite set of ordinals $\beta(1), \cdots, \beta(n)$ each less than α.

The class T of all the Z_α clearly has property (b). Let $Y_1, \cdots, Y_n$ be distinct sets in T, Y_n being the one with greatest suffix when considered as a term of (1). Then $Y_n \neq O(Y_1, \cdots, Y_{n-1})$, and

$$O(Y_1, \cdots, Y_n) \supset [O(Y_1, \cdots, Y_{n-1}) - Y_n] + [Y_n - O(Y_1, \cdots, Y_{n-1})] \neq 0.$$

Hence T also has property (a).

Also solved by R. J. Dickson, A. J. Goldman, S. H. Gould, and the Proposer.

Difference Equation

4655 [1955, 586]. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Lawrence, Mass.*

Solve the difference equation

$$F(n+1) = F(n) - 2(n+1)\ \mathrm{sg}\ \{F(n) + 3n - n^2\}, \qquad F(0) = 0,$$

where $\mathrm{sg}\ (x) = 1, 0, -1$ according as $x > 0$, $x = 0$, $x < 0$.

Solution by Leonard Carlitz, Duke University. Put $G(n) = F(n) + 3n - n^2$. Then the given equation becomes

$$G(n+1) = G(n) - 2(n+1)\ \mathrm{sg}\ G(n) - 2n + 2, \ G(0) = 0. \tag{1}$$

We shall prove by induction that

$$G(2^m + r) = 6 + 4r - 2^{m+2} \qquad (0 \leqq r < 2^m). \tag{2}$$

Note that (2) implies $G(n) < 0$ unless $n = 2^m - 1$.

Clearly (2) holds for $m = 0$. Also, (1) implies

$$G(n+1) = G(n) + 4 \qquad (G(n) < 0), \tag{3}$$

$$G(n+1) = G(n) - 4n \qquad (G(n) > 0). \tag{4}$$

Assuming the truth of (2) for fixed m and r with $r < 2^m - 1$, we get the corresponding result for m and $r+1$ by using (3); for $r = 2^m - 1$ we use (4) and get the corresponding result with $m+1$ and $r = 0$. This completes the proof of (2). Finally we have, for $n > 0$,

$$F(n) = n^2 + n + 6 - 2^{m+3},$$

where $m=[\log_2 n]$. (For $n\leqq 0$, always $F(n)=n^2+n$.)

Also solved by C. M. Ablow, R. J. Dickson, Calvin Foreman, Samuel Goldberg, A. J. Goldman, Emil Grosswald, A. R. Hyde, P. G. Kirmser, C. W. Langley, W. S. Lawton, D. C. B. Marsh, W. O. Pennell, and Chih-yi Wang.

Roots of Unity

4656 [1955, 586]. *Proposed by H. S. Shapiro, New York University*

(1) If α is a root of unity and its real part is an algebraic integer, then $\alpha^4=1$.

(2) Show that there exists an algebraic integer of absolute value 1 which is not a root of unity.

Solution by P. T. Bateman, University of Illinois. (1) Suppose α is a root of unity which is not a fourth root of unity. Write $\beta=R(\alpha)=\frac{1}{2}(\alpha+\alpha^{-1})$. If the conjugates of α are $\alpha_1=\alpha, \alpha_2, \cdots, \alpha_k$, the conjugates of β are the distinct elements among

$$\tfrac{1}{2}(\alpha_1+\alpha_1^{-1})=R(\alpha_1), \qquad \tfrac{1}{2}(\alpha_2+\alpha_2^{-1})=R(\alpha_2), \cdots,$$
$$\tfrac{1}{2}(\alpha_k+\alpha_k^{-1})=R(\alpha_k).$$

Since $\alpha_1, \alpha_2, \cdots, \alpha_k$ are roots of unity which are not fourth roots of unity, we have

$$0<|R(\alpha_j)|<1 \qquad (j=1, 2, \cdots, k).$$

Thus the product of the conjugates of β has absolute value strictly between 0 and 1 and hence is not a rational integer. Hence β is not an algebraic integer.

(2) The number $\sqrt{2}-1+i\sqrt{2\sqrt{2}-2}$, where all the square roots are understood to be positive, has the desired property, for its absolute value is 1, it satisfies the equation $(x+1)^4=8x^2$, and it does not satisfy (1).

Also solved by Leonard Carlitz, R. J. Dickson, Leopold Flatto, Fritz Herzog, J. B. Kelly, Ivan Niven, Azriel Rosenfeld, Chih-yi Wang, Hans Zassenhaus, and the Proposer.

Uniformly Distributed Measure

4657 [1955, 586]. *Proposed by Peter Swerling, the Rand Corporation, Santa Monica, California*

Consider sets of real numbers which are subsets of the closed unit interval, which is denoted by I.

(1) Show that there exists no (Lebesgue) measurable set $S\subset I$ with the properties (i) $m(S)=M>0$ and (ii) there exists a number Δ, $0<\Delta<1$, such that for any interval $J\subset I$, $m(S\cap J)\leqq\Delta m(J)$. Here m denotes Lebesgue measure.

(1′) (Corollary) Given any measurable set $S\subset I$ such that $m(S)>0$ and given any number Δ, $0<\Delta<1$, there exists an interval $J\subset I$ such that $m(S\cap J)>\Delta m(J)$.

(1″) (Corollary) There is no measurable set $S\subset I$ with $0<m(S)=M<1$ such

that S is "uniformly distributed" in I, in the sense that for every interval $J \subset I$, $m(S \cap J) = Mm(J)$.

(2) Show that statement (1′) cannot be improved in the following sense: one cannot say that for every measurable set $S \subset I$ with $m(S) > 0$, there must exist a non-degenerate interval $J \subset I$ such that $m(S \cap J) = m(J)$.

(3) Given $S \subset I$; $m(S) > 0$; and given Δ, $0 < \Delta < 1$, prove that S can be represented as the union of two disjoint sets: $S = S_1 \cup S_2$ where (i) $m(S_1) = 0$ and (ii) S_2 can be covered by a denumerable collection of disjoint non-degenerate intervals $\{I_n\} : S_2 \subset \cup_n I_n$ such that for each I_n, $m(S \cap I_n) > \Delta m(I_n)$.

Solution by Azriel Rosenfeld, Columbia University. (1) is an easy consequence of the fact that S has density 1 at almost every point of itself. (See, *e.g.*, Titchmarsh, *The Theory of Functions*, 2nd edition, p. 371, ex. 2, 3). (1′) and (1″) are immediate consequences of (1).

(2) An example is easy to construct; see, *e.g.*, Wilansky, On a nowhere-dense set, This MONTHLY, 1953, p. 411.

(3) We may use a Vitali-type covering theorem (where at each point of S at which S has density 1 we work with the nested set of intervals for which already $m(S \cap I_n) > \Delta m(I_n)$). S_1 is then the set of measure zero of points of S at which S has density < 1. We then form the union of subsets of S of measure zero which are not covered by the "Vitali" construction. Finally we let S_2 be the remainder of S.

Also solved by the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All new material for this section should be sent to the incoming editor, Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

Advanced Calculus. By A. E. Taylor, New York, Ginn and Company, 1955. xiii+786 pages. $5.00.

This book, as stated in its preface, is "aimed at the student reader", and this aim is accomplished by various means: the author starts each chapter with a clear statement of what he intends to cover in that chapter, and gives a motivation for it. He also supplies a great number of examples which illustrate the text lucidly and which should enable a good student of the subject to benefit greatly from studying the material by himself. For the most part, these examples are very well selected, though the reviewer found in a few cases (for instance, Example 2 in par. 6.3) that the difficulty of the example is too great for what it

accomplishes. The author also ought to be commended for avoiding the fanciful and highly complex notation which is found in some texts. In particular, his proofs are written in a clear and simple language, which seems just appropriate for a beginner in rigorous mathematical thinking.

In Chapter I the main topics of elementary calculus are reviewed, but in addition to this, the chapter also treats some of the fundamental notions, such as limits and continuity, in a more rigorous manner than is usually done in elementary calculus. Thus a basis is laid for much of the work later on.

The discussion of the real number system is based on the "axiom of continuity" (cuts of real numbers), referred to by Landau in his *Grundlagen der Analysis*, page 89, as Dedekind's Principal Theorem. This reviewer believes that, for a text in advanced calculus, the choice of this theorem as an axiomatic basis of the real number system is a wise choice and preferable to the attempt of some authors to begin with Dedekind cuts of rational numbers or even earlier.

It seems unnecessary here to list in detail the contents of every chapter of the book. Among the "border line" topics that are covered, one might mention vectors and vector fields, point-set theory, complex functions, and Fourier series and integrals. Special functions treated include the Gamma and Beta functions, Bessel's functions and Laguerre's polynomials.

In his treatment of functions of two variables, the author introduces the notion of differentiability of such functions (see par. 6.4), which is later shown to be equivalent to the existence of the tangent plane and requires, of course, more than the existence of the partial derivatives. This is one of a number of instances where the book tends to clarify certain misconceptions on the part of the student.

The text is very clearly written and contains, besides the above-mentioned illustrative examples, a great number of well-selected exercises. The answers to some of these are found at the end of the book.

The organization of the material is, in the reviewer's opinion, subject to one adverse criticism: the author seems to have the tendency to treat a topic in a "piecemeal" fashion, introducing it at a certain place and continuing it in one or more places in later chapters. For instance, continuous functions of one real variable are treated in Chapters I, III and XV. This is one of several examples of this sort, and it presents a certain difficulty for the instructor who, on account of the large size of the book, is forced to make a selection of topics from the book to fit a year's course in advanced calculus.

The printing is of excellent quality. One typographical error was noted in Exercise 2 of paragraph 6.1, where the second equation should read $uv = x^2 + y^2$.

FRITZ HERZOG
Michigan State University

An Introduction to Linear Algebra, L. Mirsky, Oxford (Clarendon Press), 1955. 35 s. net.

According to the preface this book is "intended mainly for students pursuing an honours course (British) in mathematics"; the reader is "not assumed to have any previous knowledge of linear algebra"; the treatment is "thoroughly concrete"; the author "has taken every opportunity of bringing to the reader's notice" the points of contact between linear algebra and geometry. Part I contains six chapters: I. Determinants, II. Vector Spaces and Linear Manifolds, III. The Algebra of Matrices, IV. Linear Operators, V. Systems of Linear Equations and Rank of Matrices, VI. Elementary Operations and the Concept of Equivalence. Part II presents: VII. The Characteristic Equation, VIII. Orthogonal and Unitary Matrices, IX. Groups, X. Canonical Forms, XI. Matrix Analysis. Part III: XII. Bilinear, Quadratic, and Hermitian Forms, XIII. Definite and Indefinite Forms. The broad scope thus indicated is filled in with meticulous attention to theoretical detail and to illustration. There is a bibliography (wherein the city, but not the house, of publication is listed) and a good index.

The straight-forward clarity of the writing is admirable. Definitions, theorems, *etc.*, are generally stated with precision (the Corollary on p. 153 is perhaps somewhat loose in wording). The format, the composition of each page, the typography, are most attractive. A reasonably close reading unveiled so few printing errors that "It is now an easy maater . . ." popped up from page 83 as an Afrikaans howler. Use of terms like *adjugate* (≡adjoint) matrix, *partitioned* (≡block) matrix, *united* (≡fixed) points, *disposable* (≡arbitrary) unknowns, is of no real concern—but the reviewer enters a small plea that soon the English-writing mathematicians agree on standard mathematical-English terminology. Bridge passages like the one (par. 3.1) culling from elementary algebra material for extension to matrix algebra, and the examples of groups (par. 9.1.2) are enticing pedagogy; and the numerous developing exercises in the body itself of the textual material work the reader into the role of expositor. The problem lists at the end of each chapter are well, if formidably, selected. Among the many extremely well-presented portions of the book, the chapter on Matrix Analysis deserves special mention; this is tricky business, and the final application to linear differential equations is a welcome illustration.

Thus far, *magna cum laude*. The reviewer must voice, however, some objections to matters of arrangement and motivation; it is understood that these will reflect his own opinions and may, in turn, be refuted in the opposing camp.

The author of a book like this one is from the start faced with deciding upon the role of determinants in developing his theory. Mirsky begins with a classical discussion of determinant theory, "the first topic in linear algebra to be studied intensively"; on this basis he treats, for example, inverse matrix, matrix-rank, and solution of systems of linear equations. In connection with this last he remarks (p. 142) on the cumbersome nature of the solution if the system is exten-

sive, but only much later (p. 180) does he obtain solutions more easily through equivalence with an echelon-form matrix. The reviewer prefers the reversed order of this procedure (as presented in Birkhoff-MacLane: *A Survey of Modern Algebra* (Macmillan) and Stoll: *Linear Algebra and Matrix Theory* (McGraw-Hill), among others). It seems simpler and more direct to motivate study of matrices and vector spaces with the problem of solving linear systems by means of the echelon reduction, which happily stems from the elimination technique already known to the student; to use the same equipment in obtaining the inverse of a square matrix and to study rank; and to introduce the determinant of the (square) matrix A as a function-value $f(A)$ satisfying certain relevant requirements (cf. Mirsky, p. 190), in order finally to relate the solutions of a system to the coefficients in the equations. Mirsky's treatment is, of course, consistent; he is merely on one side of the fence, the reviewer on the other.

A second objection may be more serious. Postponement of the basic notion of *equivalence relation* to a position fairly late in the discussion (p. 186) leads to the following inconsistency. "Equality" of vectors (p. 40) and of matrices (p. 75) as element-to-element identity are said to be obvious implications of the definitions of vector and matrix. Yet "equality" of two ordered pairs (a, b), (c, d), of integers as used in defining rational numbers is stipulated by $ad=bc$, *not* by the (obvious?) element-to-element identity $a=c$, $b=d$ (and Mirsky cites this familiar and important example (p. 188) in illustration of equivalence relation). The notion of equivalence relation is so fundamental that it should be in evidence from the start of *any* algebra. Equality of vectors and matrices is definition, not implication, and needs to be proved a proper equivalence relation.

Finally, the reviewer thinks that the author has not lived up to his prefatory promise to link linear algebra and geometry at every opportunity. Much matrix theory—inversion, similarity, congruence, characteristic equation, even the very definition of matrix multiplication—finds motivation in linear transformations. These mappings originate in (or, to be modern, define) projective geometry. Some presentation of this geometry, even of the beautiful but almost lost theorems of its golden age, could readily be abstracted into the study of linear algebra and matrix theory, the reward of abstraction being an impressive harvest of unexpected dividends in other fields as well. It is disconcerting, then, to come upon only infrequent and all too brief (in the reviewer's opinion) gestures in the direction of projective geometry, certainly not enough to serve the beginning student as a comforting prop of "concreteness" to his algebra. Indeed, the treatment is "thoroughly concrete" only in the sense that it is thoroughly illustrated—algebraically.

But let a reviewer's few disappointments not detract from the over-all excellence of the book. It is a successful and worth while and usable introduction to linear algebra, written with spirited precision.

G. M. MERRIMAN
University of Cincinnati

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

THIRD NATIONAL SYMPOSIUM ON RELIABILITY AND QUALITY CONTROL IN ELECTRONICS

The third national Symposium on Reliability and Quality Control in Electronics, sponsored by the Institute of Radio Engineers, the American Society for Quality Control, the Radio-Electronic-Television Manufacturers Association Engineering Department, and the American Institute of Electrical Engineers, will be held during January 14–16, 1957, at the Hotel Statler, Washington, D. C. There will be two concurrent sessions, each of eight periods. The program includes papers, panels, movies, plant tours, and a banquet. For further information write to: Richard M. Jacobs, General Publicity Chairman, Radio Corporation of America, Camden 2, New Jersey.

EMPLOYMENT LISTING FOR RETIRED MATHEMATICIANS

Mathematicians, either retired or about to retire, who have taught at colleges or universities and will be available next year for employment as teachers or as mathematicians in industry, are invited to be listed without charge by the Mathematical Sciences Employment Register Committee, sponsored jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics. Before January 31, they should send to the committee chairman, Professor J. S. Frame, Mathematics Department, Michigan State University, East Lansing, Michigan, the following information: Name, date of birth, highest degree and where obtained, most recent teaching position or employment, present address, and date available. They should indicate availability for academic or industrial employment or both, and mention geographical or other preferences.

After February 15, interested employers may obtain a copy of this list of available retired mathematicians by writing Professor Frame, enclosing a self-addressed stamped envelope.

SUMMER INSTITUTE ON MATHEMATICS IN SOCIAL SCIENCE

The Social Science Research Council and the Mathematical Association of America are sponsoring a Summer Institute on Mathematics in Social Science for College Teachers of Mathematics to be held at Stanford University, California, June 24–August 17, 1957.

Further information and applications for admission, which are due not later than January 7, 1957, may be obtained from Social Science Research Council, 726 Jackson Place, N.W., Washington 6, D. C., and Professor Robert M. Thrall, Department of Mathematics, University of Michigan, Ann Arbor, Michigan.

NEW MATHEMATICS PROGRAM OF CARLETON COLLEGE

Carleton College announces that the Department of Mathematics has been awarded a four-year grant for $27,300 by the National Science Foundation. Disposition of the grant is under the direction of Professor K. O. May. The award is intended to enable the Carleton Mathematics Department to give special attention to students who show unusual promise in mathematics, regardless of their field of major interest. For such students in their first two years, the department will offer special small sections in which original work will be emphasized, and for advanced students, faculty-student collaboration in study and research will be offered. Research assistantships will be available for juniors and seniors who show unusual mathematical potential. These will be one year appointments, made on the basis of interest and ability, regardless of the student's declared major.

PERSONAL ITEMS

Brigham Young University announces the following new appointments: D. W. Robinson of Case Institute of Technology and D. W. Stoddard of the Sandia Corporation, Albuquerque, New Mexico, have been appointed to assistant professorships; Kent Nielson has been appointed to an instructorship.

Canisius College reports the following: Aaron Herschfeld of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship; Associate Professor C. G. Schilling has been promoted to a professorship; Assistant Professor F. J. Arena has been promoted to an associate professorship; Acting Chairman R. F. Tidd has been appointed Chairman of the Department of Mathematics.

Case Institute of Technology announces: J. W. Dettman, formerly a member of the technical staff of Bell Telephone Laboratories, Whippany, New Jersey, and Bayard Rankin, previously a research associate at Massachusetts Institute of Technology, have been appointed to assistant professorships; Frederick Way, III, recently an analytical engineer for Babcock & Wilcox Research Center, has been appointed Instructor and Assistant Director of the Computing Center; J. D. Neff of the University of Florida, and M. A. Stephens, formerly a lecturer at London University, have been appointed to instructorships.

Central Michigan College reports: W. J. Waggoner of the University of Wyoming has been appointed to an assistant professorship; Miss Buena Wilson has been appointed to an instructorship; Associate Professor L. G. Woodby has been promoted to the position of Head of the Department of Mathematics; Professor C. C. Richtmeyer, previously Head of the Department, has been promoted to the position of Dean of Faculty.

At Colorado Agricultural and Mechanical College: T. G. Burgess of Idaho State College, M. E. Sperline of University of Kansas, and K. J. Whitcomb, formerly an instructor in the Denver Public School System, have been appointed to instructorships; Associate Professor M. L. Madison has been promoted to the chairmanship of the Department of Mathematics.

Florida State University announces the following: Assistant Professor C. W. McArthur of Alabama Polytechnic Institute has been appointed to an associate professorship; J. L. Bagg of Michigan State University has been appointed to an assistant professorship; Professor Emeritus W. T. Stratton, formerly head of the Department of Mathematics, Kansas State College, has been appointed Lecturer; Assistant Professor R. L. Plunkett has been promoted to an associate professorship.

Georgetown University makes the following announcements: A. K. Aziz, previously a graduate assistant at the University of Maryland, and Rev. G. McConnell of Scranton University have been appointed to instructorships; Acting Chairman M. W. Oliphant had a research grant for the summer of 1956 supported by the President of Georgetown.

Kansas State College reports: T. D. Oxley, formerly an assistant at Purdue University, has been appointed to an assistant professorship; Assistant Professor L. E. Fuller has been promoted to an associate professorship.

At New Mexico College of Agriculture and Mechanic Arts: Assistant Professor D. L. Boyer of Arizona State College and Assistant Professor A. V. Fend of Florida State University have been appointed to assistant professorships: A. B. Gray and W. L. Walter have been appointed to instructorships; Assistant Professor R. B. Crouch has been promoted to an associate professorship.

University of Buffalo announces: P. G. Archer, formerly a mathematician for Bell Aircraft Corporation, Niagara Falls, New York, Associate Professor L. J. Montzingo, Jr., of Roberts Wesleyan College, and H. R. Stevens, previously a teaching fellow at Purdue University, have been appointed to instructorships; Teaching Fellows E. P. Rozycki, R. T. Sandberg, and Frederick R. White have been promoted to instructorships.

University of California, La Jolla, announces that the Regents of the University of California have voted to expand over a period of years the faculty and facilities of the La Jolla campus and to provide a graduate program emphasizing science and technology with such undergraduate instruction as is essential to support the graduate program. The La Jolla campus is the site of the Scripps Institution of Oceanography, which has been part of the University since 1912.

University of Michigan reports the following: A. L. Mayerson and Oscar Wesler have been appointed to assistant professorships; Associate Professor D. G. Higman of Montana State College has been appointed Visiting Assistant Professor; W. L. Davison, formerly an assistant at the University of Virginia, and Edward Halpern have been appointed to instructorships; Associate Professors G. E. Hay and Wilfred Kaplan have been promoted to professorships; Assistant Professors D. A. Darling and R. C. Lyndon have been promoted to associate professors; F. W. Gehring has been promoted to an assistant professorship; Professor G. Y. Rainich has retired with the title of Professor Emeritus; Associate Professor R. C. Lyndon is on leave of absence and is at the University of California as Visiting Associate Professor; Professor R. L. Wilder is on sabbatical leave during 1956–57.

At the University of Minnesota, Institute of Technology: Associate Professors Fulton Koehler, A. N. Milgram, and P. C. Rosenbloom have been promoted to professorships; Assistant Professors W. B. Fulks, J. B. Serrin, and M. L. Stein have been promoted to associate professorships; Bernard Lindgren has been promoted to an assistant professorship; Seizo Ito has been reappointed a Fulbright scholar and research fellow for 1956–57; Professor P. C. Rosenbloom is on leave of absence for the year 1956–57 and is at Harvard University as Visiting Professor; Professor L. W. Green, who is also on leave of absence during 1956–57, is at the Eidgenoessische Technische Hochschule, Zurich, Switzerland, on a grant from the American Swiss Society for Scientific Exchange; Lars Hormander of the University of Lund, Sweden, lectured and served as a research consultant for a month during the summer.

University of North Carolina announces the following: Assistant Professor J. S. MacNerney has been promoted to an associate professorship; Professors A. W. Hobbs and A. S. Winsor have retired.

University of Ottawa reports the following: A. C. Smith has been appointed to an assistant professorship; Assistant Professor Viktors Linis has been promoted to an associate professorship.

University of South Carolina makes the following announcements: J. D. McKnight, a scientist for Lockheed Missile Systems Division, has been appointed to an associate professorship; Professor J. B. Jackson has retired.

University of Texas reports: Assistant Professor F. R. Yett of Long Beach State College has been appointed to an assistant professorship; Professor H. S. Vandiver has been granted a postdoctoral fellowship by the National Science Foundation; Assistant Professor E. J. Prouse is on leave of absence for the first semester.

University of Toledo announces: Assistant Professor C. W. Thomson of Tennessee Polytechnic Institute has been appointed to an assistant professorship; A. N. Craig of the College of Wooster has been appointed to an instructorship.

At the University of Toronto: Professor H. S. M. Coxeter has been elected President of Section III of the Royal Society of Canada; A. P. Dempster of Princeton University, W. A. J. Luxemburg, formerly an assistant at Delft Institute of Technology, Holland, and Derek Naylor of A. V. Roe Aircraft Company, Manchester, England, have been appointed Lecturers; Associate Professor Abraham Robinson has been promoted to a professorship; Associate Professor D. A. F. Robinson has retired.

Washington University reports the following: Associate Professor Harvey Cohn of Wayne State University has been appointed to an associate professorship; Associate Professor T. L. Downs has been promoted to a professorship.

Western Illinois State College makes the following announcements: W. J. Thomsen of the Applied Physics Laboratory, Johns Hopkins University, has been appointed to an associate professorship; Assistant Professors J. J. Stipanowich and Arnold Wendt have been promoted to associate professorships.

W. E. Andrus, Jr., formerly a staff mathematician in charge of the Statistical Section of the Scientific Computation Laboratory, International Business Machines Corporation, Endicott, New York, has been promoted to the position of Manager of the Laboratory.

H. R. Bailey of the Naval Ordnance Plant has accepted a position as a mathematician with the Ohio Oil Company, Littleton, Colorado.

H. W. Baker, formerly a teaching assistant at Purdue University, has been appointed to an instructorship at Butler University.

J. G. Bennett, formerly a graduate student at the University of Illinois, is now a computer mathematician for North American Aviation, Dynamics Group, Columbus, Ohio.

I. O. Bentsen, previously a graduate student at the University of Rochester, has been appointed to an instructorship at Hobart College.

A. A. Benvenuto, formerly a graduate assistant at the University of Illinois, has a position as a mathematician with the Rand Corporation, Lincoln, Massachusetts.

George Caldwell of North Carolina State College has been promoted to an assistant professorship.

Assistant Professor C. D. Calhoon of the University of Toledo is on leave of absence and is at the Westinghouse Electric Corporation, Baltimore, Maryland.

Associate Professor G. G. Costello of Georgetown University is now a member of the staff of Loyola College, Baltimore, Maryland.

Associate Professor J. B. Crabtree of the University of New Hampshire has been appointed to an associate professorship at Stevens Institute of Technology.

Assistant Professor Frederic Cunningham, Jr., of the University of New Hampshire has been appointed Lecturer at Bryn Mawr College.

Assistant Professor E. L. Dolney of the School of Mines and Metallurgy, University of Missouri, has been appointed to an assistant professorship at Eastern Montana College.

T. C. Fry, assistant to the President of Bell Telephone Laboratories, retired from the Laboratories on October 1, 1956, after completing more than 40 years of Bell System service. Dr. Fry plans to engage in consulting work.

R. G. Fryer of the University of Buffalo is now in the United States Army.

Assistant Professor Albert Furman of Kansas State College has accepted a position with Lockheed Aircraft Company, California.

T. M. Gallie, Jr., previously a research engineer at Houston Research Center, Humble Oil and Refining Company, Texas, has been appointed to an assistant professorship at Duke University.

Harold Glander has been appointed to an associate professorship at Carroll College, Wisconsin.

A. G. Grace, Jr., has a position as an operations research analyst for Electric Boat Company, Groton, Connecticut.

Donald Greenspan of the University of Maryland has accepted a position as a research engineer for Hughes Aircraft Company, Culver City, California.

D. S. Greenstein of the Moore School of Electrical Engineering, University of Pennsylvania, has a position as a professional engineer with the Radio Corporation of America, Camden, New Jersey.

G. R. Hagan has been appointed to an instructorship at St. John's University.

J. J. Harton, Jr., of the University of California, has accepted a position as a mathematician with Motorola, Inc., Riverside, California.

Smith Higgins, Jr., of the University of Indiana, South Bend Extension, has been appointed Director of the Kokomo Center of Indiana University.

M. H. Hoehn of the University of New Mexico has been appointed to an instructorship at Santa Rosa Junior College.

J. E. Hoffman of the University of Oklahoma has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

R. T. Hood, formerly an assistant at the University of Chicago, has been appointed to an assistant professorship at Ohio University.

N. W. Johnson, Jr., previously employed by the Container Corporation of America, Chicago, Illinois, has been appointed Lecturer at Geneva College.

B. C. Kenny, formerly a senior mathematician with the Vitro Corporation of America, Eglin Air Force Base, Florida, is employed now as a senior engineer at the Westinghouse Electric Corporation, Baltimore, Maryland.

I. I. Kolodner, recently a senior scientist at the Institute of Mathematical Sciences, New York University, has been appointed to a professorship at the University of New Mexico.

Assistant Professor Harold W. Kuhn of Bryn Mawr College has been promoted to an associate professorship.

Assistant Professor L. H. Lange of Valparaiso University has been appointed to an instructorship at the University of Notre Dame.

C. E. Lemke, previously a professional engineer with the Radio Corporation of America, Camden, New Jersey, has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Eugene Levin of the University of California, Los Angeles, has accepted a position as supervisor of the mathematical analysis group at Ramo-Wooldridge Corporation, Los Angeles, California.

Walter Littman, formerly a research assistant at the Institute for Mathematical Sciences, New York University, has been appointed to an instructorship at the University of California.

P. G. Loewner of the University of California has accepted a position as a programmer and numerical analyst in the computing department of North American Aviation, Incorporated, Los Angeles, California.

J. N. Mangnall of the University of Buffalo has accepted a position as a research mathematician at the Los Angeles Computation Center, Inglewood, California.

J. P. Menard, previously a teaching assistant at the Catholic University of

America, has a position as a mathematician at the National Bureau of Standards, Washington, D. C.

A. S. Merrill, Vice-President of Montana State University, has retired.

Michael Montalbano, formerly a mathematician with Kaiser Steel Corporation, Fontana, California, has been appointed Director of Computer Research of the Corporation, Oakland, California.

Assistant Professor J. C. Morelock of Alabama Polytechnic Institute has been appointed to a professorship at King College.

C. R. Morris, recently an associate engineer at Lockheed Aircraft Corporation, Burbank, has been appointed to an associate professorship at Detroit Institute of Technology.

Rev. T. F. Mulcrone, formerly of St. Charles College, Louisiana, has been appointed to an assistant professorship at Loyola University, Louisiana.

R. V. Nolan of the University of Buffalo is teaching now at Cattaraugus High School, New York.

G. W. O'Shaughnessy, previously a teaching assistant at the University of Minnesota, is employed as a computing engineer at Rocketdyne, Division of North American Aviation, Canoga Park, California.

E. J. Pellicciaro of Duke University has been appointed to an assistant professorship at the University of Delaware.

Brother Vincent Rathbun, formerly a graduate student at Case Institute of Technology, has been appointed to an instructorship at the University of Dayton.

T. C. Richards, previously a student at San Jose State College, is teaching now at Riverdale High School, California.

Assistant Professor Hanno Rund of the University of Toronto has been appointed Head of the Department of Mathematics of the University of Natal, Durban, South Africa.

P. T. Schaefer, formerly an assistant at the University of Rochester, has been appointed to an assistant professorship at College for Teachers at Albany, State University of New York.

C. B. Sensenig of Bucknell University has been appointed to an assistant professorship at Lafayette College.

D. T. Sigley, previously a physicist at American Machine & Foundry Company, Greenwich, Connecticut, has been appointed Chief Engineer at Firestone Tire and Rubber Company, Los Angeles, California.

F. W. Sinden has accepted a position as a member of the technical staff of Bell Telephone Laboratories, Murray Hill, New Jersey.

Assistant Professor W. A. Small of Alfred University has been appointed to an assistant professorship at Grinnell College.

Andrew Sobczyk of Los Alamos Scientific Laboratory, New Mexico, has been appointed to a professorship at the University of Florida.

W. G. Spohn, Jr., of the University of Delaware has been appointed to an instructorship at Bowling Green State University.

Associate Professor W. H. Spragens, Jr., of the University of Mississippi has been appointed to an associate professorship at the University of Louisville.

L. W. Stark, formerly a mathematician at Redstone Arsenal, Huntsville, Alabama, has been appointed to an assistant professorship at Butler University.

Associate Professor Ruth W. Stokes is on leave of absence from Syracuse University during 1956–1957 and is Visiting Lecturer at the American University of Beirut.

Adrian Struyk, a teacher at Clifton High School, New Jersey, has been promoted to the position of Head of the Department of Mathematics.

R. C. Thompson of the University of British Columbia has accepted a position as defence scientific officer with the Defence Research Board, Ottawa, Canada.

Assistant Professor Henry Tucker of Kansas State College has accepted a position at the University of Arizona.

O. L. Wadkins, Jr., formerly an electronic engineer at U. S. Naval Ordnance Test Station, Pasadena, California, is employed as an electrical engineer at Armour Research Foundation, Tucson, Arizona.

J. V. Wehausen, formerly executive director of Mathematical Reviews, has been appointed Associate Research Mathematician at the University of California.

J. C. Wilson of Central College has been appointed to an assistant professorship at Fenn College.

Professor W. E. Wilson of Pennsylvania State University has been appointed Dean of the School of Engineering, Pratt Institute.

A. W. Yonda, previously a mathematician at the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, has been appointed to an instructorship at Temple University.

David Zeitlin of the University of Minnesota has accepted a position as a mathematician at Remington Rand Univac, St. Paul, Minnesota.

N. S. Fan, an assistant at Washington University, died in October, 1955.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE SEVENTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The seventeenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 2, 1957. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not received a college degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may do so by filling out the application blank which, together with an announcement of the forthcoming competition, is mailed to various colleges and universities early in January, and returning same to Professor L. E. Bush, Director, 301 Merrill Hall, Kent State University, Kent, Ohio. If no blank is received by January 15th, please write to the director for one. All applications must be filed with the director not later than February 10, 1957. If three candidates are presented from any one college or university, they are to constitute a team. If more than three candidates are presented from any one college or university, the team of three must be named on the application. Fewer than three candidates from one college or university may compete as individual contestants. Team members also compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the director in cases where it would entail unusual inconvenience to the contestant. Sealed copies of the examinations will be sent to the supervisor in time for the examination and are not to be opened before the hour set for the examination.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman), an annual $2,500.00 William Lowell Putnam Prize Scholarship to one of the first five con-

testants, this to be available either immediately or on completion of the student's undergraduate work.

Reports of the sixteen previous competitions and examination questions will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August–September 1947, December 1948, August-September 1949, 1950, 1951, October 1952, 1953, 1954, 1955, and January 1957.

EDITORIAL

With this issue my term as editor comes to a close, and I am turning over the affairs of the MONTHLY to the very capable hands of Professor Ralph D. James. It has been a great pleasure for me to serve the MONTHLY in various capacities during the past ten years, and in my retirement I shall miss my editorial correspondence with mathematicians throughout the world. I shall, however, be able to get along very well without the headaches of the frequent deadlines of our publication schedule.

I am sure that our readers are aware that the publication of the MONTHLY is very much a cooperative affair rather than a one-man show. I should like to take this occasion to express my deep appreciation to my staff of associate editors whose only reward is the satisfaction of a job well done. Without the efforts of such a dependable staff the publication of a diversified journal like the MONTHLY would be impossible. My warm thanks should also go to Professor Harry Gehman, whose careful proof reading has often saved us from awkward blunders, and to my editorial assistant, Mrs. Helen Zuckerman, whose knowledge of mathematics and the English language has added greatly to the style and appearance of the MONTHLY.

Even with all this help, there is still plenty for the editor to do, and he has an unusual opportunity and responsibility for shaping the course of the journal. The MONTHLY has always been something of a personal enterprise, and I hope that its increasing size and complexity will not force a change in this characteristic aspect. The choice of a new editor is consequently a matter of great importance, and in retiring I am particularly happy that my duties are passing to a new editor in whom I have the utmost confidence.

With many thanks to all my friends who helped me during my term as editor, and with my very best wishes for the future to Ralph James,

CARL B. ALLENDOERFER

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: J. P. Ballantine, Y. Bar-Hillel, R. G. Bartle, A. F. Bausch, R. A. Beaumont, Leonard Carlitz, W. B. Carver, D. G. Chapman, Nathaniel Coburn, N. A. Court, H. S. M. Coxeter, Harley Flanders, H. A. Forrester, Tomlinson Fort, R. M. Foster, J. S. Frame, H. M. Gehman, H. W. Gould, G. E. Hay, L. A. Henkin, F. B. Hildebrand, Franz Hohn, P. S. Jones, J. M. Kingston,

Erwin Kleinfeld, Morris Kline, L. A. Kokoris, H. W. Kuhn, Norman Levinson, A. E. Livingston, M. D. Marcus, N. H. McCoy, E. A. Michael, W. O. J. Moser, E. P. Northrop, H. E. Salzer, John Todd, R. L. Vaught, L. M. Weiner.

Mathematical Notes: R. P. Agnew, David Blackwell, A. A. Blank, R. W. Brink, W. E. Byrne, E. A. Cameron, Leonard Carlitz, W. B. Carver, V. F. Cowling, Ky Fan, William Feller, N. J. Fine, J. E. Freund, A. W. Goodman, P. R. Halmos, Philip Hartman, N. J. Heerema, Fritz Herzog, P. G. Hoel, A. S. Householder, E. D. Jenkins, B. W. Jones, L. M. Kelly, V. L. Klee, Jr., H. W. Kuhn, H. L. Lee, D. H. Lehmer, J. H. Lehner, E. R. Lorch, N. H. McCoy, D. D. Miller, F. D. Murnaghan, Zeev Nehari, Oystein Ore, Harry Pollard, W. T. Reid, John Riordan, Arthur Rosenthal, W. S. Snyder, R. D. Stalley, Gabor Szegö, C. T. Taam, W. R. Utz, R. J. Walker, L. M. Weiner, George Whaples, W. M. Whyburn, E. M. Wright.

Classroom Notes: Reinhold Baer, J. P. Ballantine, D. H. Ballou, R. H. Bing, Herbert Buseman, S. H. Crandall, P. D. Crout, Evelyn Frank, Philip Franklin, Gordon Fuller, Abe Gelbart, H. J. Hamilton, F. B. Hildebrand, P. G. Hoel, L. N. Howard, R. E. Johnson, M. S. Klamkin, J. J. Levin, Z. A. Melzak, Max Morris, George Pólya, Walter Prenowitz, W. F. Reynolds, R. A. Rosenbaum, S. S. Saslaw, D. J. Struik, J. H. Wolfe.

CALENDAR OF FUTURE MEETINGS

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Spring, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS

KENTUCKY, Berea College, Berea, April 20, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College of William and Mary, Williamsburg, Virginia, December 1, 1956.

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, March 23, 1957.

MINNESOTA

MISSOURI, Southeast Missouri State College, Cape Girardeau, Spring, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, June 14, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado School of Mines, Golden, Spring, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

THE CARUS MATHEMATICAL MONOGRAPHS

A new Carus Monograph has been published in 1956. The title of this Monograph is:

MONOGRAPH 11: *Irrational Numbers,* by Ivan Niven.

Each member of the Association may purchase one copy of this Carus Monograph at the special price of $1.75. Orders should be addressed to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies of Monograph 11 for members and copies for nonmembers may be purchased at $3.00 from John Wiley and Sons, 440 Fourth Avenue, New York 16, New York.

Professor Niven's Monograph on *Irrational Numbers* provides an exposition of some central results on irrational, transcendental, and normal numbers. There is a complete treatment by elementary methods of the irrationality of the exponential, logarithmic, and trigonometric functions with rational arguments. The approximation of irrational numbers by rationals, up to such results as the best possible approximation of Hurwitz, is also given with elementary techniques. The last third of the monograph treats normal and transcendental numbers, including the transcendence of π and its generalization in the Lindemann theorem, and the Gelfond-Schneider theorem.

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RESEARCH MATHEMATICIAN: Ph.D. in Mathematical Physics or applied mathematics with B.S. in Physics or related discipline. To set up mathematical models for computation, assist experimentalists in setting up data reduction schemes for computer or in other methods of analyzing experimental results mathematically as well as in setting up experimental designs. Originate the application of new techniques for these purposes. Several years of experience in applied mathematics, mathmatical physics or operations research desirable but not essential for the right man. Please send resume including salary desired to R. J. Gladieux, Personnel Administrator, Metals Research Laboratories, Electro Metallurgical Company, P.O. Box 580, Niagara Falls, New York.

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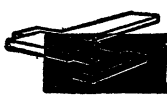
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